Asymptotic behavior of shell eigenvalue problems

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Outline

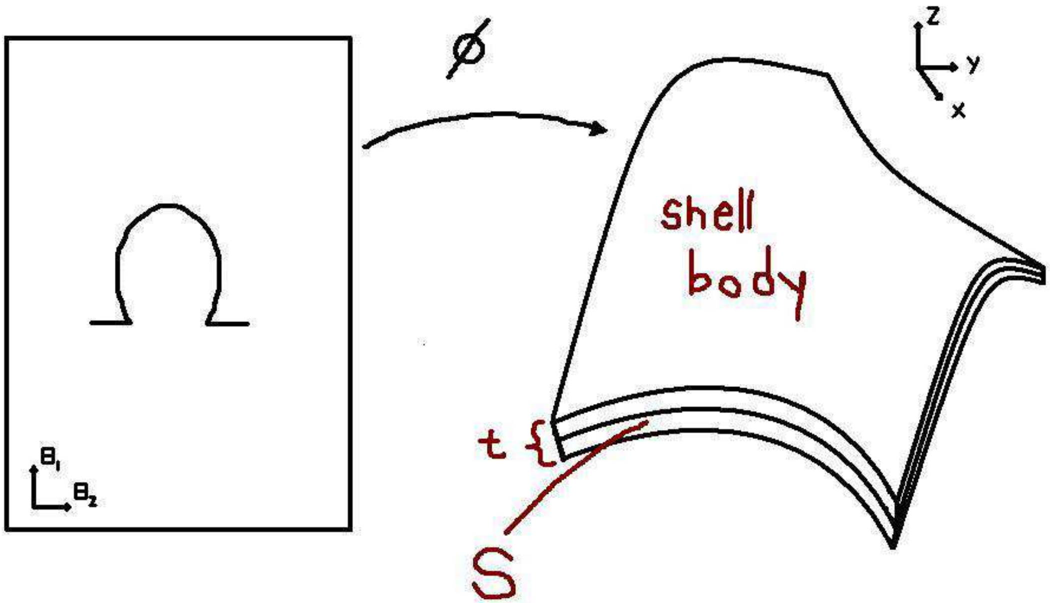
- Introduction to shell analysis [source problem]
  - Koiter model for shells
  - Asymptotic behavior of shells
- The shell eigenvalue problem
  - "Flexural shells"
  - "Non-flexural" shells
- An extended example : the clamped cylinder
  - Theoretical results
  - Numerical tests

The shell deformation problem

A shell is a material body in which one of the characteristic dimensions (the "thickness") is remarkably smaller than the other two.

Given an (homogeneous) thickness t, the shell body is uniquely determined by its middle surface $\mathcal{S} \in \mathbb{R}^3$. We assume that $\mathcal{S}$ is described as the image of a C^2 map

$$\mathbf{\Phi} : \Omega \in \mathbb{R}^2 \longrightarrow \mathcal{S} \in \mathbb{R}^3$$



The shell deformation problem

We assume an isotropic, homogeneous, linearly elastic material in the realm of small deformations.
We are interested in the displacement of the shell, given a load and some boundary conditions at the shell edges.

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In order to fix the ideas, we are going to consider a particular shell model, the well known Koiter model.

Koiter model - variational formulation

$$\begin{cases} \text{Find } \boldsymbol{u}_t \in V := [H^1_{\mathcal{BC}}(\Omega)]^2 \times H^2_{\mathcal{BC}}(\Omega) \text{ such that} \\ t^3 A_b(\boldsymbol{u}_t, \boldsymbol{v}) + t A_m(\boldsymbol{u}_t, \boldsymbol{v}) =< f, \boldsymbol{v} > \quad \forall \boldsymbol{v} \in V \end{cases}$$

where

Ω is the domain of the shell map

$$\boldsymbol{\Phi} : \Omega \in \mathbb{R}^2 \longrightarrow \mathcal{S} \in \mathbb{R}^3$$

Note that the original three dimensional domain of the problem is reduced to the bi-dimensional domain Ω.
The problem is written on the middle surface of the shell and the thickness $0 < t << |\mathcal{S}|$ becomes a parameter.

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where

V is the space of admissible displacements of the middle surface, written in the contravariant tangent basis of $\boldsymbol{\Phi}$ completed with the normal direction.

$f \in V'$ represents the applied load

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where

A_b and A_m are two symmetric and continuous bilinear forms on the space V.

A_b is the bending form, and takes into account the elastic energy related to the "flexion" (change in curvature) of the shell.

A_m is the membrane form, and takes into account the elastic energy related to the "stretching" (change in metric) of the shell.

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NOTE : due to the coercivity on V of $A_b + A_m$, there is one and only one solution of the problem for all values of the parameter $t > 0$.

Asymptotic behavior of shells

Let the space of in-extensional displacements

$$V_0 = \{\boldsymbol{v} \in V \ : \ A_m(\boldsymbol{v}, \boldsymbol{v}) = 0\} \ .$$

Moreover let W the closure of the orthogonal space

$$\{\boldsymbol{w} \in V \ : \ A_b(\boldsymbol{w}, \boldsymbol{v}) + A_m(\boldsymbol{w}, \boldsymbol{v}) = 0 \, \forall \boldsymbol{v} \in V_0\}$$

with respect to the norm weaker norm

$$\|\boldsymbol{v}\|_W = A_m(\boldsymbol{w}, \boldsymbol{w})^{1/2} \ .$$

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Then the classical theory singles out two families of shells in dependance of the asymptotic behavior ($t \to 0$):

$$\exists \boldsymbol{v} \in V_0 \ : < f, \boldsymbol{v} > \neq 0 \quad \Longrightarrow \quad \text{bending dominated}$$

$$< f, \boldsymbol{v} >= 0 \ \forall \boldsymbol{v} \in V_0 \ , \ f \in W' \quad \Longrightarrow \quad \text{membrane dominated}$$

Bending dominated shell problems

$$\exists \boldsymbol{v} \in V_0 \ : < f, \boldsymbol{v} > \neq 0$$

The solution $\boldsymbol{u}_t$ of the scaled problem

$$\begin{cases} \text{Find } \boldsymbol{u}_t \in V \text{ such that} \\ t^3 A_b(\boldsymbol{u}_t, \boldsymbol{v}) + t A_m(\boldsymbol{u}_t, \boldsymbol{v}) = t^3 < f, \boldsymbol{v} > \quad \forall \boldsymbol{v} \in V \end{cases}$$

converges in V to $\boldsymbol{u}_0$, solution of

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Moreover the bending part of the energy dominates the problem

$$\lim_{t \to 0^+} R(t) := \lim_{t \to 0^+} \frac{t^3 A_b(\boldsymbol{u}_t, \boldsymbol{u}_t)}{t^3 A_b(\boldsymbol{u}_t, \boldsymbol{u}_t) + t A_m(\boldsymbol{u}_t, \boldsymbol{u}_t)} = 1 \ .$$

Membrane dominated shell problems

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Locking in shells ("small" t)

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which are essentially singularly perturbed problems.

Shell eigenvalue problems

- Can a similar asymptotic classification be introduced for the problem of shell vibration?
- What kind of energy do we expect to dominate the problem?
- When can locking be expected in the finite element analysis of shell vibration?

The shell eigenvalue problem :

$$\begin{cases} \text{Find } \boldsymbol{u}_t \in V,\ \lambda_t \in \mathbb{R}^+ \text{ such that} \\ t^3 A_b(\boldsymbol{u}_t, \boldsymbol{v}) + t A_m(\boldsymbol{u}_t, \boldsymbol{v}) = \lambda_t (\boldsymbol{u}_t, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in V \\ ||\boldsymbol{u}_t||_{L^2(\Omega)} = 1\ . \end{cases}$$

Shell eigenvalue problems

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- What kind of energy do we expect to dominate the problem?
- When can locking be expected in the finite element analysis of shell vibration?

In the sequel we will restrict our attention to the first vibration modes (i.e. lowest eigenvalue and related eigenfunctions).

"Flexural" shells

In the case

$$V_0 \neq \{0\}$$

the lowest modes are, as expected, the "weaker" bending modes.

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PROPOSITION

There exist constants C_1 and C_2, independent of t, such that

$$C_1 t^3 \leq \lambda_t \leq C_2 t^3 \ .$$

Furthermore, it holds

$$\lim_{t \to 0^+} R(t, \boldsymbol{u}_t) = \lim_{t \to 0^+} \frac{t^3 A_b(\boldsymbol{u}_t, \boldsymbol{u}_t)}{t^3 A_b(\boldsymbol{u}_t, \boldsymbol{u}_t) + t A_m(\boldsymbol{u}_t, \boldsymbol{u}_t)} = 1 \ .$$

"Non-Flexural" shells

On the contrary, the condition

$$V_0 = \{0\}$$

is NOT sufficient to guarantee a membrane dominated behavior of the first vibration modes.

"Non-Flexural" shells

PROPOSITION *(BdV and Lovadina, submitted)*

Let

$$\alpha = \inf \left\{ 2\theta + 1 \ : \ L^2(\Omega) \subseteq (W', V')_{\theta,\infty} \right\} .$$

Then

$$\lambda_t \ \sim \ t^{\alpha} \ .$$

Furthermore, under additional reasonable assumptions, it holds

$$\lim_{t \to 0^+} R(t, \boldsymbol{u}_t) = \lim_{t \to 0^+} \frac{t^3 A_b(\boldsymbol{u}_t, \boldsymbol{u}_t)}{t^3 A_b(\boldsymbol{u}_t, \boldsymbol{u}_t) + t A_m(\boldsymbol{u}_t, \boldsymbol{u}_t)} = \frac{\alpha - 1}{2} \ .$$

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Then

$$\inf\{\beta \ : \ t^{\beta}\lambda_t^{-1} \in L^{\infty}(0,1)\} = \alpha \qquad\qquad (\lambda_t \sim t^{\alpha}) .$$

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$$\lim_{t\to 0^+} R(t, \boldsymbol{u}_t) = \lim_{t\to 0^+} \frac{t^3 A_b(\boldsymbol{u}_t, \boldsymbol{u}_t)}{t^3 A_b(\boldsymbol{u}_t, \boldsymbol{u}_t) + t A_m(\boldsymbol{u}_t, \boldsymbol{u}_t)} = \frac{\alpha - 1}{2} .$$

"Non-Flexural" shells

COROLLARY

If

$$W \subseteq L^2(\Omega) \qquad\qquad (*)$$

then the first vibration modes are membrane modes, i.e.

$$\lambda_t \sim t \,, \qquad \lim_{t \to 0^+} R(t, \boldsymbol{u}_t) = 0 \,.$$

Otherwise the first modes are expected to behave in a "mixed" way.

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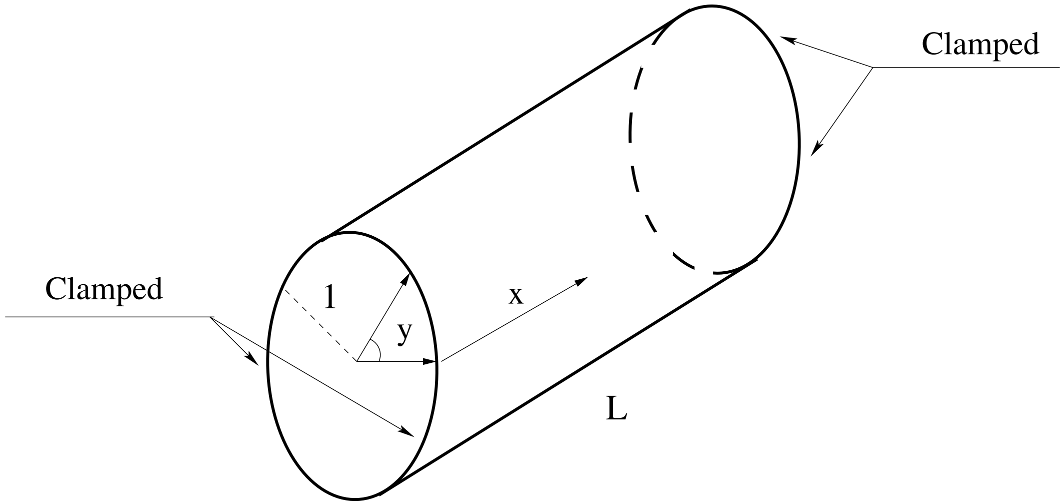
Otherwise the first modes are expected to behave in a "mixed" way.

• Note that condition $(*)$ above is in general true only for elliptic shaped shells with sufficiently strong boundary conditions.

• LOCKING may be present even if $V_0 = \{0\}$!

Clamped cylindrical shells

We consider the eigenvalue problem of a clamped cylindrical shell.



It holds

$$V_0 = \{0\} \, .$$

$$\|\boldsymbol{v}\|_W^2 := A_m(\boldsymbol{v}, \boldsymbol{v}) \simeq \|v_{1,x}\|_{L^2(\Omega)}^2 + \|v_{1,y} + v_{2,x}\|_{L^2(\Omega)}^2 + \|v_{2,y} + v_3\|_{L^2(\Omega)}^2$$

Clamped cylindrical shells

Applying the previous result and interpolation space theory, it can be proved that for the smaller eigenvalue and related eigenfunctions

$$\lambda_t \sim t^2 \,, \qquad \lim_{t \to 0^+} R(t, \boldsymbol{u}_t) = \frac{1}{2}$$

Therefore the first vibration modes present an even mix of bending and membrane behavior.

We will now analyze this classical engineering problem more in deep using different techniques.

Clamped cylindrical shells

The Euler equations of the problem are

$$\begin{cases} -\beta_{11,x} - 2\beta_{12,y} = t^{-1}\lambda\, u_1 \\ -2\beta_{12,x} - \beta_{22,y} = t^{-1}\lambda\, u_2 \\ t^2(u_{3,xxxx} + u_{3,yyyy}) + \beta_{22} = t^{-1}\lambda\, u_3\ , \end{cases}$$

where the membrane operators

$$\beta_{11} = u_{1,x}\ , \qquad \beta_{12} = \frac{1}{2}(u_{1,y} + u_{2,x})\ , \qquad \beta_{22} = u_{2,y} + \frac{u_3}{R}\ .$$

Given the particular form of the problem, it is not restrictive to assume a Fourier expansion in the angular variable:

$$\boldsymbol{u}(x,y) = \sum_{K\in\mathbb{N}} \begin{pmatrix} u_1^K(x)\cos(Ky) \\ \pm\, u_2^K(x)^{\pm}\sin(Ky) \\ u_3^K(x)\cos(Ky) \end{pmatrix}\ , \quad K\in\mathbb{N}\ .$$

Clamped cylindrical shells

Due to orthogonality properties of the above functions, the problem decouples into a sequence of one dimensional problems, each associated to a single wave number K:

$$\begin{cases} -\beta_{11}' - 2K\beta_{12} = t^{-1}\lambda_K u_1^K \\ -2\beta_{12}' + K\beta_{22} = t^{-1}\lambda_K u_2^K \\ t^2(u_3^{K\,''''} + K^4 u_3^K) + \beta_{22} = t^{-1}\lambda_K u_3^K \end{cases}$$

where now

$$\beta_{11} = u_1^{K\,\prime}\,, \qquad \beta_{12} = \frac{1}{2}(-Ku_1^K + u_2^{K\,\prime})\,, \qquad \beta_{22} = Ku_2^K + u_3^K\,.$$

Clamped cylindrical shells

Essentially with the use of scaling arguments, we can prove the following asymptotic ($t \to 0$) result:

PROPOSITION (*BdV, Hakula and Pitkäranta)*

Assuming that $\lambda \sim t^\gamma$ *for some* $\gamma \in \mathbf{R}$, *we get the following amplitude scalings*

$$\lambda \sim t^2 \,, \quad K \sim t^{-1/4} \,, \quad Lng \sim 1 \,,$$
$$u_1 \sim t^{1/2} \,, \quad u_2 \sim t^{1/4} \,, \quad u_3 \sim 1 \,,$$
$$\beta_{11} \sim t^{1/2} \,, \quad \beta_{12} \sim t^{3/4} \,, \quad \beta_{22} \sim t \,.$$

Note: the simplified eigenvalue behaves as $\lambda \sim t^3 K^4 + tK^{-4}$ *which suggests* $K \sim t^{-1/4}$ *and an even balance between the two energies.*

Presence of locking

Introducing the aforementioned scalings (in t) into the definition of the elastic energy, we get

$$1 \sim \bar{E} := t\,E \sim \|\bar{u}_{3,yy}\|^2_{L^2(\omega)} + t\|\bar{u}_{3,xx}\|^2_{L^2(\omega)} + \|\bar{u}_{1,x}\|^2_{L^2(\omega)}$$
$$+t^{-1/2}\|\bar{u}_{1,y} + \bar{u}_{2,x}\|^2_{L^2(\omega)} + t^{-1}\|\bar{u}_{2,y} + \bar{u}_3\|^2_{L^2(\omega)}\ ,$$

where all the barred quantities above

$$\bar{E}\ ,\ \bar{u}_{3,yy}\ ,\ \bar{u}_{3,xx}\ ,\ \bar{u}_{1,x}\ ,\ \ldots\ldots\ ,\ \bar{u}_3 \sim \mathbf{1}\ .$$

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$$+ t^{-1/2}\|\bar{u}_{1,y} + \bar{u}_{2,x}\|^2_{L^2(\omega)} + t^{-1}\|\bar{u}_{2,y} + \bar{u}_3\|^2_{L^2(\omega)} \; ,$$

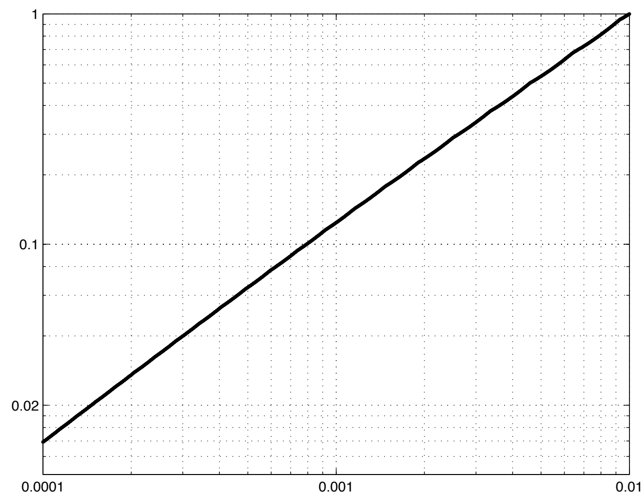
where all the barred quantities above

$$\bar{E} \; , \; \bar{u}_{3,yy} \; , \; \bar{u}_{3,xx} \; , \; \bar{u}_{1,x} \; , \; ...... \; , \; \bar{u}_3 \sim \mathbf{1} \; .$$

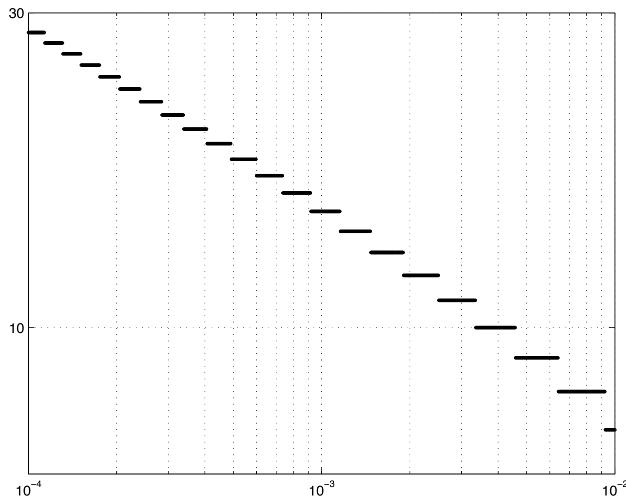
Then, the negative powers of t above represent a constraint (at the limit) and therefore a source of locking.

Note: in classical bending dominated shells the "locking factor" is typically t^{-2}.

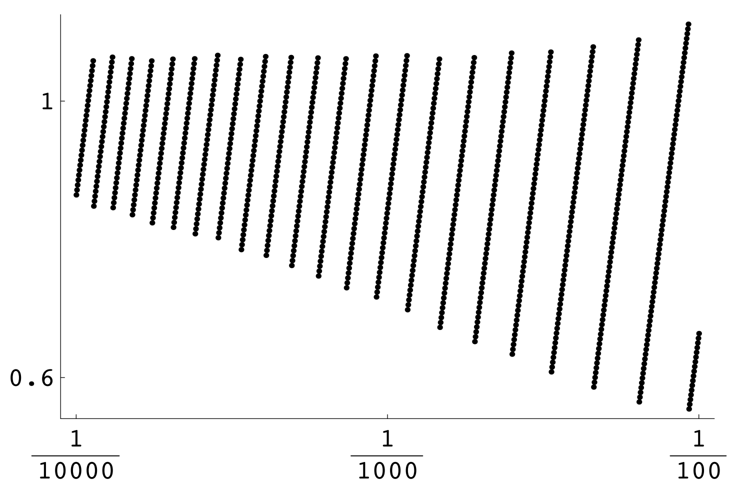
ONE DIMENSIONAL TESTS



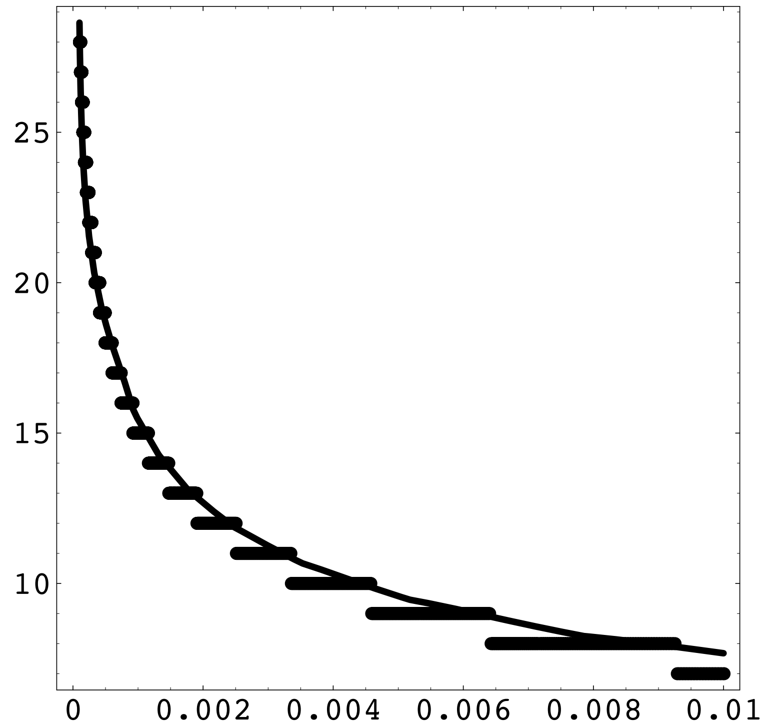
Minimum λ (divided by t) as function of t. Logarithmic scale.



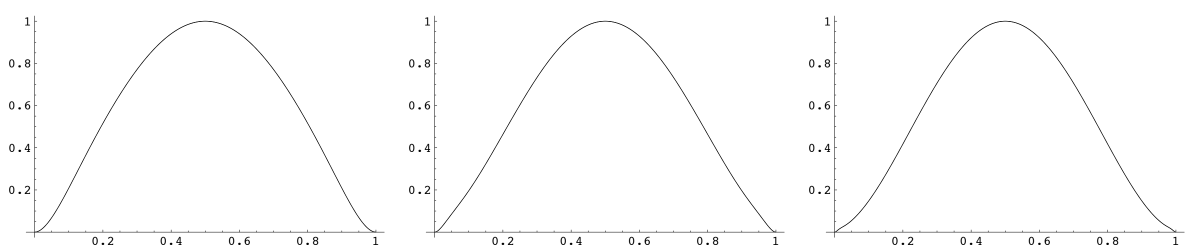
K related to the minimum λ as function of t. Logarithmic scale.



Energy ratios A_b/A_m at minimum λ as function of t. Logarithmic scale.



Energy balance curve ($A_b/A_m = 1$) and K related to the minimum λ as functions of t.



Minimum modes, transverse deflection, scaled to 1; thicknesses $t = 10^{-2}, 10^{-3}, 10^{-4}$.

TWO DIMENSIONAL TESTS

Mesh	D.o.f	K	λ
$10^* \times 70$	3905	4	4.50
$20^* \times 70$	7455	4	3.83
$30^* \times 70$	11005	4	3.72
$10^* \times 140$	7755	5	3.60
$20^* \times 140$	14805	5	2.91
$30^* \times 140$	21855	5	2.79
$10^* \times 280$	15455	6	3.06
$20^* \times 280$	29505	6	2.36
$30^* \times 280$	43555	6	2.22
Target		7	0.99

Rectangular elements, p=1. Minimum computed λ and related K. Thickness $t = 10^{-2}$.

Mesh	D.o.f	K	λ
$10^* \times 150$	8305	5	2.78
$20^* \times 150$	15855	5	1.97
$30^* \times 150$	23405	5	1.47
$10^* \times 300$	16555	7	2.16
$20^* \times 300$	31605	7	1.30
$30^* \times 300$	46655	7	1.70
$10^* \times 450$	24805	8	1.86
$20^* \times 450$	47355	8	1.04
$30^* \times 450$	69905	8	0.92
Target		15	0.12

Rectangular elements, p=1. Minimum computed λ and related K. Thickness $t = 10^{-3}$.

Mesh	D.o.f	K	λ
$10^* \times 24$	3945	6	1.33
$20^* \times 24$	7645	6	1.31
$30^* \times 24$	11345	6	1.31
$10^* \times 50$	8105	7	1.04
$20^* \times 50$	15705	7	1.04
$30^* \times 50$	23305	7	1.04
$10^* \times 70$	11305	7	1.02
$20^* \times 70$	21905	7	1.02
$30^* \times 70$	32505	7	1.02
Target		7	0.99

Rectangular elements, p=2. Minimum computed λ and related K. Thickness $t = 10^{-2}$.

Mesh	D.o.f	K	λ
$10^* \times 38$	6185	9	0.44
$20^* \times 38$	11985	10	0.42
$30^* \times 38$	17785	10	0.41
$10^* \times 75$	12105	13	0.18
$20^* \times 75$	23455	14	0.17
$30^* \times 75$	34805	14	0.17
$10^* \times 150$	24105	15	0.13
$20^* \times 150$	46705	15	0.13
$30^* \times 150$	69305	15	0.13
Target		15	0.12

Rectangular elements, p=2. Minimum computed λ and related K. Thickness $t = 10^{-3}$.