A posteriori error analysis of finite element approximation of quasi-Newtonian flows

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Targets

Our target is to obtain a reliable finite element solution of quasi-Newtonian steady incompressible flows in an efficient way.
To reach this target we need:

- computable quantities that tell us where the error in our numerical solution is large: upper bound of the error;
- computable quantities that tell us where the error in our numerical solution is small: lower bound of the error;
- an adaptive method that automatically refine the mesh where the error in the numerical solution is large and coarsen the mesh where the error is small.

Remark

- *refining* $\rightsquigarrow$ *reliability of the numerical solution;*
- *coarsening* $\rightsquigarrow$ *efficiency on the numerical method.*

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- *refining $\rightsquigarrow$ reliability of the numerical solution;*
- *coarsening $\rightsquigarrow$ efficiency on the numerical method.*

Outline

- Model problem;
- Weak formulation of the continuous problem and discrete weak formulation;
- Quasi-norm, uniform monotonicity and local Hölder continuity;
- Upper bounds of the error:
  - Upper bound in terms of residual functionals (weak residuals);
  - Upper bound in terms of computable residuals (strong residuals);
- Lower bounds of the error:
  - Lower bound in terms of residual functionals (weak residuals);
  - Lower bound in terms of computable residuals (strong residuals);
- Numerical experiments:
  - Adaptive method based on our *a posteriori* error estimates;
  - Power-law problems;
  - Carreau law problems.

Model problem

Continuous strong form

$$\begin{aligned} -\nabla \cdot \sigma &= \mathbf{f}, && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0, && \text{in } \Omega, \\ \mathbf{u} &= 0, && \text{on } \partial\Omega = \Gamma, \end{aligned}$$

where

$$\sigma(x, \mathbf{u}) = k(x, |e(\mathbf{u})|)e(\mathbf{u}) - pI$$

is the stress tensor and

$$e(\mathbf{u})_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i,j = 1, \ldots, d.$$

are the components of the strain tensor $e(\mathbf{u}) \in \mathbb{R}^{d\times d}_{\text{symm}}$.

- Ω bounded Lipschitz domain, $|\Omega| = 1$;
- $k(x, |e(\mathbf{u})|)$ is the viscosity coefficient.

Viscosity coefficient $k(x, |e(\mathbf{u})|)$

We want to deal with:

- *pseudoplastic fluids: increasing viscosity when* $|\,e(u)\,| \longrightarrow 0$;
- *dilatant fluids: decreasing viscosity when* $|\,e(u)\,| \longrightarrow 0$;

- *degenerate constitutive laws: the viscosity function can tend to* 0 *or to* ∞;
- *non-degenerate constitutive laws: the viscosity function is bounded away from* 0 *and from* ∞.

Viscosity coefficient $k(x,|e(\mathbf{u})|)$

Assumption

We assume that $k \in C(\bar{\Omega} \times (0,\infty))$ *and that, given* $r \in (1,\infty)$, *there exist constants* $\alpha \in [0,1]$ *and* $\varepsilon, K_1, K_2 > 0$ *such that, for all* $x \in \bar{\Omega}$,

(A1) $K_1[(t+s)^\alpha(1+t+s)^{1-\alpha}]^{r-2}(t-s) \le k(x,t)t - k(x,s)s$

for all $t \ge s > 0$

(A2) $k(x,t) \le K_2[t^\alpha(1+t)^{1-\alpha}]^{r-2}$ *for all* $t > 0$, *and*

$|k(x,t)t - k(x,s)s| \le K_2[(t+s)^\alpha(1+t+s)^{1-\alpha}]^{r-2}|t-s|$

for all $s,t > 0$ *satisfying* $|(s/t) - 1| \le \varepsilon$.

When $r \neq 2$, the parameter α measures the degree of degeneracy in $k(\cdot,\cdot)$ for a given value of $r \in (1,\infty)$ in the sense that the closer α is to 1 the more degenerate $k(\cdot,\cdot)$ is.

quasi-norm

Examples

(a) the classical linear Stokes equations which govern the stationary flow of a viscous incompressible Newtonian fluid:

$$k(x, |\, e(\mathbf{u}) \,|) = 2\mu,$$

corresponds to $r = 2$;

(b) the power-law model with

$$k(x, |\, e(\mathbf{u}) \,|) = 2\mu |\, e(\mathbf{u}) \,|^{r-2}$$

corresponds to $\alpha = 1$;

(c) the Carreau law

$$k(x, |\, e(\mathbf{u}) \,|) = k_\infty + (k_0 - k_\infty)(1 + \lambda |\, e(\mathbf{u}) \,|^2)^{(\theta-2)/2},$$

with $k_0 > k_\infty \geq 0$, $\lambda > 0$, $\theta \in (1, \infty)$;

- if $k_\infty = 0$, then $\alpha = 0$ with $r = \theta$;
- if $k_\infty > 0$, then $r = 2$ and $\theta \in (1, 2]$, (pseudoplastic behaviour).

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Continuous weak formulation

Find $\mathbf{u} \in \mathbf{V}$ and $p \in \mathbf{Q}$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}, \\ b(q, \mathbf{u}) &= 0 \qquad \forall q \in \mathbf{Q}, \end{aligned}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega k(x, |e(\mathbf{u})|) e(\mathbf{u}) : e(\mathbf{v}) \, d\Omega, \quad b(q, \mathbf{v}) = -\int_\Omega (\nabla \cdot \mathbf{v}) \, q \, d\Omega,$$

$\mathbf{V} = [\mathbf{W}_0^{1,r}(\Omega)]^d$, with the norm $\|\mathbf{v}\|_\mathbf{V} = \|e(\mathbf{v})\|_{\mathbf{L}^r(\Omega)}$,
$\mathbf{Q} = \mathbf{L}_0^{r'}(\Omega) = \mathbf{L}^{r'}(\Omega)/\mathbb{R}$, with the norm $\|q\|_\mathbf{Q} = \inf_{c \in \mathbb{R}} \|q + c\|_{\mathbf{L}^{r'}(\Omega)}$,
and the bilinear form $b(\cdot, \cdot)$ satisfies the following inf-sup condition:

$$\inf_{q \in \mathbf{Q}} \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(q, \mathbf{v})}{\|q\|_\mathbf{Q} \|\mathbf{v}\|_\mathbf{V}} \geq c_0 \qquad \forall q \in \mathbf{Q}.$$

Let $\mathbf{V}'$ and $\mathbf{Q}'$ denote the dual spaces of $\mathbf{V}$ and $\mathbf{Q}$.

Discrete weak formulation

Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in \mathbf{Q}_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(p_h, \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(q_h, \mathbf{u}_h) &= 0 \qquad \forall q_h \in \mathbf{Q}_h. \end{aligned}$$

- $\mathcal{T}_h$ shape-regular partition of Ω;
- $\mathbf{V}_h \subset \mathbf{V}$ continuous velocity F.E. space defined on $\mathcal{T}_h$;
- $\mathbf{Q}_h \subset \mathbf{Q}$ continuous/discontinuous pressure F.E. space on $\mathcal{T}_h$;
- the pair $(\mathbf{V}_h, \mathbf{Q}_h)$ satisfies the following inf-sup condition:

$$\inf_{q_h \in \mathbf{Q}_h} \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(q_h, \mathbf{v}_h)}{\|q_h\|_{\mathbf{Q}} \|\mathbf{v}_h\|_{\mathbf{V}}} \geq c_0', \quad c_0' \text{ independent of } h.$$

Quasi-norm in Sobolev space

To derive upper and lower bounds in the general case of a viscosity coefficient satisfying only the general assumptions described before we need to introduce a new family of tools.

For $\alpha \in [0,1]$ *and* $t \in (0,\infty)$, *let be*

$$\Xi_\alpha(r) = r^\alpha (1+r)^{1-\alpha},$$

for any $\alpha \in [0,1]$, $r \mapsto \Xi_\alpha(r)$ *is a strictly monotonic increasing and concave function of* $r \in (0,\infty)$.

Assumptions on $k(x,t)$

Quasi-norm in Sobolev space

Assumption

We assume that $k \in C(\bar{\Omega} \times (0,\infty))$ *and that, given* $r \in (1,\infty)$, *there exist constants* $\alpha \in [0,1]$ *and* $\varepsilon, K_1, K_2 > 0$ *such that, for all* $x \in \bar{\Omega}$,

(A1) $K_1[\Xi_\alpha(t+s)]^{r-2}(t-s) \leq k(x,t)t - k(x,s)s, \quad \forall t \geq s > 0,$

(A2) $k(x,t) \leq K_2[\Xi_\alpha(t)]^{r-2} \qquad \forall t > 0$, *and*

$$|k(x,t)t - k(x,s)s| \leq K_2[\Xi_\alpha(t+s)]^{r-2}|t-s|$$

for all $s,t > 0$ *satisfying* $|(s/t) - 1| \leq \varepsilon$. Monot&Cont

Definition

For all $\mathbf{v}, \mathbf{w} \in [\mathrm{W}^{1,r}(\Omega)]^d$ with $1 < r < \infty$, let us define the following quasi-norm:

$$|\mathbf{v}|^2_{(\mathbf{w},r,\alpha)} = \int_\Omega [\Xi_\alpha(|e(\mathbf{v})| + |e(\mathbf{w})|)]^{r-2}|e(\mathbf{v})|^2 \, d\Omega.$$

[Barret-Liu 1993, Barrett-Liu 1994]

Quasi-norms in Sobolev space

Proposition (part I)

Suppose that $r \in (1, \infty)$, $\alpha \in [0, 1]$ *and* $\mathbf{w} \in [\mathbf{W}^{1,r}(\Omega)]^d$;

(i) $\forall \mathbf{v} \in [\mathbf{W}^{1,r}(\Omega)]^d$, $|\mathbf{v}|_{(\mathbf{w},r,\alpha)} \geq 0$;

(ii) $\forall \mathbf{v} \in [\mathbf{W}^{1,r}(\Omega)]^d$, $|\mathbf{v}|_{(\mathbf{w},r,\alpha)} = 0$ *iff* $\mathbf{v} = \mathbf{0}$;

(iii) *(Quasi-triangle-inequality): there exists a constant* $C = C(r)$ *such that*

$$|\mathbf{v}_1 + \mathbf{v}_2|_{(\mathbf{w},r,\alpha)} \leq C \left(|\mathbf{v}_1|_{(\mathbf{w},r,\alpha)} + |\mathbf{v}_2|_{(\mathbf{w},r,\alpha)} \right)$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in [\mathbf{W}^{1,r}(\Omega)]^d$;

Quasi-norms in Sobolev space

Proposition (Part II)

(iv) *For* $1 < r \le 2$,

$$|\mathbf{v}|^{2/r}_{(\mathbf{w},r,\alpha)} \le \|e(\mathbf{v})\|_{L^r(\Omega)}$$

$$\|e(\mathbf{v})\|_{L^r(\Omega)} \le [\Xi_\alpha(\|e(\mathbf{v})\|_{L^r(\Omega)} + \|e(\mathbf{w})\|_{L^r(\Omega)})]^{(2-r)/2}\, |\mathbf{v}|_{(\mathbf{w},r,\alpha)}$$

for all $\mathbf{v} \in [W^{1,r}(\Omega)]^d$.
For $2 \le r < \infty$,

$$\|e(\mathbf{v})\|^{r/2}_{L^r(\Omega)} \le |\mathbf{v}|_{(\mathbf{w},r,\alpha)}$$

$$|\mathbf{v}|_{(\mathbf{w},r,\alpha)} \le [\Xi_\alpha(\|e(\mathbf{v})\|_{L^r(\Omega)} + \|e(\mathbf{w})\|_{L^r(\Omega)})]^{(r-2)/2}\, \|e(\mathbf{v})\|_{L^r(\Omega)}$$

for all $\mathbf{v} \in [W^{1,r}(\Omega)]^d$.

Quasi-norms in Sobolev space

Lemma

Suppose that $r \in (1,\infty)$ *and define the constants* $C_2 = 2^{-|r-2|}K_1$ *and* $C_3 = 2^{|r-2|/\max\{2,r'\}}K_2$*; then, for* $i = 1,2$*, and all* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$ *in* $\mathbf{V}$*,*

$$C_2|\mathbf{v}_1 - \mathbf{v}_2|^2_{(\mathbf{v}_i,r,\alpha)} \le a(\mathbf{v}_1, \mathbf{v}_1 - \mathbf{v}_2) - a(\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2),$$

$$|a(\mathbf{v}_1, \mathbf{w}) - a(\mathbf{v}_2, \mathbf{w})| \le C_3|\mathbf{v}_1 - \mathbf{v}_2|^{\min\{1,\frac{2}{r'}\}}_{(\mathbf{v}_i,r,\alpha)} \|\mathbf{w}\|_{\mathbf{V}} \times [\Xi_\alpha(\|\mathbf{v}_1\|_{\mathbf{V}} + \|\mathbf{v}_2\|_{\mathbf{V}})]^{\max\{0,\frac{r-2}{2}\}}.$$

Assumptions on $k(x,t)$

We are going to use these inequalities with $\mathbf{v}_1 = \mathbf{u}$ *and* $\mathbf{v}_2 = \mathbf{u}_h$*.*

Quasi-norms in Sobolev space

Lemma

Let $(\mathbf{u}, p) \in \mathbf{V} \times \mathbf{Q}$ *and* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ *denote the solutions to the continuous and discrete problems, respectively, and let* $r \in (1, \infty)$*; then*

$$\begin{aligned}
\|\mathbf{u}\|_{\mathbf{V}} &\leq G^{-1}\left(\tfrac{1}{K_1}\|\mathbf{f}\|_{\mathbf{V}'}\right),\\
\|\mathbf{u}_h\|_{\mathbf{V}} &\leq G^{-1}\left(\tfrac{1}{K_1}\|\mathbf{f}\|_{\mathbf{V}'}\right),\\
\|p\|_{\mathbf{Q}} &\leq \frac{1}{c_0}\left(\|\mathbf{f}\|_{\mathbf{V}'} + K_2 (H \circ G^{-1})\left(\tfrac{1}{K_1}\|\mathbf{f}\|_{\mathbf{V}'}\right)\right),\\
\|p_h\|_{\mathbf{Q}} &\leq \frac{1}{c_0'}\left(\|\mathbf{f}\|_{\mathbf{V}'} + K_2 (H \circ G^{-1})\left(\tfrac{1}{K_1}\|\mathbf{f}\|_{\mathbf{V}'}\right)\right),
\end{aligned}$$

where c_0 *and* c_0' *are the continuous and discrete inf-sup constants, respectively, and* G *and* H *are continuous strictly monotonic increasing functions defined on* $[0, \infty)$.

Quasi-norms in Sobolev space

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Suppose that $r \in (1, \infty)$ *and define the constants* $C_2 = 2^{-|r-2|} K_1$ *and* $C_3 = 2^{|r-2|/\max\{2,r'\}} K_2$; *then, for* $i = 1, 2$, *and all* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}$ *in* $\mathbf{V}$,

$$C_2 |\mathbf{u} - \mathbf{u}_h|^2_{(\mathbf{u},r,\alpha)} \leq a(\mathbf{u}, \mathbf{u} - \mathbf{u}_h) - a(\mathbf{u}_h, \mathbf{u} - \mathbf{u}_h),$$

$$|a(\mathbf{u}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w})| \leq C_4 |\mathbf{u} - \mathbf{u}_h|^{\min\{1, \frac{2}{r'}\}}_{(\mathbf{u},r,\alpha)} \|\mathbf{w}\|_{\mathbf{V}}$$

where $C_4 = C_3 2^{\max\{0, \frac{r-2}{2}\}} [(\Xi_\alpha \circ G^{-1})(\frac{1}{K_1} \|\mathbf{f}\|_{\mathbf{V}'})]^{\max\{0, \frac{r-2}{2}\}}$.

Upper bound in terms of residual functionals

We define $\mathbf{S}_1 \in \mathbf{V}'$ *by*

$$\langle \mathbf{S}_1, \mathbf{w}\rangle = (\mathbf{f}, \mathbf{w}) - a(\mathbf{u}_h, \mathbf{w}) - b(p_h, \mathbf{w}), \qquad \forall \mathbf{w} \in \mathbf{V}.$$

Similarly, we define $\mathbf{S}_2 \in \mathbf{Q}'$ *by*

$$\langle \mathbf{S}_2, q\rangle = -b(q, \mathbf{u}_h), \qquad \forall q \in \mathbf{Q}.$$

Theorem

Let $(\mathbf{u}, p) \in \mathbf{V} \times \mathbf{Q}$ *and* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ *denote the solutions to continuous and discrete problems, respectively. Then, there exists a positive constant* C *depending on* $K_1, K_2, c_0, c_0', r, \|\mathbf{f}\|_{\mathbf{V}'}$ *such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}}^{\mathrm{R_U}} + \|p - p_h\|_{\mathbf{Q}}^{\mathrm{R}} \leq C\left(\|\mathbf{S}_1\|_{\mathbf{V}'}^{\mathrm{R}_\mathrm{U}'} + \|\mathbf{S}_2\|_{\mathbf{Q}'}^{\mathrm{R}'}\right),$$

where $\mathrm{R_U} = \max\{r, 2\}$, $\mathrm{R} = \max\{r', 2\}$, $1/\mathrm{R_U} + 1/\mathrm{R}_\mathrm{U}' = 1$, $1/\mathrm{R} + 1/\mathrm{R}' = 1$.

lower

Computable residuals

Definition

$$\mathbf{R}_T([\mathbf{u}_h, p_h]) = \nabla \cdot (k(x, |e(\mathbf{u}_h)|)\, e(\mathbf{u}_h)) - \nabla p + \mathbf{f}\,|_T, \quad \forall T \in \mathcal{T}_h,$$

$$\mathbf{J}_E([\mathbf{u}_h, p_h]) = [\![\hat{n}_E \cdot (k\,(x, |e(\mathbf{u}_h)|)\, e(\mathbf{u}_h) - p_h\, \mathbf{I})]\!]_E, \quad \forall E \in \mathcal{E}_{h,\Omega},$$

$$\nabla \cdot \mathbf{u}_h$$

- $\Pi_T \mathbf{R}_T([\mathbf{u}_h, p_h])$ *polynomial approx. of* $\mathbf{R}_T([\mathbf{u}_h, p_h])$ *on* $T \in \mathcal{T}_h$;
- $\Pi_E \mathbf{J}_E([\mathbf{u}_h, p_h])$ *polynomial approx. of* $\mathbf{J}_E([\mathbf{u}_h, p_h])$ *on* $E \in \mathcal{E}_{h,\Omega}$.

- $\mathcal{T}_h$ shape-regular partition of Ω;
- $\forall T \in \mathcal{T}_h$, $\mathcal{E}(T)$ set of its $(d-1)$-dimensional faces;
- $\mathcal{E}_h = \bigcup_{T \in \mathcal{T}_h} \mathcal{E}(T)$ be the set of all faces of $\mathcal{T}_h$;
- $\mathcal{E}_{h,\Omega} = \{E \in \mathcal{E}_h : E \not\subset \partial\Omega\}$.

Upper bound in terms of computable residuals

Theorem

Let $(\mathbf{u}, p) \in \mathbf{V} \times \mathbf{Q}$ *and* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathbf{Q}_h$ *denote the solution to the continuous and discrete problems, respectively. Then, there exists a positive constant* C_U *depending on* $K_1, K_2, c_0, c_0', r, \|\mathbf{f}\|_{\mathbf{V}'}$ *and on the minimal angle of the triangulation such that*

$$\| \mathbf{u} - \mathbf{u}_h \|_{\mathbf{V}}^{\mathrm{R_U}} + \| p - p_h \|_{\mathbf{Q}}^{\mathrm{R}} \leq C_U \times \left[\left(\sum_{T \in \mathcal{T}_h} h_T^{r'} \| \Pi_T \mathbf{R}_T([\mathbf{u}_h, p_h]) \|_{\mathrm{L}^{r'}(T)}^{r'} \right)^{\frac{\mathrm{R}'_\mathrm{U}}{r'}} + \left(\sum_{E \in \mathcal{E}_{h,\Omega}} h_E \| \Pi_E \mathbf{J}_E([\mathbf{u}_h, p_h]) \|_{\mathrm{L}^{r'}(E)}^{r'} \right)^{\frac{\mathrm{R}'_\mathrm{U}}{r'}} + \| \nabla \cdot \mathbf{u}_h \|_{\mathrm{L}^{r'}(\Omega)}^{\mathrm{R}'} \ldots \right.$$

Upper bound in terms of computable residuals

Theorem

$$\ldots + \left(\sum_{T \in \mathcal{T}_h} h_T^{r'} \| \mathbf{R}_T([\mathbf{u}_h, p_h]) - \Pi_T \mathbf{R}_T([\mathbf{u}_h, p_h]) \|_{\mathrm{L}^{r'}(T)}^{r'} \right)^{\frac{\mathrm{R}'_{\mathrm{U}}}{r'}}$$

$$+ \left(\sum_{E \in \mathcal{E}_{h,\Omega}} h_E \| \mathbf{J}_E([\mathbf{u}_h, p_h]) - \Pi_E \mathbf{J}_E([\mathbf{u}_h, p_h]) \|_{\mathrm{L}^{r'}(E)}^{r'} \right)^{\frac{\mathrm{R}'_{\mathrm{U}}}{r'}} \Bigg] ,$$

where $\mathrm{R_U} = \max\{r, 2\}$, $\mathrm{R} = \max\{r', 2\}$, $1/\,\mathrm{R_U} + 1/\,\mathrm{R'_U} = 1$, $1/\,\mathrm{R} + 1/\,\mathrm{R}' = 1$.

Lower bound in terms of residual functionals

Theorem

Let $(\mathbf{u}, p) \in \mathbf{V} \times \mathrm{Q}$ *and* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathrm{Q}_h$ *denote the solution to the continuous and discrete problems, respectively Then, there exists a positive constant* c *depending on* $K_1, K_2, r, \|\mathbf{f}\|_{\mathbf{V}'}$ *such that*

$$c\left[\|\mathrm{S}_1\|_{\mathbf{V}'}^{\mathrm{R}_\mathrm{L}'} + \|\mathrm{S}_2\|_{\mathrm{Q}'}^{\mathrm{R}'}\right] \leq \|\,\mathbf{u}-\mathbf{u}_h\,\|_{\mathbf{V}}^{\mathrm{R}_\mathrm{L}} + \|\,p-p_h\,\|_{\mathrm{Q}}^{\mathrm{R}}$$

where $\mathrm{R}_\mathrm{L} = \min\{r, 2\}$, $\mathrm{R} = \max\{r', 2\}$, $1/\mathrm{R}_\mathrm{L} + 1/\mathrm{R}_\mathrm{L}' = 1$, $1/\mathrm{R} + 1/\mathrm{R}' = 1$, *and* S_1 *and* S_2 *are residual functionals which are computably bounded.*

- $\mathrm{R}_\mathrm{L} = \min\{r, 2\}$;
- $\mathrm{R}_\mathrm{U} = \max\{r, 2\}$.

▸ upper

Lower bound in terms of computable residuals

Theorem

Let $(\mathbf{u}, p) \in \mathbf{V} \times \mathrm{Q}$ and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \mathrm{Q}_h$ denote the solution to the continuous and discrete problems, respectively. Then, there exists a positive constant c_L depending on $K_1, K_2, r, \|\mathbf{f}\|_{\mathbf{V}'}$ and on the minimal angle of the triangulation such that

$$
c_L \left[\left(\sum_{T \in \mathcal{T}_h} h_T^{r'} \| \Pi_T \mathbf{R}_T([\mathbf{u}_h, p_h]) \|_{\mathrm{L}^{r'}(T)}^{r'} \right)^{\frac{\mathrm{R}'_\mathrm{L}}{r'}} + \left(\sum_{E \in \mathcal{E}_{h,\Omega}} h_E \| \Pi_E \mathbf{J}_E([\mathbf{u}_h, p_h]) \|_{\mathrm{L}^{r'}(E)}^{r'} \right)^{\frac{\mathrm{R}'_\mathrm{L}}{r'}} + \| \nabla \cdot \mathbf{u}_h \|_{\mathrm{L}^{r'}(\Omega)}^{\mathrm{R}'} \right]
$$
$$
\leq \| \mathbf{u} - \mathbf{u}_h \|_{\mathbf{V}}^{\mathrm{R}_\mathrm{L}} + \| p - p_h \|_{\mathrm{Q}}^{\mathrm{R}} \cdots
$$

Lower bound in terms of computable residuals

Theorem

$$+\left(\sum_{T\in\mathcal{T}_h} h_T^{r'} \parallel \mathbf{R}_T([\mathbf{u}_h,p_h]) - \Pi_T\mathbf{R}_T([\mathbf{u}_h,p_h]) \parallel_{\mathrm{L}^{r'}(T)}^{r'}\right)^{\frac{\mathrm{R}'_\mathrm{L}}{r'}}$$

$$+\left(\sum_{E\in\mathcal{E}_{h,\Omega}} h_E \parallel \mathbf{J}_E([\mathbf{u}_h,p_h]) - \Pi_E\mathbf{J}_E([\mathbf{u}_h,p_h]) \parallel_{\mathrm{L}^{r'}(E)}^{r'}\right)^{\frac{\mathrm{R}'_\mathrm{L}}{r'}}$$

where $\mathrm{R_L} = \min\{r,2\}$, $\mathrm{R} = \max\{r',2\}$, $1/\mathrm{R_L} + 1/\mathrm{R}'_\mathrm{L} = 1$, $1/\mathrm{R} + 1/\mathrm{R}' = 1$.

From two-sides bounds to the adaptive method

We have proved

- upper and lower bounds in terms of residual functionals;
- upper and lower bounds in terms of computable residuals;
- (remarks....)

now we are ready to use them in our adaptive algorithm that is based on the local information given by:

Adaptive algorithm

Definition

Foe each $T \in \mathcal{T}_h$ let us define the following elemental error indicators:

$$\begin{aligned}
\mathbf{R}_{\mathbf{mom},\mathbf{T}} &= h_T^{r'} \, \| \, \Pi_T \mathbf{R}_T([\mathbf{u}_h, p_h]) \, \|_{\mathbf{L}^{r'}(T)}^{r'} \\
&\quad + \frac{1}{2} \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,\Omega}} h_E \, \| \, \Pi_E \mathbf{J}_E([\mathbf{u}_h, p_h]) \, \|_{\mathbf{L}^{r'}(E)}^{r'} \\
&\quad + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_{h,N}} h_E \, \| \, \Pi_E \mathbf{J}_E([\mathbf{u}_h, p_h]) \, \|_{\mathbf{L}^{r'}(E)}^{r'}, \\
\mathbf{R}_{\mathbf{cont},\mathbf{T}} &= \| \, \nabla \cdot \mathbf{u}_h \, \|_{\mathbf{L}^{r}(T)}^{r} .
\end{aligned}$$

Adaptive algorithm

Algorithm

1. *Solve the problem on the given mesh;*

Adaptive algorithm

Algorithm

1. *Solve the problem on the given mesh;*
2. *compute the elemental error indicators* $\mathbf{R}_{\mathbf{mom},\mathbf{T}}$ *and* $\mathbf{R}_{\mathbf{cont},\mathbf{T}}$

Adaptive algorithm

Algorithm

1. *Solve the problem on the given mesh;*
2. *compute the elemental error indicators* $\mathbf{R}_{\mathbf{mom},\mathbf{T}}$ *and* $\mathbf{R}_{\mathbf{cont},\mathbf{T}}$
3. *sort the vectors* $\mathrm{R.mom}[\boldsymbol{T}]$, $\mathrm{R.cont}[\boldsymbol{T}]$, $\forall \boldsymbol{T} \in \mathcal{T}_h$;

Adaptive algorithm

Algorithm

1. *Solve the problem on the given mesh;*
2. *compute the elemental error indicators* $\mathbf{R_{mom,T}}$ *and* $\mathbf{R_{cont,T}}$
3. *sort the vectors* $\mathrm{R.mom}[T]$, $\mathrm{R.cont}[T]$, $\forall T \in \mathcal{T}_h$*;*
4. *mark for refinement the* $\rho\%$ *of the elements with the larger values of* $\mathrm{R.mom}$*;*

Adaptive algorithm

Algorithm

1. *Solve the problem on the given mesh;*
2. *compute the elemental error indicators* $\mathbf{R_{mom,T}}$ *and* $\mathbf{R_{cont,T}}$
3. *sort the vectors* $\mathrm{R.mom}[T]$, $\mathrm{R.cont}[T]$, $\forall T \in \mathcal{T}_h$*;*
4. *mark for refinement the* $\rho\%$ *of the elements with the larger values of* $\mathrm{R.mom}$*;*
5. *mark for refinement the* $\rho\%$ *of the elements with the larger values of* $\mathrm{R.cont}$*;*

Adaptive algorithm

Algorithm

1. *Solve the problem on the given mesh;*
2. *compute the elemental error indicators* $\mathbf{R_{mom,T}}$ *and* $\mathbf{R_{cont,T}}$
3. *sort the vectors* $\mathrm{R.mom}[T]$, $\mathrm{R.cont}[T]$, $\forall T \in \mathcal{T}_h$*;*
4. *mark for refinement the* $\rho\%$ *of the elements with the larger values of* $\mathrm{R.mom}$*;*
5. *mark for refinement the* $\rho\%$ *of the elements with the larger values of* $\mathrm{R.cont}$*;*
6. *pre-mark for coarsening the* $\gamma\%$ *of the elements with the smaller values of* $\mathrm{R.mom}$*;*

Adaptive algorithm

Algorithm

1. *Solve the problem on the given mesh;*
2. *compute the elemental error indicators* $\mathbf{R}_{\mathbf{mom},\mathbf{T}}$ *and* $\mathbf{R}_{\mathbf{cont},\mathbf{T}}$
3. *sort the vectors* $\mathrm{R.mom}[\boldsymbol{T}]$, $\mathrm{R.cont}[\boldsymbol{T}]$, $\forall \boldsymbol{T} \in \mathcal{T}_h$*;*
4. *mark for refinement the* $\rho\%$ *of the elements with the larger values of* $\mathrm{R.mom}$*;*
5. *mark for refinement the* $\rho\%$ *of the elements with the larger values of* $\mathrm{R.cont}$*;*
6. *pre-mark for coarsening the* $\gamma\%$ *of the elements with the smaller values of* $\mathrm{R.mom}$*;*
7. *pre-mark for coarsening the* $\gamma\%$ *of the elements with the smaller values of* $\mathrm{R.cont}$*;*

Adaptive algorithm

Algorithm

1. *Solve the problem on the given mesh;*
2. *compute the elemental error indicators* $\mathbf{R}_{\mathbf{mom},\mathbf{T}}$ *and* $\mathbf{R}_{\mathbf{cont},\mathbf{T}}$
3. *sort the vectors* $\mathrm{R.mom}[T]$, $\mathrm{R.cont}[T]$, $\forall T \in \mathcal{T}_h$*;*
4. *mark for refinement the* $\rho\%$ *of the elements with the larger values of* $\mathrm{R.mom}$*;*
5. *mark for refinement the* $\rho\%$ *of the elements with the larger values of* $\mathrm{R.cont}$*;*
6. *pre-mark for coarsening the* $\gamma\%$ *of the elements with the smaller values of* $\mathrm{R.mom}$*;*
7. *pre-mark for coarsening the* $\gamma\%$ *of the elements with the smaller values of* $\mathrm{R.cont}$*;*
8. *mark for coarsening the element (compatibility rule);*

Adaptive algorithm

Algorithm

1. *Solve the problem on the given mesh;*
2. *compute the elemental error indicators* $\mathbf{R_{mom,T}}$ *and* $\mathbf{R_{cont,T}}$
3. *sort the vectors* $\mathrm{R.mom}[T]$, $\mathrm{R.cont}[T]$, $\forall T \in \mathcal{T}_h$*;*
4. *mark for refinement the* $\rho\%$ *of the elements with the larger values of* $\mathrm{R.mom}$*;*
5. *mark for refinement the* $\rho\%$ *of the elements with the larger values of* $\mathrm{R.cont}$*;*
6. *pre-mark for coarsening the* $\gamma\%$ *of the elements with the smaller values of* $\mathrm{R.mom}$*;*
7. *pre-mark for coarsening the* $\gamma\%$ *of the elements with the smaller values of* $\mathrm{R.cont}$*;*
8. *mark for coarsening the element (compatibility rule);*
9. $\rho = \rho/f, \gamma = \gamma/f;$

Adaptive algorithm

Algorithm

1. *Solve the problem on the given mesh;*
2. *compute the elemental error indicators* $\mathbf{R_{mom,T}}$ *and* $\mathbf{R_{cont,T}}$
3. *sort the vectors* $\mathrm{R.mom}[T]$, $\mathrm{R.cont}[T]$, $\forall T \in \mathcal{T}_h$*;*
4. *mark for refinement the* $\rho\%$ *of the elements with the larger values of* $\mathrm{R.mom}$*;*
5. *mark for refinement the* $\rho\%$ *of the elements with the larger values of* $\mathrm{R.cont}$*;*
6. *pre-mark for coarsening the* $\gamma\%$ *of the elements with the smaller values of* $\mathrm{R.mom}$*;*
7. *pre-mark for coarsening the* $\gamma\%$ *of the elements with the smaller values of* $\mathrm{R.cont}$*;*
8. *mark for coarsening the element (compatibility rule);*
9. $\rho = \rho/f, \gamma = \gamma/f$*;*
10. *adapt the mesh and back to point 1;*

Linear test problem

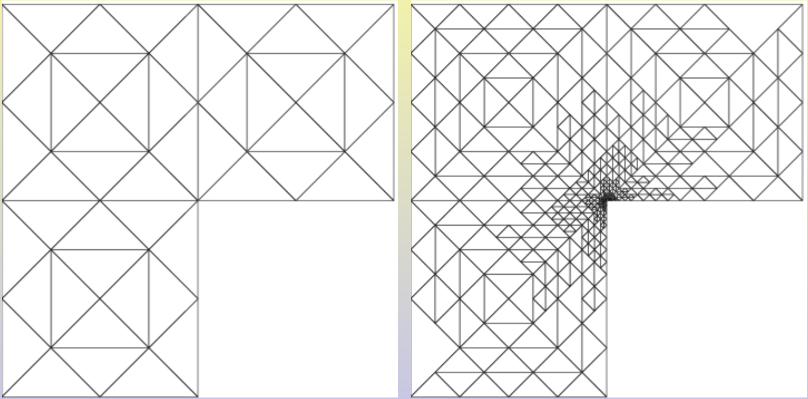
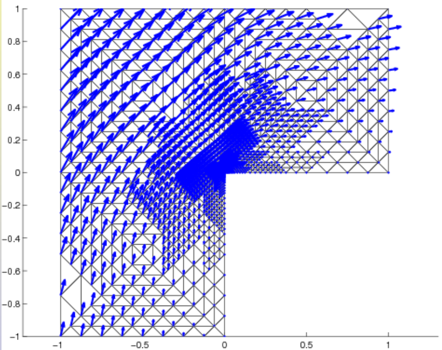


Figure: Starting and ending meshes for a linear singular problem, we use P2-P1 elements
[Verfürth 1989, Carstensen-Funken 2000]

Linear test problem



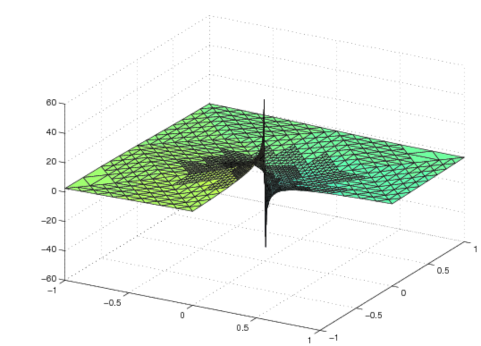


Figure: Velocity and pressure for a linear singular problem [Verfürth 1989, Carstensen-Funken 2000]

Numerical results on adapted meshes

In the following we consider:

- two different geometries Ω_1 and Ω_2;
- for each of them, a power-law model and a Carreau law model;
- for each of them, a problem with $r = 1.3$ and one with $r = 3.3$.

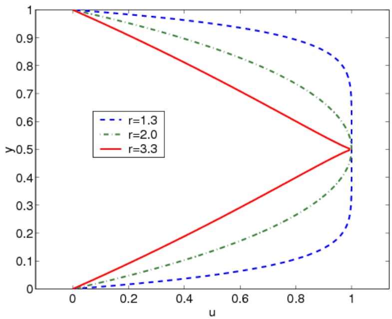


Figure: Velocity profiles for the power-law fully developed flow

meshes $r = 1.3$ meshes $r = 3.3$

Numerical results on adapted meshes

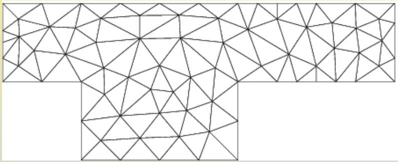


Figure: Starting mesh on the domain Ω_1

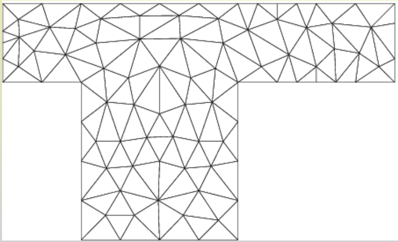


Figure: Starting mesh on the domain Ω_2

Carreau meshes $r = 1.3$

Power-law: $r = 1.3$, $\mu = 10^{-2}$

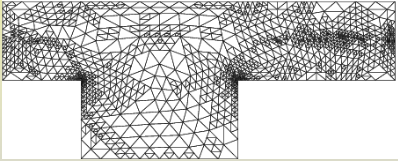


Figure: Power-law, $r = 1.3$: final mesh on the domain Ω_1

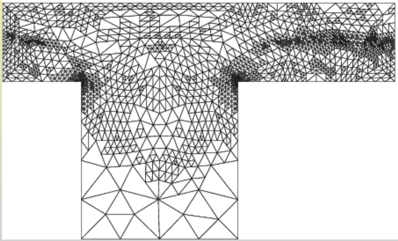


Figure: Power-law, $r = 1.3$: final mesh on the domain Ω_2

▸ profiles ▸ $u\ r = 1.3$

Power-law: $r = 1.3$, $\mu = 10^{-2}$

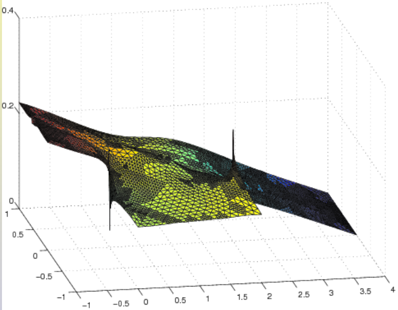


Figure: Power-law, $r = 1.3$: pressure on the final mesh on Ω_1

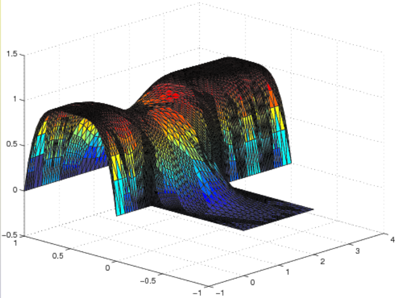


Figure: Power-law, $r = 1.3$: u component of the velocity on Ω_1

Power-law: $r = 1.3$, $\mu = 10^{-2}$

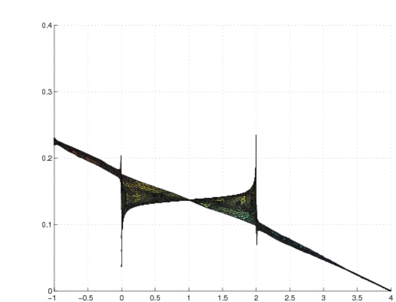


Figure: Power-law, $r = 1.3$: pressure on the final mesh on Ω_1

Figure: Power-law, $r = 1.3$: u component of the velocity on Ω_1

Power-law: $r = 1.3$, $\mu = 10^{-2}$

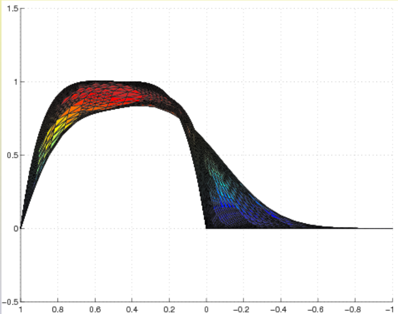


Figure: Power-law, $r = 1.3$: u profile on the final mesh on Ω_1

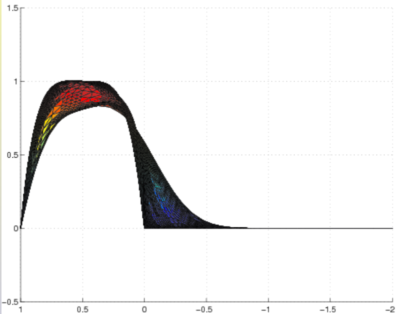


Figure: Power-law, $r = 1.3$: u profile of the velocity on Ω_2

side vortex view 33

Power-law: $r = 1.3$, $\mu = 10^{-2}$

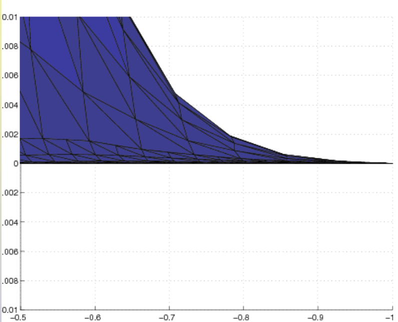


Figure: Power-law, $r = 1.3$: u profile on the final mesh on Ω_1, detail in the cavity

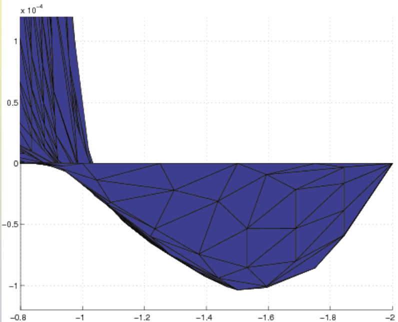


Figure: Power-law, $r = 1.3$: u profile of the velocity on Ω_2, detail in the cavity

Power-law: $r = 1.3$, $\mu = 10^{-2}$

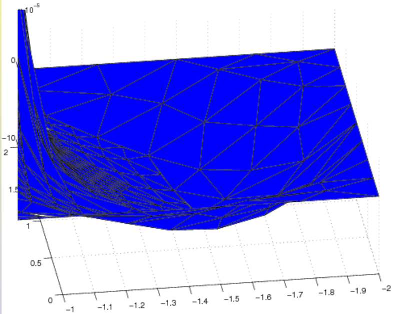


Figure: Power-law, $r = 1.3$: u component on the final mesh on Ω_2, detail in the cavity

▸ meshes $r = 1.3$ ▸ u $r = 3.3$

Monitoring the adaptive process

Definitions (Total upper residual)

$$\mathbf{tot.R.U} = \left(\sum_{T \in \mathcal{T}_h} h_T^{r'} \parallel \Pi_T \mathbf{R}_T([\mathbf{u}_h, p_h]) \parallel_{\mathbf{L}^{r'}(T)}^{r'} \right)^{\frac{\mathbf{R}'_{\mathbf{U}}}{r'}}$$
$$+ \left(\sum_{E \in \mathcal{E}_{h,\Omega,\Gamma_N}} h_E \parallel \Pi_E \mathbf{J}_E([\mathbf{u}_h, p_h]) \parallel_{\mathbf{L}^{r'}(E)}^{r'} \right)^{\frac{\mathbf{R}'_{\mathbf{U}}}{r'}}$$
$$+ \parallel \nabla \cdot \mathbf{u}_h \parallel_{L^r(\Omega)}^{\mathbf{R}'}$$

Monitoring the adaptive process

Definitions (Total lower residual)

$$\begin{aligned}\textbf{tot.R.L} &= \left(\sum_{T\in\mathcal{T}_h} h_T^{r'} \, \| \, \Pi_T \mathbf{R}_T([\mathbf{u}_h, p_h]) \, \|_{\mathbf{L}^{r'}(T)}^{r'}\right)^{\frac{\mathbf{R}_\mathbf{L}'}{r'}} \\ &+ \left(\sum_{E\in\mathcal{E}_{h,\Omega,\Gamma_N}} h_E \, \| \, \Pi_E \mathbf{J}_E([\mathbf{u}_h, p_h]) \, \|_{\mathbf{L}^{r'}(E)}^{r'}\right)^{\frac{\mathbf{R}_\mathbf{L}'}{r'}} \\ &+ \| \, \nabla \cdot \mathbf{u}_h \, \|_{L^r(\Omega)}^{\mathbf{R}'}\end{aligned}$$

Power-law: $r = 1.3$, $\mu = 10^{-2}$

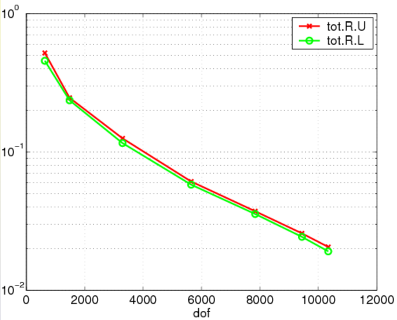


Figure: Power-law, $r = 1.3$: total residuals during the adaptive process on Ω_1

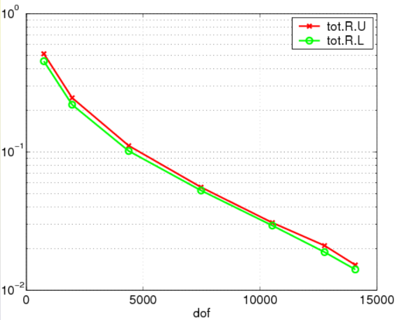


Figure: Power-law, $r = 1.3$: total residuals during the adaptive process on Ω_2

Power-law: $r = 3.3$, $\mu = 10^{-2}$

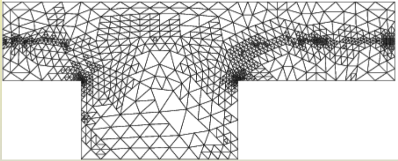


Figure: Power-law, $r = 3.3$: final mesh on the domain Ω_1

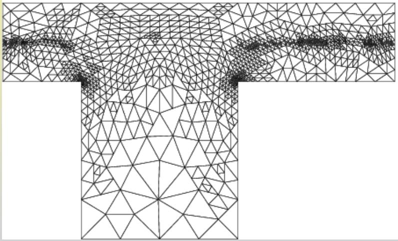


Figure: Power-law, $r = 3.3$: final mesh on the domain Ω_2

profiles

Power-law: $r = 3.3$, $\mu = 10^{-2}$

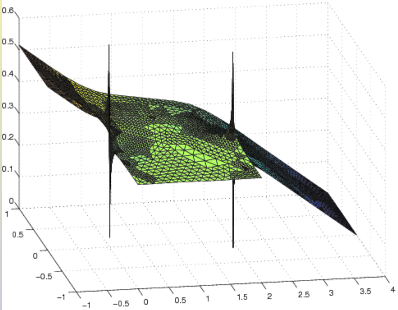


Figure: Power-law, $r = 3.3$: pressure on the final mesh on Ω_1

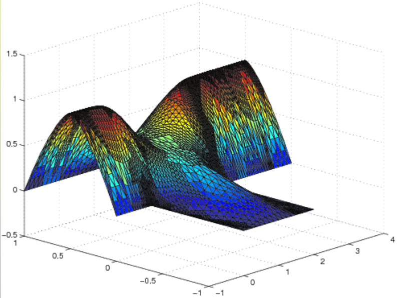


Figure: Power-law, $r = 3.3$: u component of the velocity on Ω_1

Power-law: $r = 3.3$, $\mu = 10^{-2}$

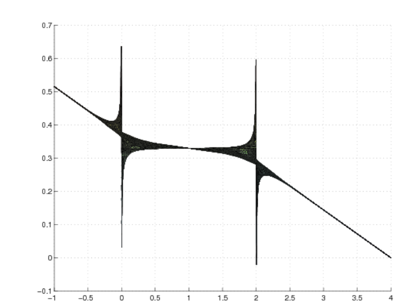


Figure: Power-law, $r = 3.3$: pressure on the final mesh on Ω_1

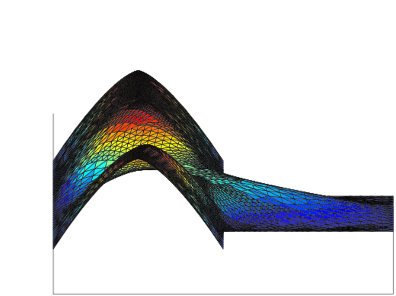


Figure: Power-law, $r = 3.3$: u component of the velocity on Ω_1

Power-law: $r = 3.3$, $\mu = 10^{-2}$

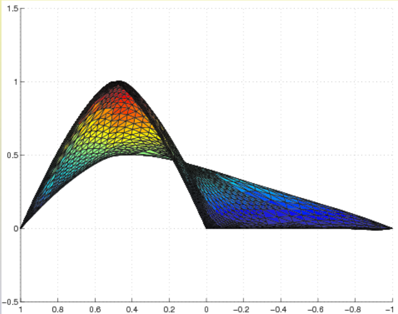


Figure: Power-law, $r = 3.3$: u profile on the final mesh on Ω_1

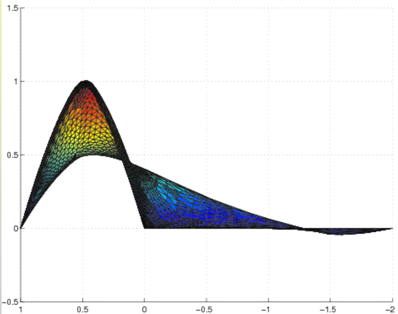


Figure: Power-law, $r = 3.3$: u profile of the velocity on Ω_2

side vortex view 13

Power-law: $r = 3.3$, $\mu = 10^{-2}$

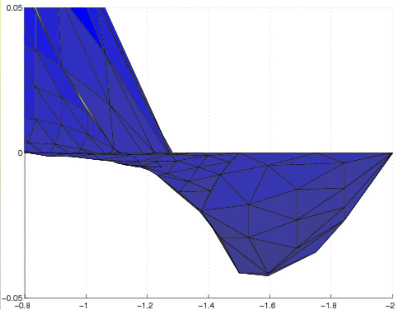


Figure: Power-law, $r = 3.3$: u profile of the velocity on Ω_2, detail in the cavity

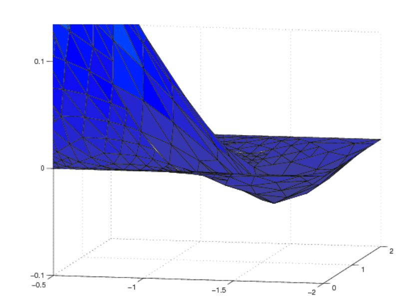


Figure: Power-law, $r = 3.3$: u component of the velocity on Ω_2, detail in the cavity

$u\ r = 1.3$

Power-law: $r = 3.3$, $\mu = 10^{-2}$

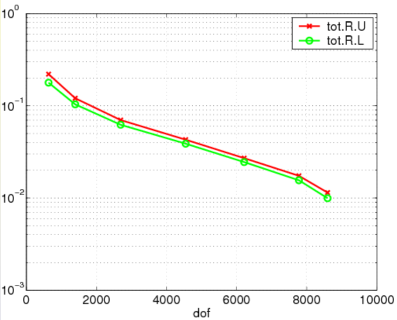


Figure: Power-law, $r = 3.3$: total residuals during the adaptive process on Ω_1

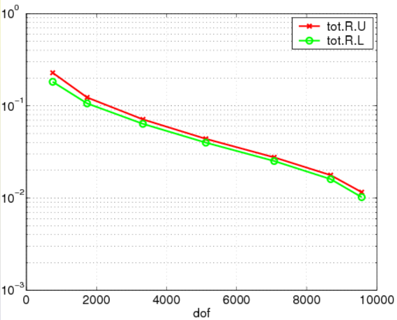


Figure: Power-law, $r = 3.3$: total residuals during the adaptive process on Ω_2

Carreau law: $\theta = r = 1.3$, $k_0 = 3 \cdot 10^{-2}$, $k_\infty = 0$, $\lambda = 10^{-4}$

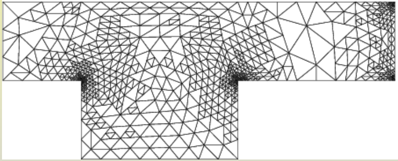


Figure: Carreau law, $r = 1.3$: final mesh on the domain Ω_1

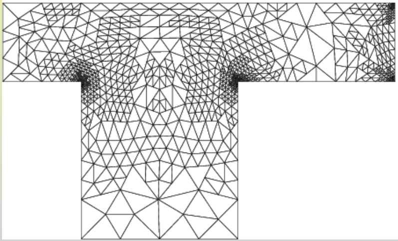


Figure: Carreau law, $r = 1.3$: final mesh on the domain Ω_2

▸ Starting meshes

Carreau law: $\theta = r = 1.3$, $k_0 = 3 \cdot 10^{-2}$, $k_\infty = 0$, $\lambda = 10^{-4}$

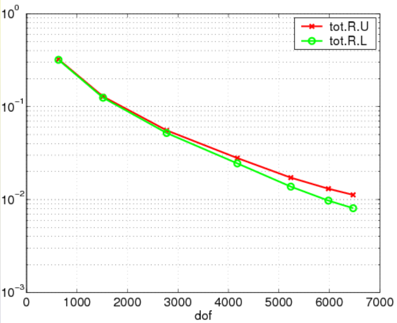


Figure: Carreau law, $r = 1.3$: total residuals during the adaptive process on Ω_1

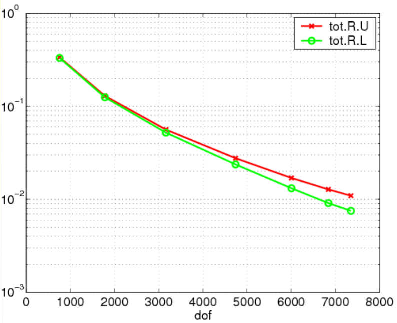


Figure: Carreau law, $r = 1.3$: total residuals during the adaptive process on Ω_2

Carreau law: $\theta = r = 3.3$, $k_0 = 3 \cdot 10^{-2}$, $k_\infty = 0$, $\lambda = 10^{-4}$

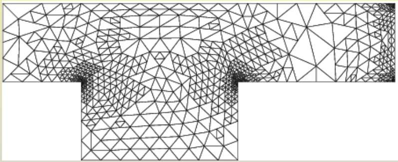


Figure: Carreau law, $r = 3.3$: final mesh on the domain Ω_1

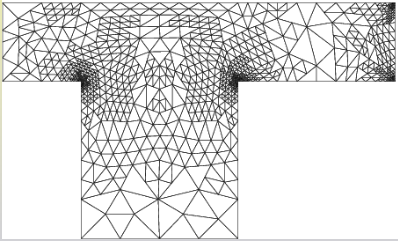


Figure: Carreau law, $r = 3.3$: final mesh on the domain Ω_2

Carreau law: $\theta = r = 3.3$, $k_0 = 3 \cdot 10^{-2}$, $k_\infty = 0$, $\lambda = 10^{-4}$

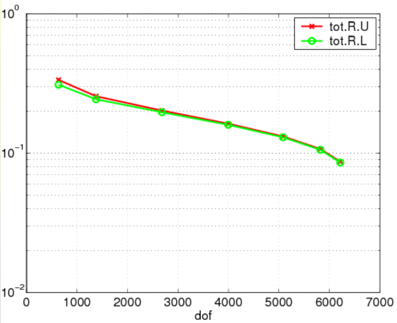


Figure: Carreau law, $r = 3.3$: total residuals during the adaptive process on Ω_1

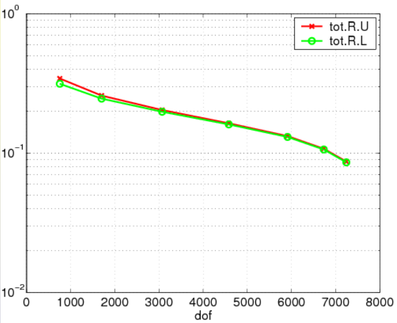


Figure: Carreau law, $r = 3.3$: total residuals during the adaptive process on Ω_2

Thank you

Thank you for your attention