A Multiscale DG Method for Convection Diffusion Problems

Annalisa Buffa

Istituto di Matematica Applicata e Tecnologie Informatiche - Pavia
National Research Council Italy

Joint project with Th. J.R. Hughes, G. Sangalli,
G. Scovazzi, P. Bochev.

Notation and Definitions

$$\begin{aligned} \mathcal{L}\phi = -\kappa\Delta\phi + \mathbf{a}\cdot\nabla\phi &= f \quad \text{in } \Omega \\ \phi &= g \quad \text{on } \Gamma \end{aligned}$$

- $\kappa \geq 0$ diffusion coefficent, **a** is the solenoidal velocity vector field; $g \in H^{1/2}(\Gamma)$ Dirichlet boundary condition
- We set:

$$\begin{aligned} \Gamma^- &= \{x \in \Gamma : \mathbf{a}(x)\cdot\mathbf{n}(x) \leq 0\} \\ \Gamma^+ &= \{x \in \Gamma : \mathbf{a}(x)\cdot\mathbf{n}(x) > 0\} \end{aligned}$$

  where **n** is the outward unit normal with respect to Γ. Γ^- will be referred to as the *inflow* boundary and Γ^+ as the *outflow* boundary;
- Convection dominated regime : $\|\mathbf{a}\| >> \kappa$

Discontinuous Galerkin discretization: generalities

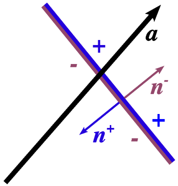
$\mathcal{T}_h$ is a mesh of Ω, $\mathcal{E}_h$ the set of edges, $\mathcal{E}_h^0$ the set of internal edges.

Finite element space :

$$V_h = \{v \in L^2(\Omega) : v|_T \in \mathcal{P}^k(T), \quad \forall T \in \mathcal{T}_h\}$$

Jumps and averages : $e \in \mathcal{E}_h^0$
T^+, T^- downwind and upwind triangles

$$\langle \nu \rangle = \frac{1}{2}(\nu^+ + \nu^-) \quad [\![\nu]\!] = \nu^+ \mathbf{n}^+ + \nu^- \mathbf{n}^-$$
$$\langle \boldsymbol{\tau} \rangle = \frac{1}{2}(\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-) \quad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+ \cdot \mathbf{n}^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^-$$



Note that, e.g., ν^+ (ν^-) are the trace values taken from the *downwind* (*upwind*) triangle, respectively.

Construction of an up-wind DG scheme

If ϕ is the solution, then, integrating by parts:

$$\int_T f\mu = \int_T \mathcal{L}\phi\,\mu = \int_T \kappa\nabla\phi\cdot\nabla\mu - \mathbf{a}\cdot\nabla\mu\phi + \int_{\Gamma_T}(\mathbf{a}\phi - \kappa\nabla\phi)\cdot\mathbf{n}_T\,\mu$$

We use following *biased* identity (take $\boldsymbol{\tau} = \mathbf{a}\phi - \kappa\nabla\phi$)

$$\sum_{T\in\mathcal{T}_h}\int_{\Gamma_T}\boldsymbol{\tau}\cdot\mathbf{n}\,\mu = \sum_{e\in\mathcal{E}_h^o}\left(\int_e \mu^+[\![\boldsymbol{\tau}]\!] + \int_e [\![\mu]\!]\cdot\boldsymbol{\tau}^-\right) + \int_\Gamma \mu\,\boldsymbol{\tau}\cdot\mathbf{n}.$$

which, together with upwind on Γ, brings to (for all $\mu \in V_h$)

$$\int_\Omega f\mu = \int_\Omega \mathcal{L}\phi\mu = \sum_{T\in\mathcal{T}_h}\int_T(\kappa\nabla\phi\cdot\nabla\mu - \mathbf{a}\cdot\nabla\mu\,\phi) + \sum_{e\in\mathcal{E}_h^0}\int_e [\![\mu]\!](\mathbf{a}\phi - \kappa\nabla\phi)^-$$

$$+\int_\Gamma \mu\,\kappa^-\nabla\phi^- + \int_{\Gamma^+}\mathbf{a}\cdot\mathbf{n}\mu\,\phi + \int_{\Gamma^-}\mathbf{a}\cdot\mathbf{n}\mu\,g$$

Construction of an up-wind DG scheme

If ϕ is the solution, then, integrating by parts:

$$\int_T f\,\mu = \int_T \mathcal{L}\phi\,\mu = \int_T \kappa\nabla\phi\cdot\nabla\mu - \mathbf{a}\cdot\nabla\mu\phi + \int_{\Gamma_T} (\mathbf{a}\phi - \kappa\nabla\phi)\cdot\mathbf{n}_T\,\mu$$

We use following *biased* identity (take $\boldsymbol{\tau} = \mathbf{a}\phi - \kappa\nabla\phi$)

$$\sum_{T\in\mathcal{T}_h}\int_{\Gamma_T}\boldsymbol{\tau}\cdot\mathbf{n}\;\mu = \sum_{e\in\mathcal{E}_h^o}\left(\int_e \mu^+[\![\boldsymbol{\tau}]\!] + \int_e [\![\mu]\!]\cdot\boldsymbol{\tau}^-\right) + \int_\Gamma \mu\;\boldsymbol{\tau}\cdot\mathbf{n}.$$

which, together with upwind on Γ, brings to (for all $\mu \in V_h$)

$$\int_\Omega f\,\mu = \int_\Omega \mathcal{L}\phi\mu = \sum_{T\in\mathcal{T}_h}\int_T(\kappa\nabla\phi\cdot\nabla\mu - \mathbf{a}\cdot\nabla\mu\,\phi) + \sum_{e\in\mathcal{E}_h^0}\int_e [\![\mu]\!](\mathbf{a}\phi - \kappa\nabla\phi)^-$$

$$+\int_\Gamma \mu\,\kappa^-\nabla\phi^- + \int_{\Gamma^+}\mathbf{a}\cdot\mathbf{n}\mu\,\phi + \int_{\Gamma^-}\mathbf{a}\cdot\mathbf{n}\mu\,g$$

Construction of an up-wind DG scheme

If ϕ is the solution, then, integrating by parts:

$$\int_T f\mu = \int_T \mathcal{L}\phi\,\mu = \int_T \kappa\nabla\phi\cdot\nabla\mu - \mathbf{a}\cdot\nabla\mu\phi + \int_{\Gamma_T}(\mathbf{a}\phi - \kappa\nabla\phi)\cdot\mathbf{n}_T\,\mu$$

We use following *biased* identity (take $\boldsymbol{\tau} = \mathbf{a}\phi - \kappa\nabla\phi$)

$$\sum_{T\in\mathcal{T}_h}\int_{\Gamma_T}\boldsymbol{\tau}\cdot\mathbf{n}\,\mu = \sum_{e\in\mathcal{E}_h^o}\left(\int_e \mu^+[\![\boldsymbol{\tau}]\!] + \int_e [\![\mu]\!]\cdot\boldsymbol{\tau}^-\right) + \int_\Gamma \mu\,\boldsymbol{\tau}\cdot\mathbf{n}.$$

which, together with upwind on Γ, brings to (for all $\mu \in V_h$)

$$\int_\Omega f\mu = \int_\Omega \mathcal{L}\phi\mu = \sum_{T\in\mathcal{T}_h}\int_T(\kappa\nabla\phi\cdot\nabla\mu - \mathbf{a}\cdot\nabla\mu\,\phi) + \sum_{e\in\mathcal{E}_h^0}\int_e [\![\mu]\!](\mathbf{a}\phi - \kappa\nabla\phi)^-$$

$$+ \int_\Gamma \mu\,\kappa^-\nabla\phi^- + \int_{\Gamma^+}\mathbf{a}\cdot\mathbf{n}\mu\,\phi + \int_{\Gamma^-}\mathbf{a}\cdot\mathbf{n}\mu\,g$$

Up-wind Discontinuous Galerkin method

$$B^{DG}(\phi^{DG},\mu) = B^{DG}_{\mathfrak{D}}(\phi^{DG},\mu) + B^{DG}_{\mathfrak{C}}(\phi^{DG},\mu) = L^{DG}(g,f;\mu) \quad (DG)$$

Bilinear Forms

$$B^{DG}_{\mathfrak{D}}(\nu,\mu) = \sum_{T\in\mathcal{T}_h}\int_T \nabla\mu\cdot\kappa\nabla\nu - \sum_{e\in\mathcal{E}_h^o}\int_e [\![\mu]\!]\cdot\kappa^-\nabla\nu^- + s\,\kappa^-\nabla\mu^-[\![\nu]\!]$$

$$+\sum_{e\in\Gamma}\int_e s\,\kappa\nabla\mu\cdot\mathbf{n}\nu - \kappa\nabla\nu\cdot\mathbf{n}\mu + \varepsilon\sum_{e\in\mathcal{E}_h}\int_e \frac{\langle\kappa\rangle}{h_\perp}[\![\mu]\!]\cdot[\![\nu]\!]$$

$$B^{DG}_{\mathfrak{C}}(\nu,\mu) = \sum_{T\in\mathcal{T}_h} -\int_T \nabla\mu\cdot\mathbf{a}\nu + \sum_{e\in\mathcal{E}_h^o}\int_e [\![\mu]\!]\cdot\mathbf{a}\nu^- + \sum_{e\in\Gamma^+}\int_e \mu\nu\mathbf{a}\cdot\mathbf{n}$$

Right hand side

$$L^{DG}(g,f;\mu) = \int_\Omega \mu f + \sum_{e\in\Gamma}\left(\varepsilon\int_e\frac{\langle\kappa\rangle}{h_\perp}\mu g + \int_e s\,\kappa\nabla\mu\cdot\mathbf{n}g\right) - \sum_{e\in\Gamma^-}\int_e \mathbf{a}\cdot\mathbf{n}\mu g.$$

Up-wind Discontinuous Galerkin method

$$B^{DG}(\phi^{DG},\mu) = B^{DG}_{\mathfrak{D}}(\phi^{DG},\mu) + B^{DG}_{\mathfrak{C}}(\phi^{DG},\mu) = L^{DG}(g,f;\mu) \quad (DG)$$

Bilinear Forms

$$\begin{aligned} B^{DG}_{\mathfrak{D}}(\nu,\mu) &= \sum_{T\in\mathcal{T}_h}\int_T \nabla\mu\cdot\kappa\nabla\nu - \sum_{e\in\mathcal{E}_h^o}\int_e [\![\mu]\!]\cdot\kappa^-\nabla\nu^- + s\,\kappa^-\nabla\mu^-[\![\nu]\!] \\ &+ \sum_{e\in\Gamma}\int_e s\,\kappa\nabla\mu\cdot\mathbf{n}\nu - \kappa\nabla\nu\cdot\mathbf{n}\mu + \varepsilon\sum_{e\in\mathcal{E}_h}\int_e \frac{\langle\kappa\rangle}{h_\perp}[\![\mu]\!]\cdot[\![\nu]\!] \\ B^{DG}_{\mathfrak{C}}(\nu,\mu) &= \sum_{T\in\mathcal{T}_h} -\int_T \nabla\mu\cdot\mathbf{a}\nu + \sum_{e\in\mathcal{E}_h^o}\int_e [\![\mu]\!]\cdot\mathbf{a}\nu^- + \sum_{e\in\Gamma^+}\int_e \mu\nu\mathbf{a}\cdot\mathbf{n} \end{aligned}$$

Right hand side

$$L^{DG}(g,f;\mu) = \int_\Omega \mu f + \sum_{e\in\Gamma}\left(\varepsilon\int_e \frac{\langle\kappa\rangle}{h_\perp}\mu g + \int_e s\,\kappa\nabla\mu\cdot\mathbf{n}g\right) - \sum_{e\in\Gamma^-}\int_e \mathbf{a}\cdot\mathbf{n}\mu g.$$

Up-wind Discontinuous Galerkin method

$$B^{DG}(\phi^{DG},\mu) = B^{DG}_{\mathfrak{D}}(\phi^{DG},\mu) + B^{DG}_{\mathfrak{C}}(\phi^{DG},\mu) = L^{DG}(g,f;\mu) \quad (DG)$$

Bilinear Forms

$$\begin{aligned} B^{DG}_{\mathfrak{D}}(\nu,\mu) &= \sum_{T\in\mathcal{T}_h}\int_T \nabla\mu\cdot\kappa\nabla\nu - \sum_{e\in\mathcal{E}^o_h}\int_e [\![\mu]\!]\cdot\kappa^-\nabla\nu^- + s\,\kappa^-\nabla\mu^-[\![\nu]\!] \\ &\quad + \sum_{e\in\Gamma}\int_e s\,\kappa\nabla\mu\cdot\mathbf{n}\nu - \kappa\nabla\nu\cdot\mathbf{n}\mu + \varepsilon\sum_{e\in\mathcal{E}_h}\int_e \frac{\langle\kappa\rangle}{h_\perp}[\![\mu]\!]\cdot[\![\nu]\!] \\ B^{DG}_{\mathfrak{C}}(\nu,\mu) &= \sum_{T\in\mathcal{T}_h} -\int_T \nabla\mu\cdot\mathbf{a}\nu + \sum_{e\in\mathcal{E}^o_h}\int_e [\![\mu]\!]\cdot\mathbf{a}\nu^- + \sum_{e\in\Gamma^+}\int_e \mu\nu\mathbf{a}\cdot\mathbf{n} \end{aligned}$$

Right hand side

$$L^{DG}(g,f;\mu) = \int_\Omega \mu f + \sum_{e\in\Gamma}\left(\varepsilon\int_e \frac{\langle\kappa\rangle}{h_\perp}\mu g + \int_e s\,\kappa\nabla\mu\cdot\mathbf{n}g\right) - \sum_{e\in\Gamma^-}\int_e \mathbf{a}\cdot\mathbf{n}\mu g.$$

Up-wind Discontinuous Galerkin method

$$B^{DG}(\phi^{DG},\mu) = B^{DG}_{\mathfrak{D}}(\phi^{DG},\mu) + B^{DG}_{\mathfrak{C}}(\phi^{DG},\mu) = L^{DG}(g,f;\mu) \quad (DG)$$

Bilinear Forms

$$\begin{aligned} B^{DG}_{\mathfrak{D}}(\nu,\mu) &= \sum_{T\in\mathcal{T}_h}\int_T \nabla\mu\cdot\kappa\nabla\nu - \sum_{e\in\mathcal{E}_h^o}\int_e [\![\mu]\!]\cdot\kappa^-\nabla\nu^- + s\,\kappa^-\nabla\mu^-[\![\nu]\!] \\ &\quad + \sum_{e\in\Gamma}\int_e s\,\kappa\nabla\mu\cdot\mathbf{n}\nu - \kappa\nabla\nu\cdot\mathbf{n}\mu + \varepsilon\sum_{e\in\mathcal{E}_h}\int_e \frac{\langle\kappa\rangle}{h_\perp}[\![\mu]\!]\cdot[\![\nu]\!] \\ B^{DG}_{\mathfrak{C}}(\nu,\mu) &= \sum_{T\in\mathcal{T}_h} -\int_T \nabla\mu\cdot\mathbf{a}\nu + \sum_{e\in\mathcal{E}_h^o}\int_e [\![\mu]\!]\cdot\mathbf{a}\nu^- + \sum_{e\in\Gamma^+}\int_e \mu\nu\mathbf{a}\cdot\mathbf{n} \end{aligned}$$

Right hand side

$$L^{DG}(g,f;\mu) = \int_\Omega \mu f + \sum_{e\in\Gamma}\left(\varepsilon\int_e \frac{\langle\kappa\rangle}{h_\perp}\mu g + \int_e s\,\kappa\nabla\mu\cdot\mathbf{n}g\right) - \sum_{e\in\Gamma^-}\int_e \mathbf{a}\cdot\mathbf{n}\mu g.$$

Up-wind Discontinuous Galerkin method

$$B^{DG}(\phi^{DG},\mu) = B^{DG}_{\mathfrak{D}}(\phi^{DG},\mu) + B^{DG}_{\mathfrak{C}}(\phi^{DG},\mu) = L^{DG}(g,f;\mu) \quad (DG)$$

Bilinear Forms

$$\begin{aligned} B^{DG}_{\mathfrak{D}}(\nu,\mu) &= \sum_{T\in\mathcal{T}_h}\int_T \nabla\mu\cdot\kappa\nabla\nu - \sum_{e\in\mathcal{E}_h^o}\int_e [\![\mu]\!]\cdot\kappa^-\nabla\nu^- + s\,\kappa^-\nabla\mu^-[\![\nu]\!] \\ &\quad + \sum_{e\in\Gamma}\int_e s\,\kappa\nabla\mu\cdot\mathbf{n}\nu - \kappa\nabla\nu\cdot\mathbf{n}\mu + \varepsilon\sum_{e\in\mathcal{E}_h}\int_e \frac{\langle\kappa\rangle}{h_\perp}[\![\mu]\!]\cdot[\![\nu]\!] \\ B^{DG}_{\mathfrak{C}}(\nu,\mu) &= \sum_{T\in\mathcal{T}_h} -\int_T \nabla\mu\cdot\mathbf{a}\nu + \sum_{e\in\mathcal{E}_h^o}\int_e [\![\mu]\!]\cdot\mathbf{a}\nu^- + \sum_{e\in\Gamma^+}\int_e \mu\nu\mathbf{a}\cdot\mathbf{n} \end{aligned}$$

Right hand side

$$L^{DG}(g,f;\mu) = \int_\Omega \mu f + \sum_{e\in\Gamma}\left(\varepsilon\int_e\frac{\langle\kappa\rangle}{h_\perp}\mu g + \int_e s\,\kappa\nabla\mu\cdot\mathbf{n}g\right) - \sum_{e\in\Gamma^-}\int_e \mathbf{a}\cdot\mathbf{n}\mu g.$$

Norms and streamline derivative

$$\|\nu\|_{DG}^2 = \|\nu\|_{\mathfrak{D}}^2 + \|\nu\|_{\mathfrak{C}}^2,$$

where we set: $(\kappa_T = \kappa|_T)$

$$\|\nu\|_{\mathfrak{D}}^2 = \sum_{T\in\mathcal{T}_h} \left(\kappa_T|\nu|_{H^1(T)}^2 + h_T^2\kappa_T|\nu|_{H^2(T)}^2\right) + \varepsilon \sum_{e\in\mathcal{E}_h} \left(h_\perp^{-1}\|\langle\kappa\rangle[\![\nu]\!]\|_{L^2(e)}^2\right),$$

$$\|\nu\|_{\mathfrak{C}}^2 = \sum_{e\in\mathcal{E}_h} \||\mathbf{a}\cdot\mathbf{n}|^{1/2}[\![\nu]\!]\|_{L^2(e)}^2.$$

Coercivity For $\varepsilon > \overline{\varepsilon}$:

$$B_{\mathfrak{C}}^{DG}(\mu,\mu) + B_{\mathfrak{D}}^{DG}(\mu,\mu) \geq \frac{1}{2}\|\mu\|_{\mathfrak{C}}^2 + \beta_1\|\mu\|_{\mathfrak{D}}^2$$

Coercivity was used to provide error estimates in the $\|\cdot\|_{DG}$ norm in Houston et al, 2001 , but this norm is *very* weak in the advection dominated regime.

Norms and streamline derivative

$$\|\nu\|_{DG}^2 = \|\nu\|_{\mathfrak{D}}^2 + \|\nu\|_{\mathfrak{C}}^2,$$

where we set: $(\kappa_T = \kappa|_T)$

$$\|\nu\|_{\mathfrak{D}}^2 = \sum_{T\in\mathcal{T}_h} \left(\kappa_T|\nu|_{H^1(T)}^2 + h_T^2\kappa_T|\nu|_{H^2(T)}^2\right) + \varepsilon \sum_{e\in\mathcal{E}_h} \left(h_\perp^{-1}\|\langle\kappa\rangle[\![\nu]\!]\|_{L^2(e)}^2\right),$$

$$\|\nu\|_{\mathfrak{C}}^2 = \sum_{e\in\mathcal{E}_h} \||\mathbf{a}\cdot\mathbf{n}|^{1/2}[\![\nu]\!]\|_{L^2(e)}^2.$$

Coercivity For $\varepsilon > \overline{\varepsilon}$:

$$B_{\mathfrak{C}}^{DG}(\mu,\mu) + B_{\mathfrak{D}}^{DG}(\mu,\mu) \geq \frac{1}{2}\|\mu\|_{\mathfrak{C}}^2 + \beta_1\|\mu\|_{\mathfrak{D}}^2$$

Coercivity was used to provide error estimates in the $\|\cdot\|_{DG}$ norm in Houston et al, 2001 , but this norm is *very* weak in the advection dominated regime.

Control on the stream-line derivative

Johnson-Pitkäranta, '86, B. Hughes, Sangalli '05

Control on the streamline derivative : $\tau_T = \tau \min\left\{ \frac{h_T}{\|\mathbf{a}\|_T}, \frac{h_T^2}{\kappa_T} \right\}$

$$\|\nu\|_{SDG}^2 = \|\nu\|_{DG}^2 + \sum_{T \in \mathcal{T}_h} \tau_T \|\mathbf{a} \cdot \nabla \nu\|_{L^2(T)}^2$$

We can prove a uniform inf-sup condition :

$$\inf_{\nu \in V_h} \sup_{\mu \in V_h} \frac{B^{DG}(\nu, \mu)}{\|\nu\|_{SDG} \|\mu\|_{SDG}} \geq \beta_{DG} > 0;$$

where β_{DG} is independent of h, κ, $\mathbf{a}$, and the domain.

Proof : Given $\nu \in V_h$, choose $\mu = \nu + \gamma \sum_{T \in \mathcal{T}_h} \tau_T (\mathbf{a} \cdot \nabla \nu)|_T \ldots$

Control on the stream-line derivative

Johnson-Pitkäranta, '86, B. Hughes, Sangalli '05

Remark : DG methods do not need stabilization (e.g., SUPG): the control of the stream-line derivative is built in it.

Error estimate : Optimal error estimates in $\|\cdot\|_{SDG}$ norm can be provided. Indeed:

$$\|\phi - \phi^{DG}\|_{SDG} \lesssim \left(\sum_{T \in \mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k} \right) |\phi|^2_{H^{k+1}(T)} \right)^{1/2} . \quad (1)$$

Main drawback of DG methods : The number of d.o.f. !

Control on the stream-line derivative

Johnson-Pitkäranta, '86, B. Hughes, Sangalli '05

Remark : DG methods do not need stabilization (e.g., SUPG): the control of the stream-line derivative is built in it.

Error estimate : Optimal error estimates in $\|\cdot\|_{SDG}$ norm can be provided. Indeed:

$$\|\phi - \phi^{DG}\|_{SDG} \lesssim \left(\sum_{T \in \mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k} \right) |\phi|^2_{H^{k+1}(T)} \right)^{1/2} . \quad (1)$$

Main drawback of DG methods : The number of d.o.f. !

Control on the stream-line derivative

Johnson-Pitkäranta, '86, B. Hughes, Sangalli '05

Remark : DG methods do not need stabilization (e.g., SUPG): the control of the stream-line derivative is built in it.

Error estimate : Optimal error estimates in $\| \cdot \|_{SDG}$ norm can be provided. Indeed:

$$\|\phi - \phi^{DG}\|_{SDG} \lesssim \left(\sum_{T \in \mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k} \right) |\phi|^2_{H^{k+1}(T)} \right)^{1/2} . \quad (1)$$

Main drawback of DG methods : The number of d.o.f. !

Multiscale Paradigm: a VMS-DG Method

Hughes, Scovazzi, Pavel, B. '05 - B., Hughes, Sangalli '06

Reduce the number of d.o.f. $\Rightarrow$ solve the DG problem on a strict subset of V_h, which is built by solving local problems in a multiscale fashion.

We introduce the space $\overline{V}_h = V_h \cap C^0(\overline{\Omega})$ and define a "interscale transfer operator" $\mathfrak{T}_h : \overline{V}_h \times L^2(\Omega) \to V_h$:

$$(\bar{\nu}, f) \in \overline{V}_h \times L^2(\Omega), \quad \text{SOLVE WITH DG} \quad \begin{cases} \mathcal{L}_T \nu &= f & T \in \mathcal{T}_h \\ \nu &= \bar{\nu} & \Gamma_T \end{cases}$$

where $\mathcal{L}_T$ is the restriction of $\mathcal{L}$ on the triangle T.

Multiscale Paradigm: a VMS-DG Method

Hughes, Scovazzi, Pavel, B. '05 - B., Hughes, Sangalli '06

Reduce the number of d.o.f. $\Rightarrow$ solve the DG problem on a strict subset of V_h, which is built by solving local problems in a multiscale fashion.

We introduce the space $\overline{V}_h = V_h \cap C^0(\overline{\Omega})$ and define a "interscale transfer operator" $\mathfrak{T}_h : \overline{V}_h \times L^2(\Omega) \to V_h$:

$$(\bar{\nu}, f) \in \overline{V}_h \times L^2(\Omega), \quad \text{SOLVE WITH DG} \qquad \begin{cases} \mathcal{L}_T \nu & = f \qquad T \in \mathcal{T}_h \\ \nu & = \bar{\nu} \qquad \Gamma_T \end{cases}$$

where $\mathcal{L}_T$ is the restriction of $\mathcal{L}$ on the triangle T.

Multiscale Paradigm: a VMS-DG Method

Hughes, Scovazzi, Pavel, B. '05 - B., Hughes, Sangalli '06

Reduce the number of d.o.f. $\Rightarrow$ solve the DG problem on a strict subset of V_h, which is built by solving local problems in a multiscale fashion.

We introduce the space $\overline{V}_h = V_h \cap C^0(\overline{\Omega})$ and define a "interscale transfer operator" $\mathfrak{T}_h : \overline{V}_h \times L^2(\Omega) \to V_h$:

$$(\bar{\nu}, f) \in \overline{V}_h \times L^2(\Omega), \quad \text{SOLVE WITH DG} \qquad \begin{cases} \mathcal{L}_T \nu & = f \qquad T \in \mathcal{T}_h \\ \nu & = \bar{\nu} \qquad \Gamma_T \end{cases}$$

where $\mathcal{L}_T$ is the restriction of $\mathcal{L}$ on the triangle T.

Remark

The affine space $\mathfrak{T}_h(\overline{V}_h, f)$ represents a subset of V_h, parametrized through the d.o.f. of $\overline{V}_h$: it is strictly contained in V_h.

Multiscale Paradigm: a VMS-DG Method

Hughes, Scovazzi, Pavel, B. '05 - B., Hughes, Sangalli '06

Reduce the number of d.o.f. $\Rightarrow$ solve the DG problem on a strict subset of V_h, which is built by solving local problems in a multiscale fashion.

We introduce the space $\overline{V}_h = V_h \cap C^0(\overline{\Omega})$ and define a "interscale transfer operator" $\mathfrak{T}_h : \overline{V}_h \times L^2(\Omega) \to V_h$:

$$(\bar{\nu}, f) \in \overline{V}_h \times L^2(\Omega), \quad \text{SOLVE WITH DG} \qquad \begin{cases} \mathcal{L}_T \nu & = f \qquad T \in \mathcal{T}_h \\ \nu & = \bar{\nu} \qquad \Gamma_T \end{cases}$$

where $\mathcal{L}_T$ is the restriction of $\mathcal{L}$ on the triangle T.

Multiscale reduction

We solve the DG problem on the set $\mathfrak{T}_h(\overline{V}_h, f)$.

Variational Multiscale method - definition

Local Problems

Given $\bar{\nu} \in \overline{V}_h$, $\nu = \mathfrak{T}_h(\bar{\nu}, f)$ is the solution of

$$b_T(\nu, \mu) = \ell_T(\bar{\nu}, f; \mu) \quad \forall\, \mu \in V_h, \quad \forall\, T \in \mathcal{T}_h \tag{LP}$$

where b_T is the DG bilinear form associated with the operator $\mathcal{L}_T$.

Global Problem

Find $\phi^{MDG} \in \mathfrak{T}_h(\overline{V}_h, f)$ such that:

$$B^{DG}(\phi^{MDG}, \mu) = L^{DG}(g, f; \mu) \quad \text{for all} \quad \mu \in \mathfrak{T}_h(\overline{V}_h, 0). \tag{MDG}$$

Questions :
1. Solvability for the local and global problems;
2. Stability of the local and global problems;
3. Approximation properties of the interscale transfer
. (affine) space $\mathfrak{T}_h(\overline{V}_h, f)$.

Variational Multiscale method - definition

Local Problems

Given $\bar{\nu} \in \overline{V}_h$, $\nu = \mathfrak{T}_h(\bar{\nu}, f)$ is the solution of

$$b_T(\nu, \mu) = \ell_T(\bar{\nu}, f; \mu) \quad \forall\, \mu \in V_h, \quad \forall\, T \in \mathcal{T}_h \tag{LP}$$

where b_T is the DG bilinear form associated with the operator $\mathcal{L}_T$.

Global Problem

Find $\phi^{MDG} \in \mathfrak{T}_h(\overline{V}_h, f)$ such that:

$$B^{DG}(\phi^{MDG}, \mu) = L^{DG}(g, f; \mu) \quad \text{for all} \quad \mu \in \mathfrak{T}_h(\overline{V}_h, 0). \tag{MDG}$$

Questions :
1. Solvability for the local and global problems;
2. Stability of the local and global problems;
3. Approximation properties of the interscale transfer
. (affine) space $\mathfrak{T}_h(\overline{V}_h, f)$.

Variational Multiscale method - algebraic structure

The algebraic structure is the following:

$$\mathfrak{T}_h(\bar{\nu}, f) \Rightarrow \mathbf{T}_u \underline{\bar{\nu}} + \mathbf{T}_f \underline{\mathrm{f}}$$

i.e., the final system is of the following type:

$$\mathbf{T}_u^\star \mathbf{B} \mathbf{T}_u \underline{\bar{\phi}} = \mathbf{T}_u^\star \underline{\mathrm{f}} - \mathbf{T}_u^\star \mathbf{B} \mathbf{T}_f \underline{\mathrm{f}}.$$

Variational Multiscale method - algebraic structure

The algebraic structure is the following:

$$\mathfrak{T}_h(\bar{\nu}, f) \Rightarrow \mathbf{T}_u \underline{\bar{\nu}} + \mathbf{T}_f \underline{\mathrm{f}}$$

i.e., the final system is of the following type:

$$\mathbf{T}_u^\star \mathbf{B} \mathbf{T}_u \underline{\bar{\phi}} = \mathbf{T}_u^\star \underline{\mathrm{f}} - \mathbf{T}_u^\star \mathbf{B} \mathbf{T}_f \underline{\mathrm{f}}.$$

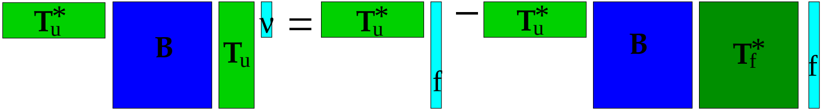


Figure: size of the final system: $N_{\overline{V}_h} \times N_{\overline{V}_h}$

Remark : The transfer matrices $\mathbf{T}_u$ $(N_{V_h} \times N_{\overline{V}_h})$ a $\mathbf{T}_f$ $(N_{V_h} \times N_{V_h})$ are build inexpensively element by element.

Variational Multiscale method - comments

Computed quantity

If ϕ^{MDG} is the *MDG* solution, i.e.,

$$B^{DG}(\phi^{MDG}, \mu) = L^{DG}(g, f; \mu) \quad \text{for all} \quad \mu \in \mathfrak{T}_h(\overline{V}_h, 0). \qquad \text{(MDG)}$$

then $\phi^{MDG} = \mathfrak{T}_h(\bar{\phi}^{MDG}, f)$, for some $\bar{\phi}^{MDG} \in \overline{V}_h$.

Indeed, we compute $\bar{\phi}^{MDG}$, and ϕ^{MDG} can be built by post processing (applying the matrix T).

We have two discrete quantities, and we can evaluate both errors:

$$\|\phi - \phi^{MDG}\| \quad \text{and} \quad \|\phi - \bar{\phi}^{MDG}\|$$

Variational Multiscale method - comments

Computed quantity

If ϕ^{MDG} is the *MDG* solution, i.e.,

$$B^{DG}(\phi^{MDG}, \mu) = L^{DG}(g, f; \mu) \quad \text{for all} \quad \mu \in \mathfrak{T}_h(\overline{V}_h, 0). \qquad \text{(MDG)}$$

then $\phi^{MDG} = \mathfrak{T}_h(\bar{\phi}^{MDG}, f)$, for some $\bar{\phi}^{MDG} \in \overline{V}_h$.

Indeed, we compute $\bar{\phi}^{MDG}$, and ϕ^{MDG} can be built by post processing (applying the matrix T).

We have two discrete quantities, and we can evaluate both errors:

$$\|\phi - \phi^{MDG}\| \quad \text{and} \quad \|\phi - \bar{\phi}^{MDG}\|$$

Local problems

$\forall \bar{\nu} \in \overline{V}_h$, we solve with an up-wind DG scheme:

$$\mathcal{L}_T \nu = f \text{ on } T \qquad \nu = \bar{\nu} \text{ on } \Gamma_T.$$

Thus, $\nu = \mathfrak{T}_h(\bar{\nu}, f)$ if $\nu \in V_h$ verifies for all $T \in \mathcal{T}_h$:

$$b_T(\nu, \mu) = \ell_T(\bar{\nu}, f; \mu) \quad \forall \mu \in V_h(T),$$

with

$$b_T(\nu, \mu) = \int_T \kappa \nabla \nu \cdot \nabla \mu - \int_{\Gamma_T} (\kappa \nabla \nu \cdot \mathbf{n} \mu - s\, \kappa \nabla \mu \cdot \mathbf{n} \nu) + \varepsilon \int_{\Gamma_T} \frac{\kappa}{h_\perp} \mu \nu$$
$$- \int_T \nabla \mu \cdot \mathbf{a} \nu + \int_{\Gamma_T^+} (1 + \delta)\, \mu \nu \mathbf{a} \cdot \mathbf{n},$$

$$\ell_T(\bar{\nu}, f; \mu) = \varepsilon \int_{\Gamma_T} \frac{\kappa}{h_\perp} \mu \bar{\nu} + \int_{\Gamma_T} s\, \kappa \nabla \mu \cdot \mathbf{n} \bar{\nu} + \int_T f\, \mu$$
$$- \int_{\Gamma_T^-} \mu \bar{\nu} \mathbf{a} \cdot \mathbf{n} + \delta \int_{\Gamma_T^+} \mu \bar{\nu} \mathbf{a} \cdot \mathbf{n}$$

Local problems

$\forall \bar{\nu} \in \overline{V}_h$, we solve with an up-wind DG scheme:

$$\mathcal{L}_T \nu = f \text{ on } T \qquad \nu = \bar{\nu} \text{ on } \Gamma_T.$$

Thus, $\nu = \mathfrak{T}_h(\bar{\nu}, f)$ if $\nu \in V_h$ verifies for all $T \in \mathcal{T}_h$:

$$b_T(\nu, \mu) = \ell_T(\bar{\nu}, f; \mu) \quad \forall \mu \in V_h(T),$$

with

$$b_T(\nu, \mu) = \int_T \kappa \nabla \nu \cdot \nabla \mu - \int_{\Gamma_T} (\kappa \nabla \nu \cdot \mathbf{n} \mu - s\, \kappa \nabla \mu \cdot \mathbf{n} \nu) + \varepsilon \int_{\Gamma_T} \frac{\kappa}{h_\perp} \mu \nu$$
$$- \int_T \nabla \mu \cdot \mathbf{a} \nu + \int_{\Gamma_T^+} (1 + \delta)\, \mu \nu \mathbf{a} \cdot \mathbf{n},$$

$$\ell_T(\bar{\nu}, f; \mu) = \varepsilon \int_{\Gamma_T} \frac{\kappa}{h_\perp} \mu \bar{\nu} + \int_{\Gamma_T} s\, \kappa \nabla \mu \cdot \mathbf{n} \bar{\nu} + \int_T f\, \mu$$
$$- \int_{\Gamma_T^-} \mu \bar{\nu} \mathbf{a} \cdot \mathbf{n} + \delta \int_{\Gamma_T^+} \mu \bar{\nu} \mathbf{a} \cdot \mathbf{n}$$

Local problems

$\forall \bar{\nu} \in \overline{V}_h$, we solve with an up-wind DG scheme:

$$\mathcal{L}_T \nu = f \text{ on } T \qquad \nu = \bar{\nu} \text{ on } \Gamma_T.$$

Thus, $\nu = \mathfrak{T}_h(\bar{\nu}, f)$ if $\nu \in V_h$ verifies for all $T \in \mathcal{T}_h$:

$$b_T(\nu, \mu) = \ell_T(\bar{\nu}, f; \mu) \quad \forall \mu \in V_h(T),$$

with

$$b_T(\nu, \mu) = \int_T \kappa \nabla \nu \cdot \nabla \mu - \int_{\Gamma_T} (\kappa \nabla \nu \cdot \mathbf{n} \mu - s\, \kappa \nabla \mu \cdot \mathbf{n} \nu) + \varepsilon \int_{\Gamma_T} \frac{\kappa}{h_\perp} \mu \nu$$
$$- \int_T \nabla \mu \cdot \mathbf{a} \nu + \int_{\Gamma_T^+} (1 + \delta)\, \mu \nu \mathbf{a} \cdot \mathbf{n},$$

$$\ell_T(\bar{\nu}, f; \mu) = \varepsilon \int_{\Gamma_T} \frac{\kappa}{h_\perp} \mu \bar{\nu} + \int_{\Gamma_T} s\, \kappa \nabla \mu \cdot \mathbf{n} \bar{\nu} + \int_T f\, \mu$$
$$- \int_{\Gamma_T^-} \mu \bar{\nu} \mathbf{a} \cdot \mathbf{n} + \delta \int_{\Gamma_T^+} \mu \bar{\nu} \mathbf{a} \cdot \mathbf{n}$$

Solvability and Stability for the local problems

The *local problems* read: $\forall \bar{\nu} \in \overline{V}_h$, find $\nu \in V_h$:

$$b_T(\nu, \mu) = \ell_T(\bar{\nu}, f; \mu) \quad \forall \mu \in V_h, \forall T \in \mathcal{T}_h$$

We construct local norms (with control on the stream-line derivative)

$$\|\nu\|^2_{SDG(T)} := \kappa_T |\nu|^2_{H^1(T)} + h_T^2 \kappa_T |\nu|^2_{H^2(T)} + \varepsilon h_T^{-1} \kappa_T \|\nu\|^2_{L^2(\Gamma_T)} + \tau_T \|\mathbf{a} \cdot \nabla \nu\|^2_{L^2(T)} + \||\mathbf{a} \cdot \mathbf{n}|^{1/2} \nu\|^2_{L^2(\Gamma_T)}.$$

Stability and Solvability

There exists positive $\bar{\varepsilon}$ and $\bar{\delta}$ such that for all $\varepsilon \geq \bar{\varepsilon}$ and $\delta \leq \bar{\delta}$,

$$\inf_{\nu \in V_h(T)} \sup_{\mu \in V_h(T)} \frac{b_T(\nu, \mu)}{\|\nu\|_{SDG(T)} \|\mu\|_{SDG(T)}} \geq \beta_b > 0, \qquad \forall T \in \mathcal{T}_h,$$

Solvability and Stability for the local problems

The *local problems* read: $\forall \bar{\nu} \in \overline{V}_h$, find $\nu \in V_h$:

$$b_T(\nu, \mu) = \ell_T(\bar{\nu}, f; \mu) \quad \forall\, \mu \in V_h, \forall\, T \in \mathcal{T}_h$$

We construct local norms (with control on the stream-line derivative)

$$\|\nu\|^2_{SDG(T)} := \kappa_T |\nu|^2_{H^1(T)} + h_T^2 \kappa_T |\nu|^2_{H^2(T)} + \varepsilon h_T^{-1} \kappa_T \|\nu\|^2_{L^2(\Gamma_T)} + \tau_T \|\mathbf{a} \cdot \nabla \nu\|^2_{L^2(T)} + \||\mathbf{a} \cdot \mathbf{n}|^{1/2} \nu\|^2_{L^2(\Gamma_T)}.$$

Stability and Solvability

There exists positive $\bar{\varepsilon}$ and $\bar{\delta}$ such that for all $\varepsilon \geq \bar{\varepsilon}$ and $\delta \leq \bar{\delta}$,

$$\inf_{\nu \in V_h(T)} \sup_{\mu \in V_h(T)} \frac{b_T(\nu, \mu)}{\|\nu\|_{SDG(T)} \|\mu\|_{SDG(T)}} \geq \beta_b > 0, \qquad \forall T \in \mathcal{T}_h,$$

Solvability and Stability for the local problems

The *local problems* read: $\forall \bar{\nu} \in \overline{V}_h$, find $\nu \in V_h$:

$$b_T(\nu, \mu) = \ell_T(\bar{\nu}, f; \mu) \quad \forall \mu \in V_h, \forall T \in \mathcal{T}_h$$

We construct local norms (with control on the stream-line derivative)

$$\|\nu\|^2_{SDG(T)} := \kappa_T |\nu|^2_{H^1(T)} + h_T^2 \kappa_T |\nu|^2_{H^2(T)} + \varepsilon h_T^{-1} \kappa_T \|\nu\|^2_{L^2(\Gamma_T)} + \tau_T \|\mathbf{a} \cdot \nabla \nu\|^2_{L^2(T)} + \||\mathbf{a} \cdot \mathbf{n}|^{1/2} \nu\|^2_{L^2(\Gamma_T)}.$$

Stability and Solvability

There exists positive $\bar{\varepsilon}$ and $\bar{\delta}$ such that for all $\varepsilon \geq \bar{\varepsilon}$ and $\delta \leq \bar{\delta}$,

$$\inf_{\nu \in V_h(T)} \sup_{\mu \in V_h(T)} \frac{b_T(\nu, \mu)}{\|\nu\|_{SDG(T)} \|\mu\|_{SDG(T)}} \geq \beta_b > 0, \qquad \forall T \in \mathcal{T}_h,$$

Solvability and Stability for the local problems

The *local problems* read: $\forall \bar{\nu} \in \overline{V}_h$, find $\nu \in V_h$:

$$b_T(\nu, \mu) = \ell_T(\bar{\nu}, f; \mu) \quad \forall \mu \in V_h, \forall T \in \mathcal{T}_h$$

We construct local norms (with control on the stream-line derivative)

$$\|\nu\|^2_{SDG(T)} := \kappa_T |\nu|^2_{H^1(T)} + h_T^2 \kappa_T |\nu|^2_{H^2(T)} + \varepsilon h_T^{-1} \kappa_T \|\nu\|^2_{L^2(\Gamma_T)} + \tau_T \|\mathbf{a} \cdot \nabla \nu\|^2_{L^2(T)} + \||\mathbf{a} \cdot \mathbf{n}|^{1/2} \nu\|^2_{L^2(\Gamma_T)}.$$

Stability and Solvability

There exists positive $\overline{\varepsilon}$ and $\overline{\delta}$ such that for all $\varepsilon \geq \overline{\varepsilon}$ and $\delta \leq \overline{\delta}$,

$$\inf_{\nu \in V_h(T)} \sup_{\mu \in V_h(T)} \frac{b_T(\nu, \mu)}{\|\nu\|_{SDG(T)} \|\mu\|_{SDG(T)}} \geq \beta_b > 0, \qquad \forall T \in \mathcal{T}_h,$$

Approximation properties for the transfer spaces

Let ϕ be the solution of the continuous problem:

$$\mathcal{L}\phi = f \quad \Omega\,, \quad \phi = g \quad \partial\Omega$$

then there exists $\nu \in \mathfrak{T}_h(\overline{V}_h, f)$ such that

$$|||\phi - \nu|||^2_{SDG} \lesssim \sum_{T\in\mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k}\right) |\phi|^2_{H^{k+1}(T)}$$

Proof : Select $\nu = \mathfrak{T}_h(\phi^I, f)$, ϕ^I being the interpolant of ϕ, use the stability of local problems and an *ad-hoc* Poincaré inequality:

$$\tau_T^{-1}\|\nu\|^2_{L^2(T)} \lesssim \|\nu\|^2_{SDG(T)} \quad \tau_T = \tau \min\left\{\frac{h_T}{a_T}, \frac{h_T^2}{\kappa_T}\right\}.$$

Approximation properties for the transfer spaces

Let ϕ be the solution of the continuous problem:

$$\mathcal{L}\phi = f \quad \Omega\,, \quad \phi = g \quad \partial\Omega$$

then there exists $\nu \in \mathfrak{T}_h(\overline{V}_h, f)$ such that

$$|\!|\!|\phi - \nu|\!|\!|^2_{SDG} \lesssim \sum_{T\in\mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k}\right) |\phi|^2_{H^{k+1}(T)}$$

Proof : Select $\nu = \mathfrak{T}_h(\phi^I, f)$, ϕ^I being the interpolant of ϕ, use the stability of local problems and an *ad-hoc* Poincaré inequality:

$$\tau_T^{-1}\|\nu\|^2_{L^2(T)} \lesssim |\!|\!|\nu|\!|\!|^2_{SDG(T)} \quad \tau_T = \tau \min\left\{\frac{h_T}{a_T}, \frac{h_T^2}{\kappa_T}\right\}.$$

Solvability and Stability for the global problem

Find $\phi^{MDG} \in \mathfrak{T}_h(\overline{V}_h, f)$ such that:

$$B^{DG}(\phi^{MDG}, \mu) = L^{DG}(g, f; \mu) \quad \text{for all} \quad \mu \in \mathfrak{T}_h(\overline{V}_h, 0). \qquad \text{(MDG)}$$

√ Coercivity in "weak" norm: $B^{DG}(\phi^{MDG}, \phi^{MDG}) \geq \alpha \|\phi^{MDG}\|^2_{DG}$

+ Inf-sup condition

$$\inf_{\nu \in \mathfrak{T}_h(\overline{V}_h, 0)} \sup_{\mu \in \mathfrak{T}_h(\overline{V}_h, 0)} \frac{B^{DG}(\nu, \mu)}{\|\nu\|_{SDG} \|\mu\|_{SDG}} \geq \beta_{MDG} > 0;$$

If we had the inf-sup condition (or if we add a SUPG stabilization), then

$$\|\phi - \phi^{MDG}\|^2_{SDG} \lesssim \sum_{T \in \mathcal{T}_h} (a_T h_T^{2k+1} + \kappa_T h_T^{2k}) |\phi|^2_{H^{k+1}(T)}$$

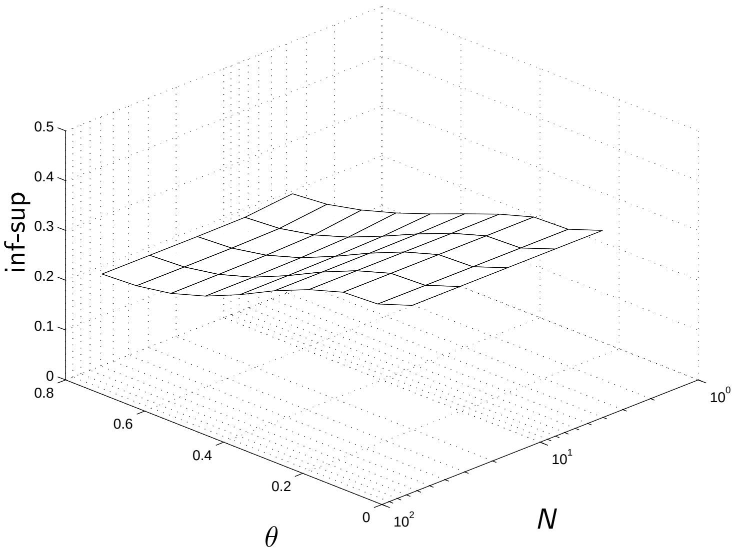


Figure: Inf-sup constant of the MDG method vs. $\mathbf{a} = [\cos\theta, \sin\theta]$ and N.

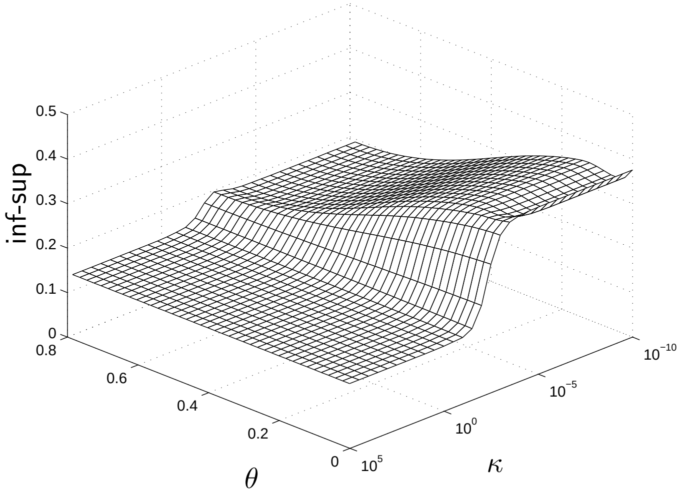


Figure: Inf-sup constant of the MDG method vs. $\mathbf{a} = [\cos\theta, \sin\theta]$ and κ on a 10×10 mesh

1-D numerical tests

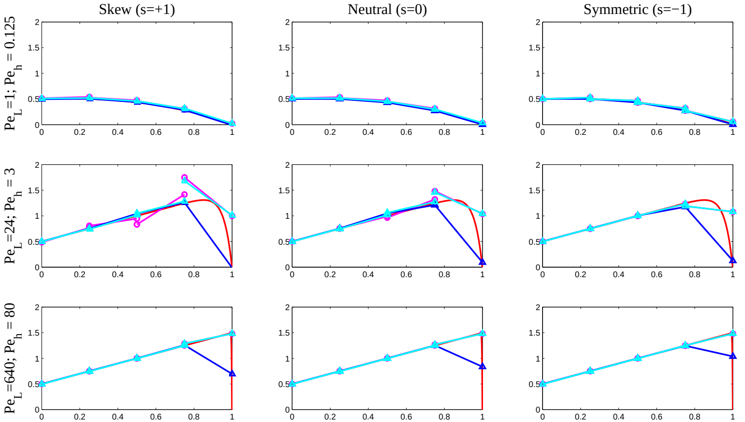


Figure: blue=MDG (cont), L-blue= MDG (disc), magenta = donor DG

1-D numerical tests, convergence (neutral)

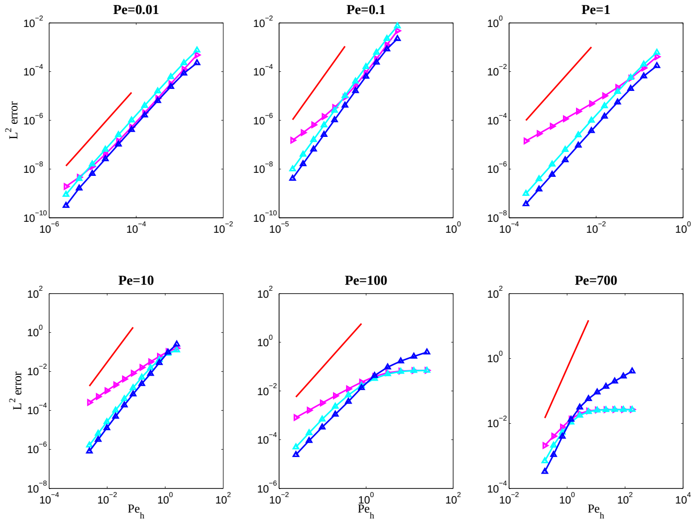


Figure: blue=MDG (cont), L-blue= MDG (disc), magenta = donor DG

1-D numerical tests, convergence (skew-symmetric)

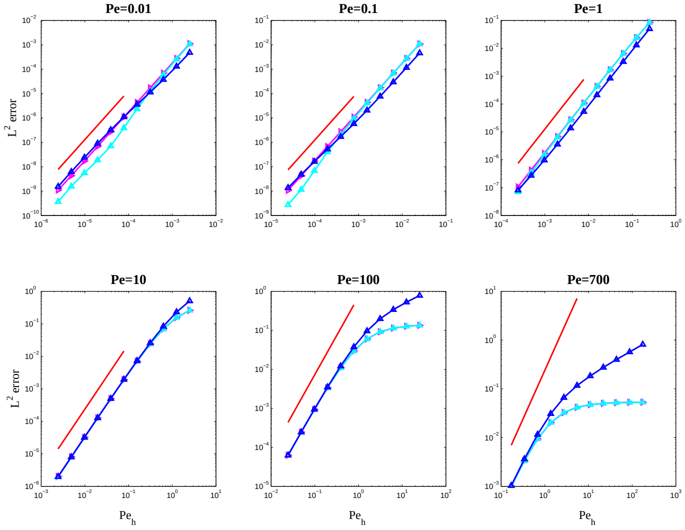
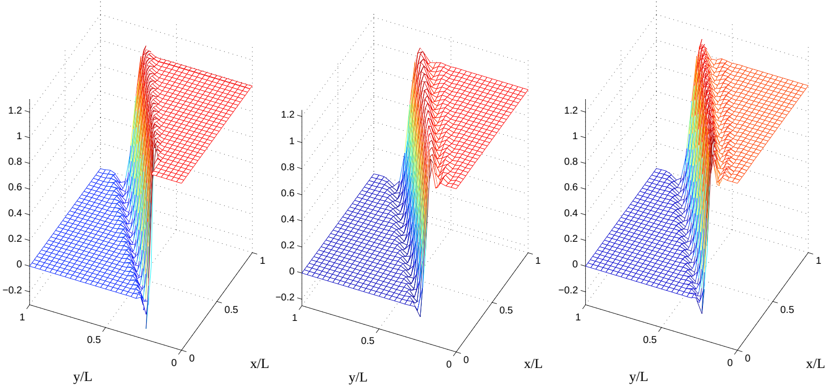


Figure: blue=MDG (cont), L-blue= MDG (disc), magenta = donor DG

2-D numerical tests, internal layer



(a) donor DG (b) MDG (cont) (c) MDG (tot)

2-D numerical tests, section

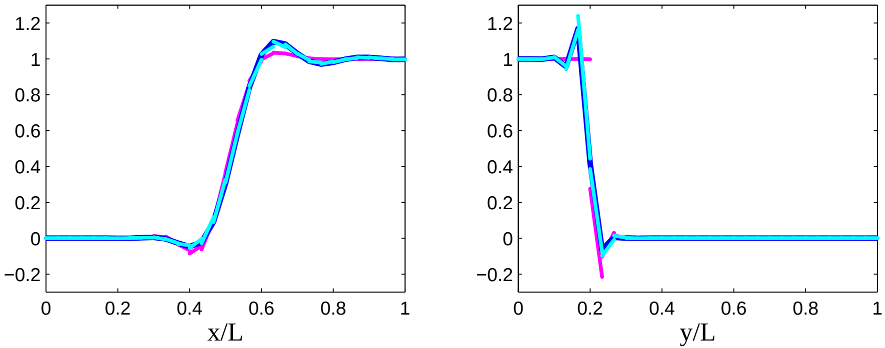


Figure: blue=MDG (cont), L-blue= MDG (disc), magenta = donor DG

Multiscale interpretation and comparison with RFB

Multiscale local problem

Given $\nu = \mathfrak{T}_h(\bar{\nu}, f)$, with $\bar{\nu} \in \overline{V}_h$, then $\nu = \bar{\nu} + \nu'$, $\nu' =$ fast scale

When $k = 1$ (to fix ideas), the fast scale ν' solves the local problems:

$$b_T(\nu', \mu) = \int_T (f - \mathcal{L}_T \bar{\nu})\mu \,, \quad \forall \mu \in V_h(T)$$

Thus, ν' is a DG discretization for the *Residual Free Bubble* :

$$\mathcal{L}_T \nu^{bubble} = f - \mathcal{L}_T \bar{\nu} \text{ on } T \,, \qquad \nu^{bubble}|_{\Gamma_T} = 0.$$

Multiscale global problem

Compute $\bar{\phi} \in \overline{V}_h$

$$B^{DG}(\bar{\phi} + \phi'(\bar{\phi}, f), \bar{\mu} + \mu'(\bar{\phi}, 0)) = L^{DG}(g, f; \bar{\mu} + \mu'(\bar{\phi}, 0)) \quad \text{for all} \quad \bar{\mu} \in \overline{V}_h.$$

The MDG method can be interpreted as a discontinuous stabilization method for standard finite elements. The relation between MDG and *bubble* type stabilization deserves further investigation.

Multiscale interpretation and comparison with RFB

Multiscale local problem

Given $\nu = \mathfrak{T}_h(\bar{\nu}, f)$, with $\bar{\nu} \in \overline{V}_h$, then $\nu = \bar{\nu} + \nu'$, $\nu' =$ fast scale
When $k = 1$ (to fix ideas), the fast scale ν' solves the local problems:

$$b_T(\nu', \mu) = \int_T (f - \mathcal{L}_T \bar{\nu})\mu\,, \quad \forall \mu \in V_h(T)$$

Thus, ν' is a DG discretization for the *Residual Free Bubble* :

$$\mathcal{L}_T \nu^{bubble} = f - \mathcal{L}_T \bar{\nu} \text{ on } T\,, \qquad \nu^{bubble}|_{\Gamma_T} = 0.$$

Multiscale global problem

Compute $\bar{\phi} \in \overline{V}_h$

$$B^{DG}(\bar{\phi} + \phi'(\bar{\phi}, f), \bar{\mu} + \mu'(\bar{\phi}, 0)) = L^{DG}(g, f; \bar{\mu} + \mu'(\bar{\phi}, 0)) \quad \text{for all} \quad \bar{\mu} \in \overline{V}_h.$$

The MDG method can be interpreted as a discontinuous stabilization method for standard finite elements. The relation between MDG and *bubble* type stabilization deserves further investigation.

Multiscale interpretation and comparison with RFB

Multiscale local problem

Given $\nu = \mathfrak{T}_h(\bar{\nu}, f)$, with $\bar{\nu} \in \overline{V}_h$, then $\nu = \bar{\nu} + \nu'$, $\nu' =$ fast scale
When $k = 1$ (to fix ideas), the fast scale ν' solves the local problems:

$$b_T(\nu', \mu) = \int_T (f - \mathcal{L}_T \bar{\nu})\mu \,, \quad \forall \mu \in V_h(T)$$

Thus, ν' is a DG discretization for the *Residual Free Bubble* :

$$\mathcal{L}_T \nu^{bubble} = f - \mathcal{L}_T \bar{\nu} \text{ on } T \,, \qquad \nu^{bubble}|_{\Gamma_T} = 0.$$

Multiscale global problem

Compute $\bar{\phi} \in \overline{V}_h$

$$B^{DG}(\bar{\phi} + \phi'(\bar{\phi}, f), \bar{\mu} + \mu'(\bar{\phi}, 0)) = L^{DG}(g, f; \bar{\mu} + \mu'(\bar{\phi}, 0)) \quad \text{for all} \quad \bar{\mu} \in \overline{V}_h.$$

The MDG method can be interpreted as a discontinuous stabilization method for standard finite elements. The relation between MDG and *bubble* type stabilization deserves further investigation.

Multiscale interpretation and comparison with RFB

Multiscale local problem

Given $\nu = \mathfrak{T}_h(\bar{\nu}, f)$, with $\bar{\nu} \in \overline{V}_h$, then $\nu = \bar{\nu} + \nu'$, ν' = fast scale
When $k = 1$ (to fix ideas), the fast scale ν' solves the local problems:

$$b_T(\nu', \mu) = \int_T (f - \mathcal{L}_T \bar{\nu})\mu\,, \quad \forall \mu \in V_h(T)$$

Thus, ν' is a DG discretization for the *Residual Free Bubble* :

$$\mathcal{L}_T \nu^{bubble} = f - \mathcal{L}_T \bar{\nu} \text{ on } T\,, \qquad \nu^{bubble}|_{\Gamma_T} = 0.$$

Multiscale global problem

Compute $\bar{\phi} \in \overline{V}_h$

$$B^{DG}(\bar{\phi} + \phi'(\bar{\phi}, f), \bar{\mu} + \mu'(\bar{\phi}, 0)) = L^{DG}(g, f; \bar{\mu} + \mu'(\bar{\phi}, 0)) \quad \text{for all} \quad \bar{\mu} \in \overline{V}_h.$$

The MDG method can be interpreted as a discontinuous stabilization method for standard finite elements. The relation between MDG and *bubble* type stabilization deserves further investigation.

Conclusions

We presented the first results on the analysis of a Multiscale DG method. The method can be interpreted as:

- a strategy to reduce the number of d.o.f for DG methods while preserving the main features such flexibility and stability;
- a multiscale stabilization mechanism for convection dominated problems & standard conforming finite elements.

Our approach is related to the *hybridization technique of DG methods* recently introduced by Cockburn *et al*: further investigations are due.

Thanks for the attention!

http://www.imati.cnr.it/annalisa

Conclusions

We presented the first results on the analysis of a Multiscale DG method.
The method can be interpreted as:

- a strategy to reduce the number of d.o.f for DG methods while preserving the main features such flexibility and stability;
- a multiscale stabilization mechanism for convection dominated problems & standard conforming finite elements.

Our approach is related to the *hybridization technique of DG methods* recently introduced by Cockburn *et al*: further investigations are due.

Thanks for the attention!

http://www.imati.cnr.it/annalisa

Conclusions

We presented the first results on the analysis of a Multiscale DG method. The method can be interpreted as:

- a strategy to reduce the number of d.o.f for DG methods while preserving the main features such flexibility and stability;
- a multiscale stabilization mechanism for convection dominated problems & standard conforming finite elements.

Our approach is related to the *hybridization technique of DG methods* recently introduced by Cockburn *et al*: further investigations are due.

Thanks for the attention!

http://www.imati.cnr.it/annalisa

SUPG stabilization

Recalling $\tau_T = \tau \min\left\{\frac{h_T}{\|\mathbf{a}\|_T}, \frac{h_T^2}{\kappa_T}\right\}$, we set:

back

$$B^{SDG}(\nu,\mu) = B^{DG}(\nu,\mu) + \sum_{T\in\mathcal{T}_h} \tau_T \int_T (\mathcal{L}_T\nu)(\mathbf{a}\cdot\nabla\mu),$$

$$L^{SDG}(g,f;\mu) = L^{DG}(g,f;\mu) + \sum_{T\in\mathcal{T}_h} \tau_T \int_T f(\mathbf{a}\cdot\nabla\mu),$$

and solve: Find $\phi^{SMDG} \in \mathfrak{T}_h(\overline{V}_h, f)$ such that

$$B^{SDG}(\phi^{SMDG},\mu) = L^{SDG}(g,f;\mu) \quad \text{for all} \quad \mu \in \mathfrak{T}_h(\overline{V}_h, 0). \qquad \text{(SMDG)}$$

Coercivity: : $B^{SDG}(\nu,\nu) \geq \alpha\|\nu\|_{SDG}^2 \quad \forall \nu \in \mathfrak{T}_h(\overline{V}_h, 0).$

Error estimate : for $\tau \leq \bar{\tau}$:

$$\|\phi - \phi^{SMDG}\|_{SDG}^2 \lesssim \sum_{T\in\mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k}\right) |\phi|_{H^{k+1}(T)}^2$$

SUPG stabilization

Recalling $\tau_T = \tau \min \left\{ \frac{h_T}{\|\mathbf{a}\|_T}, \frac{h_T^2}{\kappa_T} \right\}$, we set:

back

$$B^{SDG}(\nu, \mu) = B^{DG}(\nu, \mu) + \sum_{T \in \mathcal{T}_h} \tau_T \int_T (\mathcal{L}_T \nu)(\mathbf{a} \cdot \nabla \mu),$$

$$L^{SDG}(g, f; \mu) = L^{DG}(g, f; \mu) + \sum_{T \in \mathcal{T}_h} \tau_T \int_T f(\mathbf{a} \cdot \nabla \mu),$$

and solve: Find $\phi^{SMDG} \in \mathfrak{T}_h(\overline{V}_h, f)$ such that

$$B^{SDG}(\phi^{SMDG}, \mu) = L^{SDG}(g, f; \mu) \quad \text{for all} \quad \mu \in \mathfrak{T}_h(\overline{V}_h, 0). \qquad \text{(SMDG)}$$

Coercivity: : $B^{SDG}(\nu, \nu) \geq \alpha \|\nu\|_{SDG}^2 \quad \forall \nu \in \mathfrak{T}_h(\overline{V}_h, 0).$

Error estimate : for $\tau \leq \bar{\tau}$:

$$\|\phi - \phi^{SMDG}\|_{SDG}^2 \lesssim \sum_{T \in \mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k} \right) |\phi|_{H^{k+1}(T)}^2$$

SUPG stabilization

Recalling $\tau_T = \tau \min\left\{\frac{h_T}{\|\mathbf{a}\|_T}, \frac{h_T^2}{\kappa_T}\right\}$, we set:

back

$$B^{SDG}(\nu,\mu) = B^{DG}(\nu,\mu) + \sum_{T\in\mathcal{T}_h} \tau_T \int_T (\mathcal{L}_T\nu)(\mathbf{a}\cdot\nabla\mu),$$

$$L^{SDG}(g,f;\mu) = L^{DG}(g,f;\mu) + \sum_{T\in\mathcal{T}_h} \tau_T \int_T f(\mathbf{a}\cdot\nabla\mu),$$

and solve: Find $\phi^{SMDG} \in \mathfrak{T}_h(\overline{V}_h, f)$ such that

$$B^{SDG}(\phi^{SMDG},\mu) = L^{SDG}(g,f;\mu) \quad \text{for all} \quad \mu \in \mathfrak{T}_h(\overline{V}_h, 0). \qquad \text{(SMDG)}$$

Coercivity: : $B^{SDG}(\nu,\nu) \geq \alpha\|\nu\|_{SDG}^2 \quad \forall\, \nu \in \mathfrak{T}_h(\overline{V}_h, 0)$.

Error estimate : for $\tau \leq \bar{\tau}$:

$$\|\phi - \phi^{SMDG}\|_{SDG}^2 \lesssim \sum_{T\in\mathcal{T}_h} \left(a_T h_T^{2k+1} + \kappa_T h_T^{2k}\right) |\phi|_{H^{k+1}(T)}^2$$