A minimal stabilization procedure for discontinuous Galerkin approximations of the advection-reaction equation

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The model problem

Find $u : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \beta \cdot \nabla u + \mu u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega^- . \end{aligned}$$

- Let Ω be an open bounded and connected set in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, let $\partial\Omega^{\pm} = \{x \in \partial\Omega;\ \pm\beta(x){\cdot}n(x) > 0\}$.
- Let $\mathcal{T}$ be a conforming triangulation of the domain Ω. Let h denote the mesh size.
- Let $\mathcal{F}_i$ denote the set of interior faces of the mesh. The sets $\mathcal{F}_{\pm}$ denote the faces that are included in $\partial\Omega^{\pm}$ respectively and denote $\mathcal{F} = \mathcal{F}_i \cup \mathcal{F}_+ \cup \mathcal{F}_-$.

Some notation

- W denotes the graph space ($\|v\|_W^2 = \|v\|^2 + \|\beta \cdot \nabla v\|^2$),
- The average of v at $F = \kappa_1 \cap \kappa_2$: $\{v\}|_F = v|_{\kappa_1} + v|_{\kappa_2}$,
- The jump of u at $F = \kappa_1 \cap \kappa_2$: $[\![v]\!]|_F = v|_{\kappa_1} n_{\kappa_1} + v|_{\kappa_2} n_{\kappa_2}$.
- $(\cdot,\cdot)_{\mathcal{F}_i} := \sum_{F \in \mathcal{F}_i} (\cdot,\cdot)_F$.
- $\beta_n = \|\beta \cdot n\|_{\infty,F} + \epsilon \|\beta \times n\|_{\infty,F}$, with $\epsilon \geq 0$,

The bilinear form

On $W \times W$ define the discontinuous Galerkin bilinear form

$$a(v,w) = ((\mu - \nabla\cdot\beta)v, w) - (v, \beta\cdot\nabla w)_{h,\Omega} + (\{\beta v\}, [\![w]\!])_{\mathcal{F}_i \cup \mathcal{F}_+},$$

We assume

- $(\mu - \frac{1}{2}\nabla\cdot\beta) > c_0 > 0$ (Coercivity)
- β Lipschitz. (For the model analysis we assume β constant).

For sufficiently smooth v, w we define the jump penalty operators

$$\begin{aligned} b_0(v,w) &= (\gamma_0 \beta_n [\![v]\!], [\![w]\!])_{\mathcal{F}_i}, \\ b_1(v,w) &= (\gamma_1 h^2 \beta_n [\![\nabla v]\!], [\![\nabla w]\!])_{\mathcal{F}_i} \end{aligned}$$

The discrete problem

- The discontinuous finite element approximation:

Find $u_d \in W_h^p$ such that
$a(u_d, v_d) + b_0(u_d, v_d) + b_1(u_d, v_d) = (f, v_d), \quad \forall v_d \in W_h^p$

for $\gamma_0 \geq 0$ and $\gamma_1 \geq 0$. W_h^p denotes the space of piecewise polynomial *discontinuous* functions of polynomial order p.

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- The continuous finite element approximation:

Find $u_c \in V_h^p$ such that

$$a(u_c, v_c) + b_1(u_c, v_c) = (f, v_c), \quad \forall v_c \in V_h^p$$

for $\gamma_1 \geq 0$. V_h^p denotes the space of piecewise polynomial *continuous* functions of polynomial order p.

Why do we need stabilized methods, CG, $\gamma_1 = 0$

Find $u_c \in V_h^p$ such that

$$a(u_c, v_c) = (f, v_c), \quad \forall v_c \in V_h^p$$

A priori error estimates are based on (Cea's lemma), let $e_h = u_h - \pi_h u$ (with $\pi_h u$ the L^2-projection of u onto V_h^p).

- Coercivity L^2-norm: $\|e_h\|^2 \lesssim a(e_h, e_h)$
- Galerkin orthogonality: $\|e_h\|^2 \lesssim a(u - \pi_h u, e_h)$
- Continuity H^1-norm: $\|e_h\|^2 \lesssim \|u - \pi_h u\| \left(\|e_h\| + \|\beta \cdot \nabla e_h\|\right)$
- Bounding the H^1-norm of the continuity by an L^2 norm: the inverse inequality leads to the loss of a power of h.
- Result: solution wildly oscillating at layers.

Why stabilized methods, DG, $\gamma_0 = \gamma_1 = 0$

Find $u_d \in W_h^p$ such that

$$a(u_d, v_d) = (f, v_d), \quad \forall v_d \in W_h^p$$

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- Coercivity L^2-norm: $\|e_h\|^2 \lesssim a(e_h, e_h)$
- Galerkin orthogonality: $(u - \pi_h u, \beta \cdot \nabla e_h) = 0$

$$\|e_h\|^2 \lesssim a(u-\pi_h u, e_h) = (u-\pi_h u, e_h) + (\{\beta(u-\pi_h u)\}, [\![e_h]\!])_{\mathcal{F}_i}$$

- Continuity discrete H^1-norm:

  $$\|e_h\|^2 \lesssim (\|u - \pi_h u\|_\Omega + \|u - \pi_h u\|_{\mathcal{F}_i}) (\|e_h\| + \|[e_h]\|_{\mathcal{F}_i})$$

- Same problem as in the continuous case using a trace inequality.

L^2-estimates, reconquering $h^{\frac{1}{2}}$

In order to optimize our estimates:

- Add stabilization, $\gamma_0 > 0$ for DG and $\gamma_1 > 0$ for CG
- perform analysis in the triple norm

$$|||e_h|||^2 = \|e_h\|^2 + b_0(e_h, e_h) + b_1(e_h, e_h)$$

- Continuity in the continuous case, (recall $\pi_h u$ the L^2-projection):

$$(u - \pi_h u, \beta \cdot \nabla e_h) = \min_{\xi_h \in V_h^p} (u - \pi_h u, \beta \cdot \nabla e_h - \xi_h)$$

$$\lesssim \|h^{-\frac{1}{2}}(u - \pi_h u)\| \min_{\xi_h \in V_h^p} \|h^{\frac{1}{2}}(\beta \cdot \nabla e_h - \xi_h)\|$$

$$\lesssim \|h^{-\frac{1}{2}}(u - \pi_h u)\| b_1(e_h, e_h)^{\frac{1}{2}}$$

L^2-estimates, reconquering $h^{\frac{1}{2}}$

- We recall that for DG

$$(u - \pi_h u, \beta \cdot \nabla e_h) = 0$$

- and hence
  $a(u - \pi_h u, e_h) = (u - \pi_h u, e_h) + (\{\beta(u - \pi_h u)\}, [\![e_h]\!])_{\mathcal{F}_i}$
- Continuity in the discontinuous case:

$$(\{\beta(u - \pi_h u)\}, [\![e_h]\!])_{\mathcal{F}_i} \leq \|\beta\|_\infty \|u - \pi_h u\|_{\mathcal{F}_i} b_0(e_h, e_h)^{\frac{1}{2}}$$

L^2-estimates, reconquering $h^{\frac{1}{2}}$

L^2-norm convergence now follows by

1. Coercivity in the triple norm
2. Galerkin orthogonality
3. Modified continuity: $a(u - \pi_h u, e_h) \leq \|u - \pi_h u\|_* |||e_h|||$ where $\|u - \pi_h u\|_* \leq c\, h^{p+\frac{1}{2}} \|u\|_{p+1,\Omega}$

A more general convergence result

- Let $\|u\|_{\mathcal{K}}^2 = \sum_{\kappa \in \mathcal{K}} \|u\|_{\kappa}^2$.
- By proving an *inf-sup* condition we may include $\|h^{\frac{1}{2}} \beta \cdot \nabla u\|_{\mathcal{K}}$ in the triple norm
- We then have the following stronger result

Theorem: *The continuous method with $\gamma_1 > 0$ and the discontinuous method with $\gamma_0 > 0$ both have the same order of convergence in h. If the exact solution u satisfies $u \in H^{p+1}(\Omega)$, then:*

$$\|u - u_h\|_{L^2(\Omega)} + \|h^{\frac{1}{2}} \beta \cdot \nabla (u - u_h)\|_{\mathcal{K}} + B(u_h, u_h) \leq c\, h^{p+\frac{1}{2}} \|u\|_{p+1,\Omega}$$

where $B(u_h, u_h) = \sum_{i=1}^{2} b_i(u_h, u_h)^{\frac{1}{2}}$, $c > 0$ is a constant independent of h.

Some numerical results

- Let Ω be the sector defined by

$$\Omega = \{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ |\, 0.1 \le \sqrt{x^2+y^2} \le 1\}.$$

- The problem consists of seeking u such that

$$\begin{cases} \mu u + \beta \cdot \nabla u = 0, \\ u|_{\partial\Omega^-} = g(y). \end{cases}$$

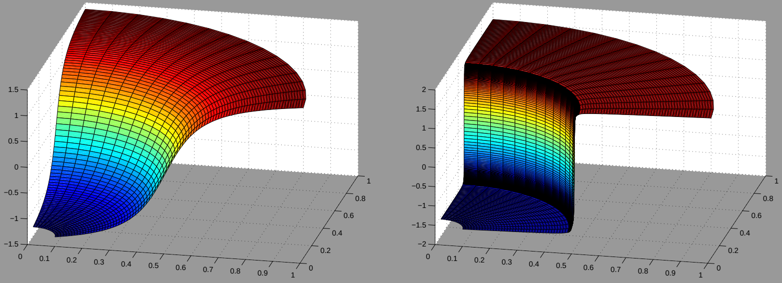
where

$$\beta(x,y) = \begin{pmatrix} y \\ -x \end{pmatrix} \frac{1}{\sqrt{x^2+y^2}} \quad \text{and} \quad g(y) = \arctan\left(\frac{y-0.5}{\varepsilon}\right).$$

Then, the solution writes

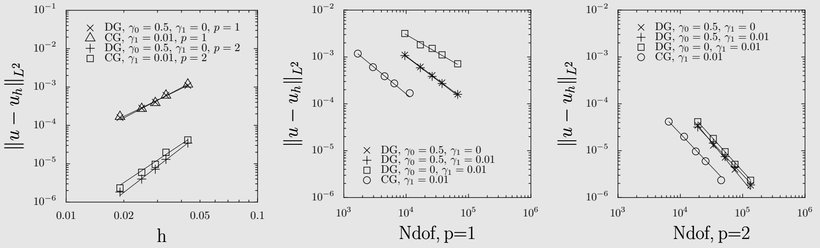
$$u(x,y) = e^{\mu\sqrt{x^2+y^2}\arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right)} \arctan\left(\frac{\sqrt{x^2+y^2}-0.5}{\varepsilon}\right).$$

Numerical results - the model problem in pictures



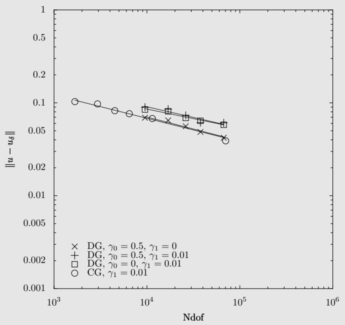
- Left: exact solution with $\epsilon = 0.1$
- Right: exact solution with $\epsilon = 0.001$

Numerical example: the smooth solution



- Note that for $p = 2$ the DG method stabilized using the gradient jumps yields optimal convergence. Min. stab. for DG

Numerical example: the non-smooth solution



- Left: L^2-error against degrees of freedom, P1
- Right: L^2-error against degrees of freedom, P2

Asymptotic limit $\gamma_0 \to \infty$:

Theorem

Let u_d and u_c be the solutions of the method using discontinuous resp. continuous approximation respectively with $\gamma_1 \geq 0$. Let $u \in H^{p+1}(\Omega)$, with $p \geq 1$, solve the model problem. Then

$$u_d \to u_c \quad as \quad \gamma_0 \to \infty.$$

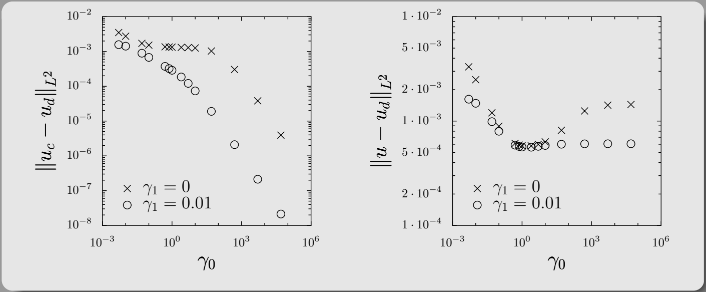
More precisely there is a constant $c > 0$, independent of γ_0, such that

$$\|u_c - u_d\|_{L^2(\Omega)} \leq \frac{c}{\gamma_0}.$$

Proof.

The proof follows a similar result for the elliptic problem by Larson and Niklasson, modified to account for the nonsymmetry of the transport operator. □

Numerical results

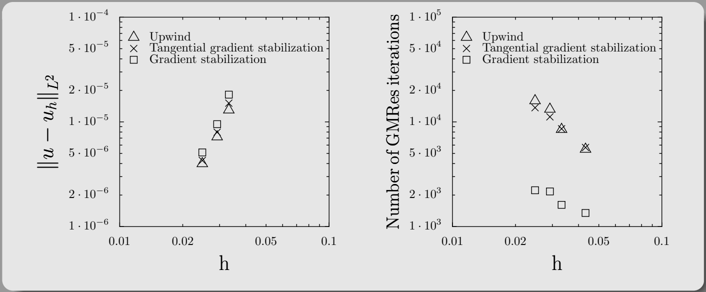


(a) (b)

(a) Difference between the solutions of the discontinuous method and of the continuous method when $\gamma_0 \to \infty$.

(b) Difference between the exact solution and the solution of the discontinuous method when $\gamma_0 \to \infty$, min. value taken for $\gamma_0 = 2.5$.

Different stabilization operators for DG, $p = 2$



(a) (b)

(a) L^2-error for various stabilization terms.

(b) Number of GMRes iterations needed to solve the linear system for various stabilization terms (without preconditionner).

Local mass conservation for DG-methods

- The exact solution u of the model problem satisfies the following local mass conservation property:

$$\int_{\partial K} \beta \cdot n_K u = \int_K f. \qquad (1)$$

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- Discontinuous Galerkin stabilized with $\gamma_0 > 0$ and $\gamma_1 \geq 0$:

$$\int_{\partial K} \Sigma^d_{K,\gamma_0}(u_d) = \int_K f,$$

with the numerical flux defined by

$$\Sigma^d_{K,\gamma_0}(w) = \begin{cases} \beta \cdot n_K \{w\} + \gamma_0 \beta_n [\![w]\!] & \text{on } \mathcal{F}_i \cap \partial K \\ \beta \cdot n_K w & \text{on } \mathcal{F}_+ \cap \partial K \\ 0 & \text{on } \mathcal{F}_- \cap \partial K \end{cases}$$

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- Discontinuous Galerkin stabilized with $\gamma_0 = 0$, $\gamma_1 > 0$ and $p \geq 2$:

$$\int_{\partial K} \beta \cdot n_K \{u_d\} = \int_K f.$$

This property can be considered as generalization of (1) for functions which are double valued on faces.

Some conclusions so far

- Similar a priori error estimates for CG and DG.
- For smooth solutions the CG method requires much less degrees of freedom than the DG method for a given precision.
- For non-smooth solution both method require the same amount of degrees of freedom.
- $u_h^d \to u_h^c$ as $\gamma_0 \to \infty$.
- For $p = 2$ numerical experiments show optimal convergence order for DG stabilized using the gradient jumps only.
- Reduced stabilization $\to$ improved local mass conservation.

Reducing the stabilization by projection

- Question: How can the stabilization of the gradient jumps give control of the solution jumps? recall graphics.
- Answer: We can use some additional stability obtained by the term $(\{\beta v\}, [\![w]\!])_{\mathcal{F}_i \cup \mathcal{F}_+}$
- If the projection satisfies $(\pi_h u, 1)_{\mathcal{F}_i} = (u, 1)_{\mathcal{F}_i}$ we may use the continuity (compare CG/IP !)

$$\begin{aligned}
(\{\beta(u - \pi_h u)\}, [\![e_h]\!])_{\mathcal{F}_i} &= \min_{r \in \mathbb{R}} (\{\beta(u - \pi_h u)\}, [\![e_h]\!] - r)_{\mathcal{F}_i} \\
&\leq \|u - \pi_h u\|_{\mathcal{F}_i} \min_{r \in \mathbb{R}} \|[\![e_h]\!] - r\|_{\mathcal{F}_i} \\
&\leq \|u - \pi_h u\|_{\mathcal{F}_i} \|h [\![\nabla e_h \times n]\!]\|_{\mathcal{F}_i}
\end{aligned}$$

Generalization: a discontinuous Galerkin method

We propose a method where we only penalize the projection of the solution jumps onto the upper part of the polynomial spectrum:

Find $u_h \in W_h^p$, with $p \geq 2$, such that

$$a(u_h, v_h) + j(u_h, v_h) = (f, v_h), \quad \forall v_h \in W_h^p,$$

where $j(v, w) = \gamma_s(|\beta \cdot n|_\infty (I - P_\lambda)[v], (I - P_\lambda)[w])_{\mathcal{F}_i}$

- L^2-projection on face: $P_\lambda : L^2(F) \to P_\lambda(F)$
- $\lambda = \lfloor \frac{p+1}{3} \rfloor - 1$
- Local mass conservation holds independently of γ_s.

Recovering the solution jumps

- Observe that $\|(I - P_0)[\![u_h]\!]\|_{\mathcal{F}_i} \lesssim \|h[\![\nabla u_h \times n]\!]\|_{\mathcal{F}_i}$
- Control of solution jumps $\rightarrow$ Poincaré inequality
- We want graph-norm convergence, i.e. in the triple norm:

$$|||v|||^2 = \|v\|_{\mathcal{K}}^2 + \|h^{\frac{1}{2}}\beta\cdot\nabla v\|_{\mathcal{K}}^2 + \||\beta\cdot n|^{\frac{1}{2}}[\![v]\!]\|_{\mathcal{F}}^2.$$

- We need to prove an inf-sup condition to recover control of

$$\|h^{\frac{1}{2}}\beta\cdot\nabla v\|_{\mathcal{K}}^2 \text{ and } \||\beta|^{\frac{1}{2}}[\![v]\!]\|_{\mathcal{F}}^2$$

An *inf-sup* condition

Theorem: *Assume that β is Lipschitz continuous, then there exists a constant $c > 0$, independent of the mesh size h, such that for $p \geq 2$*

$$c\,|||v_h||| \leq \sup_{v'_h \in W_h^p} \frac{a(v_h, v'_h) + j(v_h, v'_h)}{|||v'_h|||} \qquad \forall v_h \in W_h^p,$$

where the stabilization operator is defined by

$$j(v,w) = \gamma_s(|\beta \cdot n|_\infty (I - P_\lambda)[\![v]\!], (I - P_\lambda)[\![w]\!])_{\mathcal{F}_i},$$

and $0 \leq \lambda \leq \lfloor \frac{p+1}{3} \rfloor - 1.$

Some comments on the proof of the *inf-sup* condition

1. Prove: For all $v_h \in W_h^p$ there exists $v_h' \in W_h^p$ and $c > 0$ such that
$$|||v_h|||^2 \lesssim a(v_h, v_h') + j(v_h, v_h').$$
2. Prove: Fix $v_h \in W_h^p$ and let $v_h' \in W_h^p$ be the function defined in the previous point, then there exists a constant $c > 0$ such that
$$|||v_h'||| \lesssim |||v_h|||.$$
3. $$c\,|||v_h||| \lesssim \frac{a(v_h, v_h') + j(v_h, v_h')}{|||v_h|||} \lesssim \frac{a(v_h, v_h') + j(v_h, v_h')}{|||v_h'|||}$$

A projection operator in 2D on triangles

Theorem: Let $v_1 \in L^2(\Omega)$ and $v_2 \in L^2(\mathcal{F})$, then there exists a projection $\Pi_h = \Pi_h(v_1, v_2) \in W_h^p(\mathcal{K})$ such that

$$\begin{aligned} \int_{\mathcal{K}} (\Pi_h - v_1)\, w_h &= 0 \qquad \forall w_h \in W_h^{p-1}(\mathcal{K}), \\ \int_{\mathcal{F}} (\{\Pi_h\} - v_2)\, z_h &= 0 \qquad \forall z_h \in W_h^{\lambda}(\mathcal{F}), \end{aligned}$$

for all $0 \leq \lambda \leq \lfloor \frac{p+1}{3} \rfloor - 1$.

- Orthogonality against polynomials of order $p-1$ on elements
- Orthogonality against polynomials of order λ on faces

The first step of the proof, (β constant)

For every v_h find v_h' such that

$$|||v_h|||^2 \lesssim a(v_h, v_h') + j(v_h, v_h').$$

1. Testing with $w_{0,h} = u_h$:
   $\|u_h\|^2 + j(u_h, u_h) = a(u_h, w_{0,h}) + j(u_h, w_{0,h})$
2. Testing with $w_{1,h} = \Pi_h(0, [\![u_h]\!])$:
   $\||\beta\cdot n|P_\lambda [\![u_h]\!]\|^2_{\mathcal{F}_i} - \|u_h\|^2 - j(u_h, u_h) \lesssim a(u_h, w_{1,h}) + j(u_h, w_{1,h})$
3. Testing with $w_{2,h} = h\beta\cdot\nabla u_h$:
   $$\|h^{\frac{1}{2}}\beta\cdot\nabla u_h\|^2 - \|u_h\|^2 - \||\beta\cdot n|[\![u_h]\!]\|^2_{\mathcal{F}_i} \lesssim a(u_h, w_{2,h}) + j(u_h, w_{2,h})$$
4. Take $v_h' = \sum_{i=1}^3 c_i w_{i,h}$ with carefully chosen c_i.

The second step of the proof

Fix v_h, for v_h' given in point one, show that

$$|||v_h'||| \lesssim |||v_h|||$$

1. $|||v_h'||| = |||\sum_{i=1}^{3} c_i w_{i,h}||| \lesssim \sum_{i=1}^{3} |||w_{i,h}|||$
2. $|||w_{i,h}||| \lesssim |||v_h|||$ for $i = 1, 2, 3$ by inverse inequalities, trace inequalities and by the stability of the projection.

Some conclusions:

- We derive the a priori error estimate

$$|||\pi_h u - u_h||| \lesssim h^{k+\frac{1}{2}} \|u\|_{k+1,\Omega}$$

  in the standard fashion.
- The method will work whenever the projection Π_h is well defined.
- So far only triangles and $p \geq 2$ are OK, (proof very technical)
- Thetrahedra are expected to work, possibly with a different definition of λ.

Some remarks:

- As the polynomial order increases an increasingly large portion of the polynomial spectrum is in the kernel of the stabilization operator.
  Example of λ for $2 \leq p \leq 17$

p	2 - 4	5 - 7	8 - 10	11 - 13	14 - 16
λ	0	1	2	3	4

- Stability constant non-uniform in λ:

$$\|\Pi_h(0, u)\|_{\partial\kappa} \leq c(p, \lambda)\, 2^{(2\lambda+1)}\, \|u\|_{\partial\kappa}$$

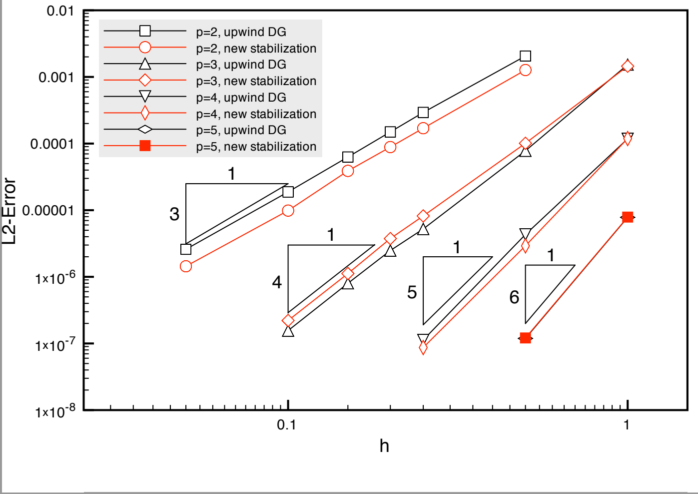
- Could cause degeneration for high λ.

A model case with smooth solution

$$\begin{aligned} \beta \cdot \nabla u + \mu u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega^{-}. \end{aligned}$$

- $\Omega = (-1, 1)^2$
- $\beta = (1, 0)$
- $\mu = 0.01$
- $f = 0$
- $g(y) = \sin(\frac{\pi}{2} y)$
- $u(x, y) = \exp(-\mu x) g(y) \in C^\infty(\Omega)$

A model case with smooth solution, $\lambda = \lfloor \frac{p+1}{3} \rfloor - 1$

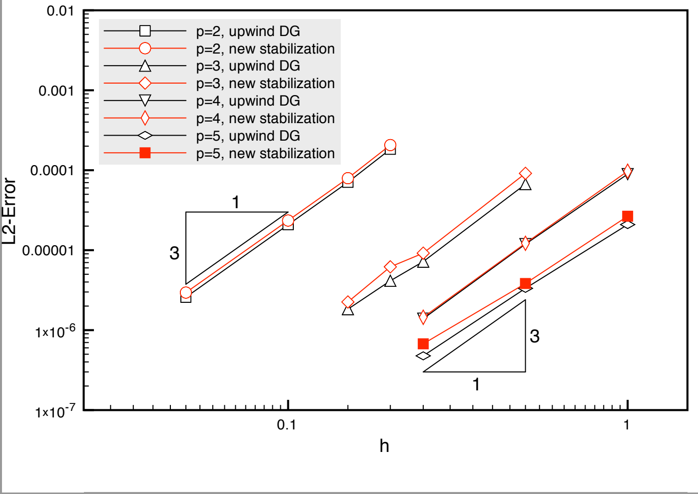


A model case with non-smooth solution

$$\begin{aligned} \beta \cdot \nabla u + \mu u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega^-. \end{aligned}$$

- $\Omega = (-1, 1)^2$
- $\beta = (1, 0)$
- $\mu = 1$
- $g = 1$
- f chosen to get $u(x, y)$
- $u(x, y) = \exp(x + 1) + (x + 1)^{2.5} \in H^{3-\epsilon}(\Omega)$

A model case with non-smooth solution, $\lambda = \lfloor \frac{p+1}{3} \rfloor - 1$

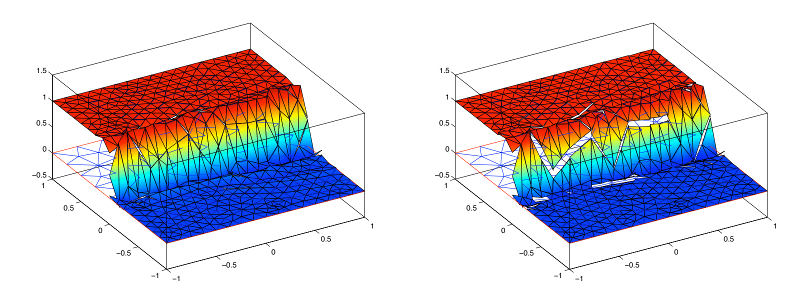


A discontinuous case

$$\begin{aligned} \beta \cdot \nabla u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega^-. \end{aligned}$$

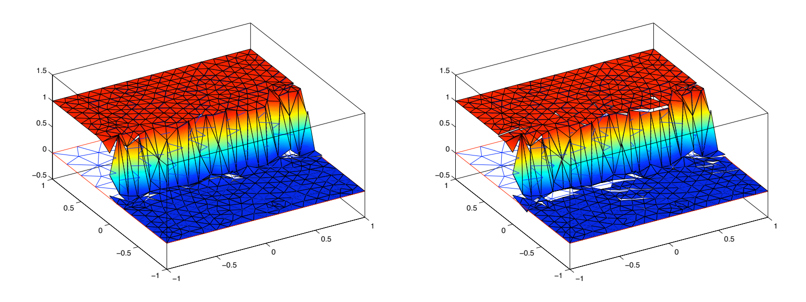
- $\Omega = (-1, 1)^2$
- $\beta = (1, 0)$
- $f = 0$, $\mu = 0$
- $g(y) = \begin{cases} 1, & y > 0 \\ 0, & y < 0 \end{cases}$
- $u(x, y) = \begin{cases} 1, & y > 0 \\ 0, & y < 0 \end{cases}$

A discontinuous case, P2: upwind vs. filtered fluxes



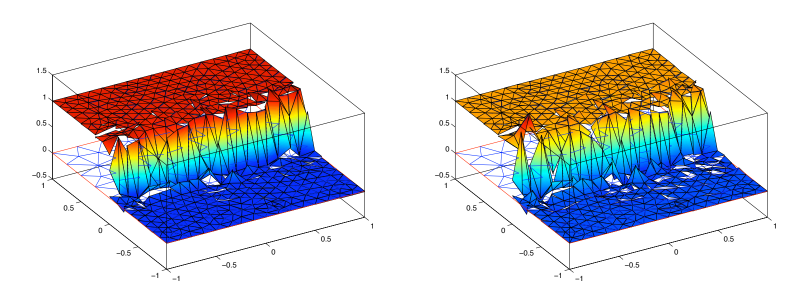
- Left: $p = 2$, standard upwind
- Right: $p = 2$, $\lambda = 0$, $\gamma_s = 0.5$

A discontinuous case, P5: upwind vs. filtered fluxes



- Left: $p = 5$, standard upwind
- Right: $p = 5$, $\lambda = 0$, $\gamma_s = 0.5$

A discontinuous case, P5: violating the limit for λ



- Left: $p = 5$, $\lambda = 1$, $\gamma_s = 0.5$
- Right: $p = 5$, $\lambda = 2$, $\gamma_s = 0.5$

Concluding remarks

- It seems to be possible to shift the stabilization to the high polynomial modes for high order DG methods.
- This leads to improved local mass conservation.
- Extension of the analysis to first order systems straightforward.
- In general: many open questions.

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