The finite element immersed boundary method: model, stability, and numerical results

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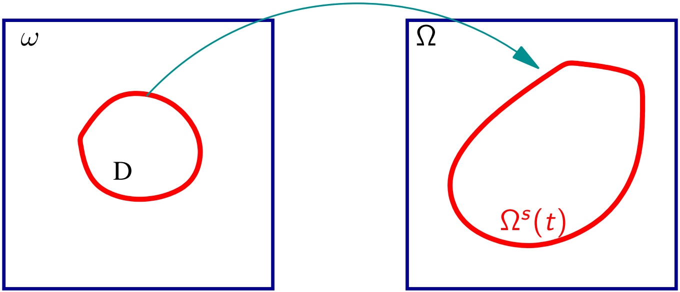
Outline

1. The immersed boundary method
2. Mathematical formulation
3. Space discretization by finite elements
4. Stability of the discrete problem
5. Numerical results

History

- Introduced by Peskin
  - Flow patterns around heart valves: a digital computer method for solving the equations of motion, PhD Thesis, 1972
  - *Numerical analysis of blood flow in the heart*, J. Comput. Phys., 1977
- Extended by Peskin and McQueen since '83 to simulate the blood flow in a three dimensional model of heart and great vessels
- Review article by Peskin
  The immersed boundary method, Acta Numerica, 2002
- Several application in biology, e.g. simulation of a flapping flexible filament in a flowing soap film, insect flight, computational model of the cochlea, ...

The immersed boundary method

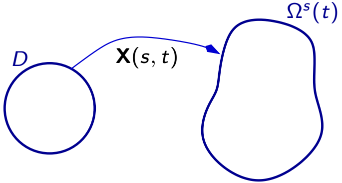


Ω fluid and solid domain
$\Omega \in \mathbb{R}^d$ with $d = 2, 3$
$\mathbf{x}$ Eulerian variables in Ω

$\mathbf{u}(\mathbf{x}, t)$ fluid velocity
$p(\mathbf{x}, t)$ fluid pressure

$\Omega^s(t)$ deformable structure domain
$\Omega^s(t) \in \mathbb{R}^m$ with $m = d, d - 1$
s Lagrangian variables in D
D reference domain
$\mathbf{X}(s, t)$ position of the elastic body

Elastic materials

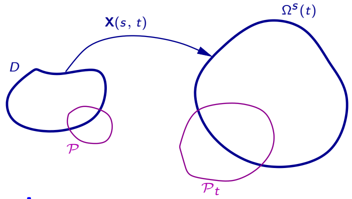


Trajectory of a material point: $\mathbf{X} : D \times [0, T] \to \Omega^s(t)$

Deformation gradient: $\mathbb{D}(s,t) = \dfrac{\partial \mathbf{X}}{\partial s}(s,t)$ with $|\mathbb{D}| = \det(\mathbb{D}) > 0$

Velocity field: $\mathbf{u}(\mathbf{x},t) = \dfrac{\partial \mathbf{X}}{\partial t}(s,t)$ where $\mathbf{x} = \mathbf{X}(s,t)$

Cauchy stress tensor



From conservation of momenta, in absence of external forces, it holds

$$\rho \dot{\mathbf{u}} = \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\, \mathbf{u} \right) = \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \quad \text{in } \Omega.$$

Cauchy stress tensor

Incompressible fluid: $\boldsymbol{\sigma} = \boldsymbol{\sigma}_f = -p\mathbb{I} + \mu(\boldsymbol{\nabla}\, \mathbf{u} + (\boldsymbol{\nabla}\, \mathbf{u})^T)$

Visco elastic materials: $\boldsymbol{\sigma} = \boldsymbol{\sigma}_f + \boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_s$ elastic part of the stress.

$\boldsymbol{\sigma}_s$ can be expressed in Lagrangian variables by means of the Piola-Kirchhoff stress tensor by:

$$\int_{\partial \mathcal{P}_t} \boldsymbol{\sigma}_s \mathbf{n} da = \int_{\partial \mathcal{P}} \mathbb{P}\mathbf{N}\, dA \quad \text{for all } \mathcal{P}_t;$$

so that $\mathbb{P}(s,t) = |\mathbb{D}(s,t)| \boldsymbol{\sigma}_s(\mathbf{X}(s,t),t) \mathbb{D}^{-T}(s,t)$, $s \in D$.

Mathematical formulation

Cauchy stress tensor $\boldsymbol{\sigma} = \begin{cases} \boldsymbol{\sigma}_f & \text{in } \Omega^f(t) \\ \boldsymbol{\sigma}_f + \boldsymbol{\sigma}_s & \text{in } \Omega^s(t) \end{cases}$

Virtual work principle

$$\int_\Omega \rho \dot{\mathbf{u}} \mathbf{v} d\mathbf{x} + \int_\Omega \boldsymbol{\sigma}_f : \boldsymbol{\nabla} \mathbf{v} d\mathbf{x} = -\int_{\Omega^s(t)} \boldsymbol{\sigma}_s : \boldsymbol{\nabla} \mathbf{v} d\mathbf{x}$$

for any smooth $\mathbf{v}$ with compact support in Ω.
Use the Lagrangian variables in the solid region:

$$\int_\Omega \rho \dot{\mathbf{u}} \mathbf{v} d\mathbf{x} + \int_\Omega \boldsymbol{\sigma}_f : \boldsymbol{\nabla} \mathbf{v} d\mathbf{x} = -\int_D \mathbb{P} : \nabla_s \mathbf{v}(\mathbf{X}(s,t)) ds$$

Next, integrate by parts:

$$\int_\Omega (\rho \dot{\mathbf{u}} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_f) \mathbf{v} d\mathbf{x} = \int_D (\nabla_s \cdot \mathbb{P}) \cdot \mathbf{v}(\mathbf{X}(s,t)) ds - \int_{\partial D} \mathbb{P}\mathbf{N} \cdot \mathbf{v}(\mathbf{X}(s,t)) dA,$$

Mathematical formulation (cont'd)

An Implicit Change of Variables using the Dirac Delta distribution

$$\mathbf{v}(\mathbf{X}(s,t)) = \int_{\Omega} \mathbf{v}(\mathbf{x})\delta(\mathbf{x} - \mathbf{X}(s,t))d\mathbf{x}$$

$$\int_{\Omega} (\rho\dot{\mathbf{u}} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_f)\cdot\mathbf{v}d\mathbf{x} = \int_{D}(\nabla_s\cdot\mathbb{P})\cdot\mathbf{v}(\mathbf{X}(s,t))ds - \int_{\partial D}\mathbb{P}\mathbf{N}\cdot\mathbf{v}(\mathbf{X}(s,t))dA$$

Mathematical formulation (cont'd)

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$$\int_\Omega (\rho\dot{\mathbf{u}} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_f)\cdot\mathbf{v}d\mathbf{x} = \int_D (\nabla_s\cdot\mathbb{P})\cdot\int_\Omega \mathbf{v}\delta(\mathbf{x}-\mathbf{X}(s,t))d\mathbf{x}ds - \int_{\partial D}\mathbb{P}\mathbf{N}\cdot\int_\Omega \mathbf{v}\delta(\mathbf{x}-\mathbf{X}(s,t))d\mathbf{x}dA$$

Mathematical formulation (cont'd)

An Implicit Change of Variables using the Dirac Delta distribution

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$$\int_\Omega (\rho\dot{\mathbf{u}} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_f)\cdot\mathbf{v}d\mathbf{x} = \int_\Omega \int_D (\nabla_s \cdot \mathbb{P})\delta(\mathbf{x}-\mathbf{X}(s,t))ds\cdot\mathbf{v}d\mathbf{x} - \int_\Omega \int_{\partial D} \mathbb{P}\mathbf{N}\delta(\mathbf{x}-\mathbf{X}(s,t))dA\cdot\mathbf{v}d\mathbf{x}$$

Mathematical formulation (cont'd)

Since $\mathbf{v}$ is arbitrary, we get

$$\rho\dot{\mathbf{u}} - \nabla\cdot\boldsymbol{\sigma}_f = \int_D \nabla_s \cdot \mathbb{P}\delta(\mathbf{x} - \mathbf{X}(s,t))ds - \int_{\partial D} \mathbb{P}\mathbf{N}\delta(\mathbf{x} - \mathbf{X}(s,t))dA$$

Inner force density

$$\mathbf{g}(\mathbf{x},t) = \int_D \nabla_s \cdot \mathbb{P}(s,t)\delta(\mathbf{X}(s,t) - \mathbf{x})ds.$$

Transmission force density

$$\mathbf{t}(\mathbf{x},t) = -\int_{\partial D} \mathbb{P}(s,t)\mathbf{N}(s)\delta(\mathbf{X}(s,t) - \mathbf{x})ds.$$

Remark

If $m = d - 1$, D is either a curve in 2D or a surface in 3D, then $\partial D = \emptyset$.

Problem setting. Four ingredients.

1. The Navier-Stokes equations

Ω: domain containing the fluid and the elastic structure

$\mathbf{x}$: Eulerian variables

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\,\mathbf{u}\right) - \mu\Delta\mathbf{u} + \nabla p = \mathbf{g} + \mathbf{t} \qquad \text{in } \Omega\times\,]0,T[$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega\times\,]0,T[$$

- ρ: fluid density (constant)
- μ: fluid viscosity (constant)
- $\mathbf{u}(\mathbf{x},t)$, $p(\mathbf{x},t)$: velocity and pressure
- $\mathbf{g}$: inner force density
- $\mathbf{t}$: transmission force density

Problem setting (cont'd)

2. Force densities

$$\mathbf{g}(\mathbf{x},t) = \int_D \nabla_s \cdot \mathbb{P}(s,t)\delta(\mathbf{x}-\mathbf{X}(s,t))ds, \text{ in } \Omega\times]0,T[$$
$$\mathbf{t}(\mathbf{x},t) = -\int_{\partial D} \mathbb{P}(s,t)\mathbf{N}(s)\delta(\mathbf{x}-\mathbf{X}(s,t))ds, \text{ in } \Omega\times]0,T[$$

3. Motion of the immersed structure

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \text{in } D\times]0,T[$$

4. Initial and boundary conditions

$$\mathbf{u}(\mathbf{x},t) = 0 \qquad \text{on } \partial\Omega\times]0,T[$$
$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \text{ in } \Omega, \quad \mathbf{X}(s,0) = \mathbf{X}_0(s) \text{ in } D$$

Equivalence with standard formulation

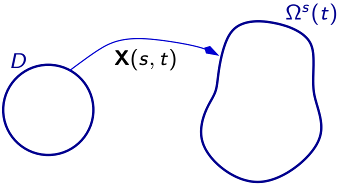
The definition of $\mathbf{g}$ and $\mathbf{t}$ implies that

$$\mathbf{g}(\mathbf{x},t)=0 \quad \text{for } \mathbf{x}\neq\Omega_s(t), \qquad \mathbf{t}(\mathbf{x},t)=0 \quad \text{for } \mathbf{x}\neq\partial\Omega_s(t).$$

The given Navier-Stokes equations are equivalent to:

$$\begin{aligned}
&\rho\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\,\mathbf{u}\right)-\mu\Delta\mathbf{u}+\nabla\,p=0 && \text{in } (\Omega\setminus\Omega_s(t))\times\,]0,T[\\
&\rho\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\,\mathbf{u}\right)-\mu\Delta\mathbf{u}+\nabla\,p=\mathbf{g} && \text{in } \Omega_s(t)\times\,]0,T[\\
&\boldsymbol{\sigma}_f^+\mathbf{n}^+ + \boldsymbol{\sigma}_f^-\mathbf{n}^- = |\mathbb{D}|^{-1}\mathbb{P}\mathbf{N} && \text{on } \partial\Omega_s(t)\times\,]0,T[
\end{aligned}$$

Elastic materials



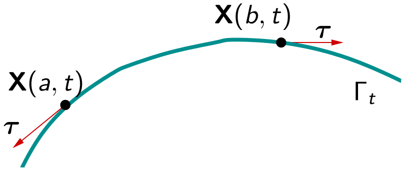
Trajectory of a material point
$\mathbf{X} : D \times [0, T] \to \Omega^s(t)$
Deformation gradient:
$$\mathbb{D}(s,t) = \frac{\partial \mathbf{X}}{\partial s}(s,t)$$

Wide class of elastic materials are characterized by:

- **potential energy density** $W(\mathbb{D}(s,t))$
- **Piola-Kirchoff stress tensor** $\mathbb{P}(\mathbb{D}(s,t)) = \frac{\partial W}{\partial \mathbb{D}}(s,t)$
- **elastic potential energy** $E\left(\mathbf{X}(t)\right) = \int_D W(\mathbb{D}(s,t))ds$
- **force density** $\mathbf{f}(s,t) = \nabla_s \cdot \left(\frac{\partial W}{\partial \mathbb{D}}(\mathbb{D}(s,t))\right) = \nabla_s \cdot \mathbb{P}(\mathbb{D}(s,t))$

  can be obtained taking the Fréchet derivative of E

Example - thin fibers (1D) immersed in a fluid (2D)



$$\boxed{\mathbb{P} = T\tau}$$

- $T = \varphi\left(\left|\frac{\partial \mathbf{X}}{\partial s}\right|; s, t\right)$ Hooke's law for the boundary tension;
- $\boldsymbol{\tau} = \frac{\partial \mathbf{X}/\partial s}{|\partial \mathbf{X}/\partial s|}$ unit tangent
- $\mathbf{f} = \frac{\partial}{\partial s}(T\boldsymbol{\tau})$

Easiest possible case

$W(\mathbb{D}) == \frac{\kappa}{2}|\mathbb{D}|^2$, $T = \kappa|\mathbb{D}|$, $\tau = \mathbb{D}/|\mathbb{D}|$, $\mathbf{f} = \kappa\frac{\partial^2 \mathbf{X}}{\partial s^2}$ κ stiffness of the fiber

Variational formulation

Lemma: Variational definition of the source term

Assume that, for all $t \in [0, T]$, $\partial\Omega_s(t)$ is $\mathbf{C}^1$ regular and that $\mathbb{P}$ is $W^{1,\infty}$. Then for all $t \in]0, T[$, the force density $\mathbf{F} = \mathbf{g} + \mathbf{t}$ is a distribution function belonging to $H^{-1}(\Omega)^d$ defined as follows: for all $\mathbf{v} \in H_0^1(\Omega)^d$

$$_{H^{-1}}< \mathbf{F}(t), \mathbf{v} >_{H_0^1} = -\int_D \mathbb{P}(\mathbb{D}(s,t)) : \nabla_s \mathbf{v}(\mathbf{X}(s,t))\, ds \quad \forall t \in]0, T[\,.$$

Variational formulation

Navier-Stokes equations

$$\rho \frac{d}{dt}(\mathbf{u}(t), \mathbf{v}) + a(\mathbf{u}(t), \mathbf{v}) + b(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) - (\nabla\cdot \mathbf{v}, p(t)) = < \mathbf{F}(t), \mathbf{v} > \quad \forall \mathbf{v} \in H_0^1(\Omega)^d$$

$$(\nabla\cdot \mathbf{u}(t), q) = 0 \qquad \forall q \in L_0^2(\Omega)$$

where $a(\mathbf{u}, \mathbf{v}) = \mu(\boldsymbol{\nabla}\, \mathbf{u}, \boldsymbol{\nabla}\, \mathbf{v})$, $b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \rho(\mathbf{u} \cdot \boldsymbol{\nabla}\, \mathbf{v}, \mathbf{w})$.

$$< \mathbf{F}(t), \mathbf{v} > = -\int_D \mathbb{P}(\mathbb{D}(s,t)) : \nabla_s \mathbf{v}(\mathbf{X}(s,t))\, ds, \quad \forall \mathbf{v} \in H_0^1(\Omega)^d$$

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t), t) \quad \forall s \in [0, L]$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \ \forall \mathbf{x} \in \Omega, \quad \mathbf{X}(s, 0) = \mathbf{X}_0(s) \ \forall s \in [0, L],$$

Stability property

Lemma - Energy estimate

$$\frac{\rho}{2}\frac{d}{dt}||\mathbf{u}(t)||_0^2 + \mu||\nabla\mathbf{u}(t)||_0^2 + \frac{d}{dt}E(\mathbf{X}(t)) = 0.$$

Proof Take $\mathbf{v} = \mathbf{u}$, use the divergence free constraint and the equation of motion of $\mathbf{X}$:

$$\rho\frac{d}{dt}(\mathbf{u}(t),\mathbf{u}(t)) + a(\mathbf{u}(t),\mathbf{u}(t)) = -\int_D \mathbb{P}(\mathbb{D}(s,t))\frac{\partial}{\partial s}\frac{\partial\mathbf{X}(s,t)}{\partial t}ds$$

Use the definition of the potential energy density W and the elastic potential energy E:

$$\int_D \frac{\partial W}{\partial\mathbb{D}}(\mathbb{D}(s,t))\frac{\partial}{\partial t}\left(\frac{\partial\mathbf{X}}{\partial s}(s,t)\right)ds = \int_D \frac{\partial W}{\partial\mathbb{D}}(\mathbb{D}(s,t))\frac{\partial\mathbb{D}}{\partial t}(s,t)ds$$
$$= \frac{d}{dt}\int_D W(\mathbb{D}(s,t))ds = \frac{d}{dt}\left(E\left(\mathbf{X}(t)\right)\right)$$

Finite element spaces

$\mathcal{T}_h$ subdivision of Ω into elements with meshsize h_x.
Let $\mathbf{V}_h \subseteq H_0^1(\Omega)^d$ and $Q_h \subseteq L_0^2(\Omega)$ be finite dimensional.
We choose a pair of spaces $\mathbf{V}_h$ and Q_h which satisfies the inf-sup condition.

Example

$$\mathbf{V}_h = \{\mathbf{v} \in H_0^1(\Omega)^d : \mathbf{v} \text{ continuous piecewise biquadratic}\} \qquad Q_2$$

$$Q_h = \{q \in L_0^2(\Omega) : q \text{ discontinuous piecewise linear}\} \qquad P_1$$

$\mathcal{S}_h$ subdivision of D into elements with meshsize h_s.

$$\mathbf{S}_h = \{\mathbf{Y} \in C^0(D;\Omega) : \mathbf{Y} \text{ continuous piecewise linear}\}$$

Notation

- T_k $k = 1, \dots, M_e$ elements of $\mathcal{S}_h$
- $\mathbf{s}_j$, $j = 1, \dots, M$ vertices of $\mathcal{S}_h$
- $\mathcal{E}_h$ set of the edges e of $\mathcal{S}_h$

Discrete source term

We have to discretize:

$$< \mathbf{F}(t), \mathbf{v} >= -\int_D \mathbb{P}(\mathbb{D}(s,t)) : \nabla_s \mathbf{v}(\mathbf{X}(s,t)) ds$$

$\mathbf{X}$ piecewise linear $\Rightarrow \mathbb{D}$ piecewise constant;
$\mathbb{P}$ depends only on $\mathbb{D} \Rightarrow \mathbb{P}$ piecewise constant.
After integration by parts we get

$$< \mathbf{F}_h(t), \mathbf{v} >_h = -\sum_{k=1}^{M_e} \int_{T_k} \mathbb{P} : \nabla_s \mathbf{v}(\mathbf{X}(s,t)) ds = -\sum_{k=1}^{M_e} \int_{\partial T_k} \mathbb{P}\mathbf{N}\mathbf{v}(\mathbf{X}(s,t)) dA$$

that is

$$< \mathbf{F}_h(t), \mathbf{v} >_h = -\sum_{e \in \mathcal{E}_h} \int_e [\![\mathbb{P}]\!] \cdot \mathbf{v}(\mathbf{X}(s,t)) dA$$

$[\![\mathbb{P}]\!] = \mathbb{P}^+\mathbf{N}^+ + \mathbb{P}^-\mathbf{N}^-$ jump of $\mathbb{P}$ across e for internal edges
$[\![\mathbb{P}]\!] = \mathbb{P}\mathbf{N}$ when $e \subset \partial D$

The finite element immersed boundary method

Find $(\mathbf{u}_h, p_h) :]0, T[\rightarrow \mathbf{V}_h \times Q_h$ and $\mathbf{X}_h : [0, T] \rightarrow \mathbf{S}_h$ such that

$$\mathbf{NS}_h \begin{cases} \rho \dfrac{d}{dt}(\mathbf{u}_h(t), \mathbf{v}) + a(\mathbf{u}_h(t), \mathbf{v}) + b_h(\mathbf{u}_h(t), \mathbf{u}_h(t), \mathbf{v}) \\ \quad -(\nabla \cdot \mathbf{v}, p_h(t)) = -\displaystyle\sum_{e \in \mathcal{E}_h} \int_e [\![\mathbb{P}]\!] \cdot \mathbf{v}(\mathbf{X}(s,t)) dA & \forall \mathbf{v} \in \mathbf{V}_h \\ (\nabla \cdot \mathbf{u}_h(t), q) = 0 & \forall q \in Q_h \end{cases}$$

$$\frac{d\mathbf{X}_{hi}}{dt}(t) = \mathbf{u}_h(\mathbf{X}_{hi}(t), t) \quad \forall i = 1, \ldots, M$$

$$\mathbf{u}_h(0) = \mathbf{u}_{0h} \text{ in } \Omega, \qquad \mathbf{X}_{hi}(0) = \mathbf{X}_0(s_i) \; \forall i = 1, \ldots, M,$$

where $b_h(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{\rho}{2} ((\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{v}, \mathbf{w}) - (\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{w}, \mathbf{v}))$.

Stability property

Lemma - Semidiscrete energy estimate

$$\frac{\rho}{2}\frac{d}{dt}||\mathbf{u}_h(t)||_0^2 + \mu||\nabla \mathbf{u}_h(t)||_0^2 + \frac{d}{dt}E(\mathbf{X}_h(t)) = 0.$$

Proof The proof is the same as in the continuous case.

Fully discrete problem - Backward Euler scheme

Notation: Δt time step, $t_n = n\Delta t$, $\mathbf{u}_h^n \approx \mathbf{u}_h(t_n)$, $\mathbf{X}_{hi}^n \approx \mathbf{X}_{hi}(t_n)$.
Initial data: $\mathbf{u}_h^0 = \mathbf{u}_{0h}$, $\mathbf{X}_{hi}^0 = \mathbf{X}_0(s_i)$ $\forall i = 1, \cdots, m$.

Backward Euler scheme - **BE**

Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{V}_h \times Q_h$ and $\mathbf{X}_h^{n+1} \in \mathbf{S}_h$, such that

$$< \mathbf{F}_h^{n+1}, \mathbf{v} >_h = -\sum_{e \in \mathcal{E}_h} \int_e [\![\mathbb{P}]\!]^{n+1} \cdot \mathbf{v}(\mathbf{X}_h^{n+1}(s,t))dA \qquad \forall \mathbf{v} \in \mathbf{V}_h;$$

$$\mathbf{LNS}_h \begin{cases} \rho\left(\dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}\right) + a(\mathbf{u}_h^{n+1}, \mathbf{v}) - (\nabla\cdot \mathbf{v}, p_h^{n+1}) = < \mathbf{F}_h^{n+1}, \mathbf{v} >_h \\ \hfill \forall \mathbf{v} \in \mathbf{V}_h \\ (\nabla\cdot \mathbf{u}_h^{n+1}, q) = 0 \qquad \forall q \in Q_h; \end{cases}$$

$$\frac{\mathbf{X}_{hi}^{n+1} - \mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^{n+1}) \quad \forall i = 1, \cdots, M.$$

Fully discrete problem - Modified Backward Euler scheme

Notation: Δt time step, $t_n = n\Delta t$, $\mathbf{u}_h^n \approx \mathbf{u}_h(t_n)$, $\mathbf{X}_{hi}^n \approx \mathbf{X}_{hi}(t_n)$.
Initial data: $\mathbf{u}_h^0 = \mathbf{u}_{0h}$, $\mathbf{X}_{hi}^0 = \mathbf{X}_0(s_i)$ $\forall i = 1, \cdots, m$.

Modified backward Euler scheme - MBE

Step 1. $< \mathbf{F}_h^n, \mathbf{v} >_h = -\sum_{e\in\mathcal{E}_h} \int_e [\![\mathbb{P}]\!]^n \cdot \mathbf{v}(\mathbf{X}_h^n(s,t))dA \qquad \forall \mathbf{v} \in \mathbf{V}_h;$

Step 2. find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{V}_h \times Q_h$, such that

$$\mathbf{LNS}_h \begin{cases} \rho\left(\dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}\right) + a(\mathbf{u}_h^{n+1}, \mathbf{v}) + b_h(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}) \\ \qquad -(\nabla\cdot \mathbf{v}, p_h^{n+1}) = < \mathbf{F}_h^n, \mathbf{v} >_h & \forall \mathbf{v} \in \mathbf{V}_h \\ (\nabla\cdot \mathbf{u}_h^{n+1}, q) = 0 & \forall q \in Q_h; \end{cases}$$

Step 3. $\dfrac{\mathbf{X}_{hi}^{n+1} - \mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^n) \quad \forall i = 1, \cdots, M.$

Discrete Energy Estimate

Set

$$\mathbb{H}_{i\alpha j\beta} = \frac{\partial^2 W}{\partial \mathbb{D}_{i\alpha} \partial \mathbb{D}_{j\beta}}(\mathbb{D})$$

Assumption There exist $\kappa_{min} > 0$ and $\kappa_{max} > 0$ s.t. for all tensors $\mathbb{E}$

$$\kappa_{min}\mathbb{E}^2 \le \mathbb{E} : \mathbb{H} : \mathbb{E} \le \kappa_{max}\mathbb{E}^2$$

Artificial Viscosity Theorem

Let $\mathbf{u}_h^n$, p_h^n and $\mathbf{X}_h^n$ be a solution to the FE-IBM, then

$$\frac{\rho}{2\Delta t}\left(\|\mathbf{u}_h^{n+1}\|_0^2 - \|\mathbf{u}_h^n\|_0^2\right) + (\mu + \mu_a)\|\nabla \mathbf{u}_h^{n+1}\|_0^2 + \frac{1}{\Delta t}\left(E\left[\mathbf{X}_h^{n+1}\right] - E\left[\mathbf{X}_h^n\right]\right) \le 0$$

CFL Condition: $\mu + \mu_a > 0$

Artificial Viscosity

BE scheme

$$\mu_a = \kappa_{min} C \frac{h_s^{(m-2)} \Delta t}{h_x^{(d-1)}} L^{n+1}$$

MBE scheme

$$\mu_a = -\kappa_{max} C \frac{h_s^{(m-2)} \Delta t}{h_x^{(d-1)}} L^{n}$$

$$L^n := \max_{T_k \in \mathbf{S}_h} \left\{ \max_{\mathbf{s}_j, \mathbf{s}_i \in V(T_k)} |\mathbf{X}_h^n(\mathbf{s}_j) - \mathbf{X}_h^n(\mathbf{s}_i)| \right\}$$

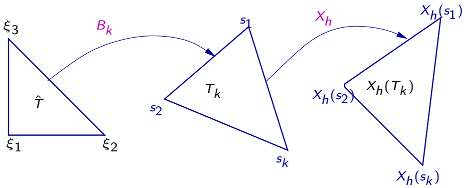
Proof

BE: $\tilde{n} = n+1$, MBE: $\tilde{n} = n$

Take $\mathbf{v} = \mathbf{u}_h^{n+1}$ in the FE-IBM formulation

$$
\begin{aligned}
\frac{1}{2\Delta t}(\|\mathbf{u}_h^{n+1}\|_0^2 - \|\mathbf{u}_h^{n}\|_0^2) + \mu\|\boldsymbol{\nabla}\,\mathbf{u}_h^{n+1}\|_0^2 &\le -\sum_{k=1}^{M_e}\int_{T_k}\mathbb{P}^{\tilde{n}} : \nabla_s\,\mathbf{u}_h^{n+1}(\mathbf{X}_h^{\tilde{n}})ds \\
&= -\sum_{k=1}^{M_e}\int_{T_k}\mathbb{P}^{\tilde{n}} : \nabla_s\left(\frac{\mathbf{X}_h^{n+1}-\mathbf{X}_h^{n}}{\Delta t}\right)ds \\
&= -\frac{1}{\Delta t}\sum_{k=1}^{M_e}\int_{T_k}\mathbb{P}^{\tilde{n}} : (\mathbb{D}^{n+1}-\mathbb{D}^{n})ds \\
&= -\frac{1}{\Delta t}\sum_{k=1}^{M_e}|T_k|\,\mathbb{P}^{\tilde{n}} : (\mathbb{D}^{n+1}-\mathbb{D}^{n}).
\end{aligned}
$$

Proof (cont'd)
BE: $\tilde{n} = n+1$, MBE: $\tilde{n} = n$



- $-\mathbb{P}^{\tilde{n}} : (\mathbb{D}^{n+1} - \mathbb{D}^n) \leq -(W^{n+1} - W^n) + \tilde{\kappa}|\mathbb{D}^{n+1} - \mathbb{D}^n|^2$
  where $\tilde{\kappa} = \kappa_{max}$ for MBE and $\tilde{\kappa} = -\kappa_{min}$ for BE
- $|\mathbb{D}^{n+1} - \mathbb{D}^n|^2 \leq C\Delta t^2 h_s^{-2} \sum_{j=1}^{m} |\mathbf{u}_h^{n+1}(\mathbf{X}_{hj}^{\tilde{n}}) - \mathbf{u}_h^{n+1}(\mathbf{X}_{hk}^{\tilde{n}})|^2$
- $|\mathbf{u}_h^{n+1}(\mathbf{X}_{hj}^{\tilde{n}}) - \mathbf{u}^{n+1}(\mathbf{X}_{hk}^{\tilde{n}})|^2 \leq C h_x^{-(d-1)} \left|\mathbf{X}_{hj}^{\tilde{n}} - \mathbf{X}_{hk}^{\tilde{n}}\right| \|\boldsymbol{\nabla}\, \mathbf{u}_h^{n+1}\|_{0,\hat{T}_k}^2$

Concluding:

$$-\frac{1}{\Delta t}\sum_{k=1}^{M_e} |T_k| \mathbb{P}^{\tilde{n}} : (\mathbb{D}^{n+1} - \mathbb{D}^n) \leq C\frac{h_s^{m-2}\Delta t}{h_x^{d-1}} \sum_{k=1}^{M_e}\sum_{j=1}^{m} \left|\mathbf{X}_{hj}^{\tilde{n}} - \mathbf{X}_{hk}^{\tilde{n}}\right| \|\boldsymbol{\nabla}\, \mathbf{u}_h^{n+1}\|_{0,\hat{T}_k}^2$$

Main result

BE

$$\mu_a = \kappa_{min} C \frac{h_s^{(m-2)} \Delta t}{h_x^{(d-1)}} L^{n+1}$$

MBE

$$\mu_a = -\kappa_{max} C \frac{h_s^{(m-2)} \Delta t}{h_x^{(d-1)}} L^{n}$$

$$L^n := \max_{T_k \in \mathbf{S}_h} \left\{ \max_{\mathbf{s}_j, \mathbf{s}_i \in V(T_k)} \left| \mathbf{X}^n(\mathbf{s}_j) - \mathbf{X}^n(\mathbf{s}_i) \right| \right\}$$

If we use BE and $\kappa_{min} > 0$, then the method is unconditionally stable.

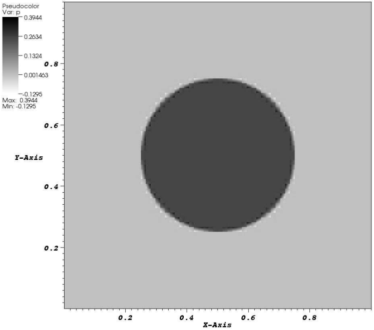
CFL conditions for MBE

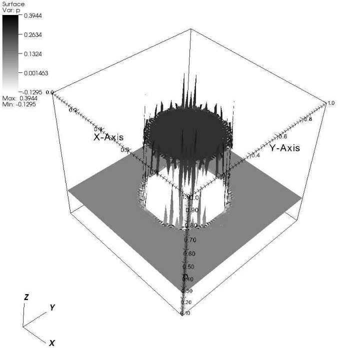
$$\mu_a = -\kappa_{max} C \frac{h_s^{(m-2)} \Delta t}{h_x^{(d-1)}} L^n$$

space dim.	solid dim.	CFL condition
2	1	$L^n \Delta t \leq C h_x h_s$
2	2	$L^n \Delta t \leq C h_x$
3	2	$L^n \Delta t \leq C h_x^2$
3	3	$L^n \Delta t \leq C h_x^2 / h_s$

Static circle
$d = 2,\ m = 1$

Energy density: $W = \kappa \frac{1}{2} |\mathbb{D}|^2$





Volume loss
Percentage

m	20	40	80	160	320
$N = 8$	16.43	16.00	15.85	15.82	15.80
$N = 16$	12.92	5.77	5.42	5.32	5.29
$N = 32$	33.15	5.66	1.86	1.70	1.65

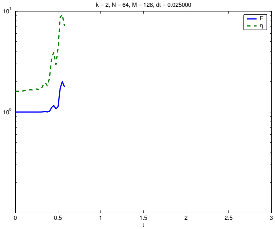
Optimal choice: $h_s \leq h_x/2$

Stability analysis

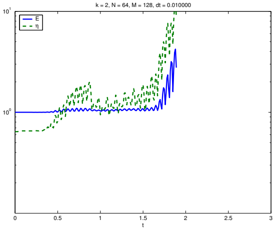
$d = 2,\ m = 1$

Plot: $-\mu_a/\mu$ VS $E[\mathbf{X}] + 1/2\rho\|\mathbf{u}\|_0^2$

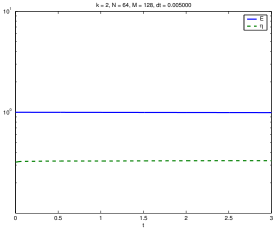
$$\mu_a = -\kappa C \frac{\Delta t}{h_s h_x} L^n$$



$\kappa = 2$, N=64, M=128,
$\Delta t = 0.025$

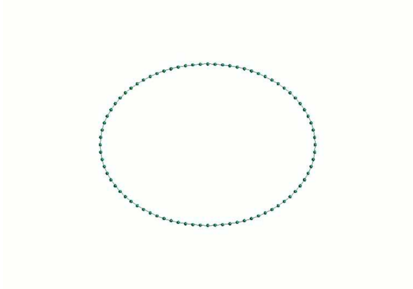


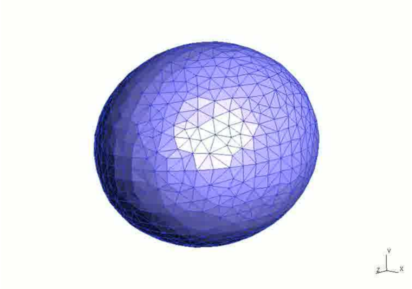
$\kappa = 2$, N=64, M=128,
$\Delta t = 0.010$



$\kappa = 2$, N=64, M=128,
$\Delta t = 0.005$

Examples of instability

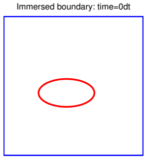




Ellipse immersed in a static fluid
$d = 2,\ m = 1$

Fluid initially at rest: $\mathbf{u}_{0h} = 0$

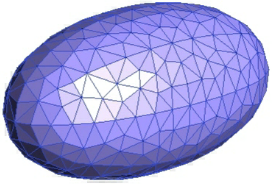
$$\mathbf{X}_0(s) = \begin{pmatrix} 0.35\cos(2\pi s) + 0.5 \\ 0.25\sin(2\pi s) + 0.5 \end{pmatrix} \quad s \in [0,1],$$

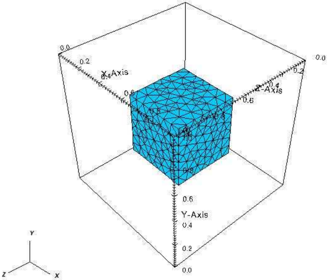


Surface immersed in a static fluid

$d = 3,\ m = 2$

Initial immersed boundary

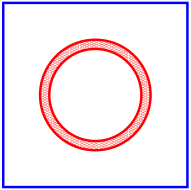




Static fluid: $\mathbf{u}_{0h} = 0$

Two-dimensional visco-elastic cell: static case

$d = 2,\ m = 2$



$$D = [0, 2\pi R] \times [0, t], \quad \text{periodic in } s_1$$

$$\mathbf{X}_0 = \begin{pmatrix} R(1+s_2)\cos(s_1/R) + 0.5 \\ R(1+s_2)\sin(s_1/R) + 0.5 \end{pmatrix}$$

Anisotropic material

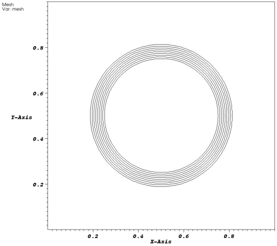
$$W = \frac{\kappa}{2t}\left|\frac{\partial \mathbf{X}}{\partial s_1}\right|^2 = \frac{\kappa}{2t}\left(\frac{\partial X_1}{\partial s_1}^2 + \frac{\partial X_2}{\partial s_1}^2\right); \quad \mathbb{P} = \frac{\kappa}{2t}\begin{pmatrix} \frac{\partial X_1}{\partial s_1} & \frac{\partial X_2}{\partial s_1} \\ 0 & 0 \end{pmatrix}$$

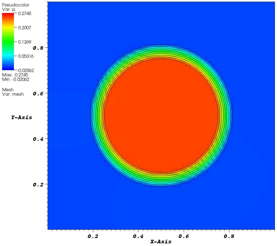
Inner force density: $\nabla_s \cdot \mathbb{P} = \frac{\kappa}{t}\frac{\partial^2 \mathbf{X}}{\partial s_1^2} = -\frac{\kappa}{t}\frac{1+s_2}{R}\mathbf{r}$

Transmission stress density: $-\mathbb{P}\mathbf{N} = 0$

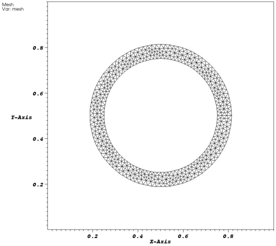
Computed pressure

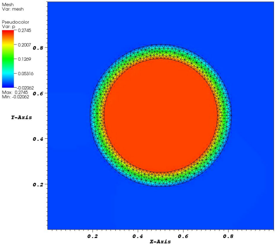
Collection of fibers





Standard finite element mesh



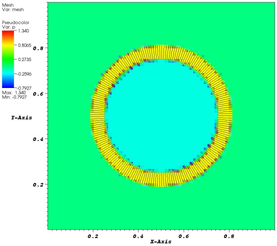


Anistropic material II

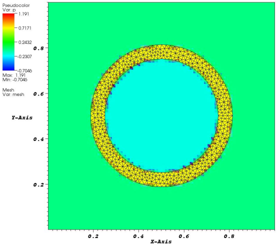
$d = 2,\ m = 2$

$$W = \frac{\kappa}{2t}\left|\frac{\partial \mathbf{X}}{\partial s_2}\right|^2; \quad \mathbb{P} = \frac{\kappa}{2t}\begin{pmatrix} 0 & 0 \\ \frac{\partial X_1}{\partial s_2} & \frac{\partial X_2}{\partial s_2} \end{pmatrix}$$

Collection of fibers



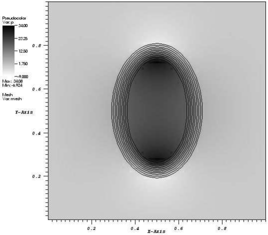
Standard finite element mesh



Two-dimensional visco-elastic cell: dynamic case

$d = 2,\ m = 2$

Initial configuration



$$\mathbf{X}_0 = \begin{pmatrix} R(1+s_2)\cos(s_1/R) + 0.5 \\ R(1+\gamma+s_2)\sin(s_1/R) + 0.5 \end{pmatrix}$$

Collection of fibers

Standard finite element mesh

Conclusions

- The Immersed Boundary Method is extended to the treatment of thick materials modeled by hyper-elastic constitutive laws.
- The finite element approach is efficient and can easily handle the case of thick materials also.
- Stability analysis of the space-time discretization is provided.