Discontinuous Galerkin methods for advection-diffusion-reaction problems

Donatella Marini

Dipartimento di Matematica, Università di Pavia, Italy
& IMATI- C.N.R., Pavia, Italy

Joint work with Blanca Ayuso + ideas from works with Arnold, Brezzi, Cockburn, Hughes, Süli...

INdAM Workshop on Multiscale Problems: Modeling, Adaptive Discretization, Stabilization, Solvers
Cortona, September 18-22, 2006

A bit of History

Introduced for purely hyperbolic problems (Reed-Hill 70's, Lesaint-Raviart)

Used for second order elliptic (Douglas-Dupont school, mid 70's) and for fourth order (Baker).

Abandoned because of the big size of the final system.

Great revival some 10 years ago (mainly by Cockburn-Shu) also for applications to problems where the elliptic part is present but it is not dominant.

(example: strongly advection-dominated equations, very thin Reissner-Mindlin plates)

Outline

Model Problem

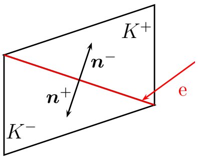
$\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ bounded convex (polygonal or polyhedral)

$$\begin{cases} A\,u \equiv -\mathrm{div}(\kappa \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

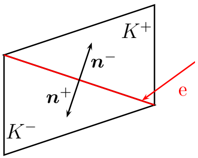
$$\boldsymbol{\sigma} = -\kappa \nabla u$$

$$\begin{cases} \kappa^{-1}\boldsymbol{\sigma} + \nabla u = 0 & \text{in } \Omega \\ \mathrm{div}\boldsymbol{\sigma} = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

Averages & Jumps



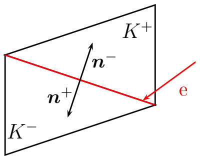
Averages & Jumps

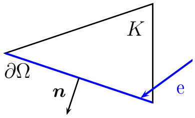


$$\{v\} = \frac{v^+ + v^-}{2}; \qquad [\![v]\!] = v^+\mathbf{n}^+ + v^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{internal edges}$$

$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}; \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+\mathbf{n}^+ + \boldsymbol{\tau}^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{internal edges}$$

Averages & Jumps



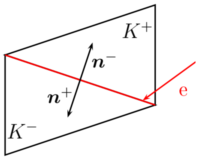


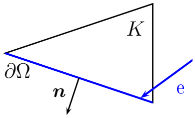
$$\{v\} = \frac{v^+ + v^-}{2}; \qquad [\![v]\!] = v^+\mathbf{n}^+ + v^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{internal edges}$$

$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}; \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+\mathbf{n}^+ + \boldsymbol{\tau}^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{internal edges}$$

On boundary edges $[\![v]\!] = v\mathbf{n}$; $\qquad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}$

Averages & Jumps





$$\{v\} = \frac{v^+ + v^-}{2}; \qquad [\![v]\!] = v^+\mathbf{n}^+ + v^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{internal edges}$$

$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}; \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+\mathbf{n}^+ + \boldsymbol{\tau}^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{internal edges}$$

On boundary edges $[\![v]\!] = v\mathbf{n}$; $\{\boldsymbol{\tau}\} = \boldsymbol{\tau}$

Crucial Formula

$$\sum_{K\in\mathscr{T}_h} \int_{\partial K} v\boldsymbol{\tau}\cdot\mathbf{n}_T = \sum_{e\in\mathcal{E}_h} \int_e [\![v]\!]\cdot\{\boldsymbol{\tau}\} + \sum_{e\in\mathcal{E}_h^\circ} \int_e \{v\}[\![\boldsymbol{\tau}]\!]$$

Original Derivation of DG Methods

In the beginning DG methods were derived in a simple way (see e.g. Douglas-Dupont, M.F. Wheeler, D.N. Arnold).
Taking the equation $-\mathrm{div}(\kappa\nabla u) = f$, we multiply it by a piecewise smooth function v, and integrate by parts:

$$\int_\Omega \kappa\nabla u \cdot \nabla_h v - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \kappa\nabla u \cdot \mathbf{n}\, v = \int_\Omega f v.$$

Original Derivation of DG Methods

In the beginning DG methods were derived in a simple way (see e.g. Douglas-Dupont, M.F. Wheeler, D.N. Arnold).
Taking the equation $-\mathrm{div}(\kappa\nabla u) = f$, we multiply it by a piecewise smooth function v, and integrate by parts:

$$\int_\Omega \kappa\nabla u\cdot\nabla_h v - \sum_{K\in\mathcal{T}_h}\int_{\partial K}\kappa\nabla u\cdot\mathbf{n}\, v = \int_\Omega fv.$$

Rearranging terms with the crucial formula we have

$$\int_\Omega \kappa\nabla u\cdot\nabla_h v - \sum_{e\in\mathcal{E}_h}\int_e [\![v]\!]\cdot\{\kappa\nabla u\} - \sum_{e\in\mathcal{E}_h^\circ}\int_e [\![\kappa\nabla u]\!]\cdot\{v\} = \int_\Omega fv.$$

Original Derivation of DG Methods

In the beginning DG methods were derived in a simple way (see e.g. Douglas-Dupont, M.F. Wheeler, D.N. Arnold).
Taking the equation $-\mathrm{div}(\kappa\nabla u) = f$, we multiply it by a piecewise smooth function v, and integrate by parts:

$$\int_\Omega \kappa\nabla u \cdot \nabla_h v - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \kappa\nabla u \cdot \mathbf{n}\, v = \int_\Omega fv.$$

Rearranging terms with the crucial formula we have

$$\int_\Omega \kappa\nabla u \cdot \nabla_h v - \sum_{e\in\mathcal{E}_h} \int_e [\![v]\!] \cdot \{\kappa\nabla u\} - \sum_{e\in\mathcal{E}_h^\circ} \int_e [\![\kappa\nabla u]\!] \cdot \{v\} = \int_\Omega fv.$$

Then, since u is smooth, $[\![\kappa\nabla u]\!]$ is zero and we can forget about it:

$$\int_\Omega \kappa\nabla u \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\kappa\nabla u\} = \int_\Omega fv.$$

The Original Derivation of DG methods

We had

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\kappa \nabla u\} = \int_\Omega f v.$$

The Original Derivation of DG methods

We had

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\kappa \nabla u\} = \int_\Omega f v.$$

Then, always since u is smooth, also $[\![u]\!] = 0$, and can add a term *"to restore symmetry"*

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\kappa \nabla u\} - \sum_e \int_e [\![u]\!] \cdot \{\kappa \nabla_h v\} = \int_\Omega f v.$$

The Original Derivation of DG methods

We had

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\kappa \nabla u\} = \int_\Omega fv.$$

Then, always since u is smooth, also $[\![u]\!] = 0$, and can add a term *"to restore symmetry"*

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\kappa \nabla u\} - \sum_e \int_e [\![u]\!] \cdot \{\kappa \nabla_h v\} = \int_\Omega fv.$$

Then we realize that the method is unstable! However, since $[\![u]\!] = 0$, we can add a stabilizing term

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v - \int_{\mathcal{E}_h} [\![v]\!] \cdot \{\kappa \nabla u\} - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{\kappa \nabla_h v\} + \sum_e \frac{\gamma}{|e|} \int_e [\![u]\!] \cdot [\![v]\!] = \int_\Omega fv.$$

The Original Derivation of DG methods

We had

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\kappa \nabla u\} = \int_\Omega f v.$$

Then, always since u is smooth, also $[\![u]\!] = 0$, and can add a term *"to restore symmetry"*

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\kappa \nabla u\} - \sum_e \int_e [\![u]\!] \cdot \{\kappa \nabla_h v\} = \int_\Omega f v.$$

Then we realize that the method is unstable! However, since $[\![u]\!] = 0$, we can add a stabilizing term

$$\int_\Omega \kappa \nabla u \cdot \nabla_h v - \int_{\mathcal{E}_h} [\![v]\!] \cdot \{\kappa \nabla u\} - \int_{\mathcal{E}_h} [\![u]\!] \cdot \{\kappa \nabla_h v\} + \sum_e \frac{\gamma}{|e|} \int_e [\![u]\!] \cdot [\![v]\!] = \int_\Omega f v.$$

$\Longrightarrow$ This is "IP" (Interior Penalty).

DG Methods as Weighted Residuals (Brezzi-Cockburn-M.-Süli, CMAME 2006)

If you allow, a priori, your solution to be discontinuous, then the equations to be required are:

- $A\,u - f \;=\; 0$ in each element
- $[\![u]\!] = 0$ on each edge
- $[\![\boldsymbol{\sigma}]\!] = 0$ on each internal edge

(remember that $\boldsymbol{\sigma} = -\kappa \nabla u$).

Starting Point

We define $H^2(\mathscr{T}_h) := \{v \in L^2(\Omega) :\ v_{|K} \in H^2(K)\ \forall K \in \mathscr{T}_h\}$.

We take three operators B_0, B_1, B_2 from $H^2(\mathscr{T}_h)$ to $L^2(\mathscr{T}_h)$, $[L^2(\mathcal{E}_h)]^d$ and $L^2(\mathcal{E}_h^\circ)$ respectively.

Then we consider the following *variational* formulation

$$\text{find } u \in H^2(\mathscr{T}_h) \text{ such that, } \forall v \in H^2(\mathscr{T}_h) :$$

$$(A\,u - f,\ B_0\,(v))_h + < [\![u]\!],\ B_1(v) >_h + < [\![\boldsymbol{\sigma}]\!],\ B_2(v) >_h^0 = 0,$$

where

$$(u,\ v)_h = \sum_{K \in \mathscr{T}_h} \int_K u\,v\,\mathrm{d}x \qquad < u,\ v >_h = \sum_{e \in \mathcal{E}_h} \int_e u\,v\,\mathrm{d}s$$

and $< u,\ v >_h^0$ runs only on internal edges

Conditions on the Operators B_i

$$(A\,u - f,\, B_0\,(v))_h + < [\![u]\!],\, B_1(v) >_h + < [\![\boldsymbol{\sigma}]\!],\, B_2(v) >_h^0 = 0 \quad \forall v$$

The above equation gives back the original three equations on u (that is $A\,u = f$, $[\![u]\!] = 0$, and $[\![\boldsymbol{\sigma}]\!] = 0$) **if** (and, essentially, only if)

- $\forall K \in \mathscr{T}_h$ and $\forall \varphi \in C_0^\infty(K)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_0 v = \varphi \text{ in } K, \quad B_0 v = 0 \text{ in } \mathscr{T}_h \setminus K, \quad B_1 v \equiv 0, \quad B_2 v \equiv 0$$

Conditions on the Operators B_i

$$(A\,u - f,\, B_0\,(v))_h + < [\![u]\!],\, B_1(v) >_h + < [\![\boldsymbol{\sigma}]\!],\, B_2(v) >_h^0 = 0 \quad \forall v$$

The above equation gives back the original three equations on u (that is $A\,u = f$, $[\![u]\!] = 0$, and $[\![\boldsymbol{\sigma}]\!] = 0$) **if** (and, essentially, only if)

- $\forall K \in \mathscr{T}_h$ and $\forall \varphi \in C_0^\infty(K)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_0 v = \varphi \text{ in } K, \quad B_0 v = 0 \text{ in } \mathscr{T}_h \setminus K, \quad B_1 v \equiv 0, \quad B_2 v \equiv 0$$

- $\forall e \in \mathcal{E}_h$ and $\forall \boldsymbol{\psi} \in C_0^\infty(e)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_1 v = \boldsymbol{\psi} \text{ on } e, \quad B_1 v = 0 \text{ on } \mathcal{E}_h \setminus e \quad B_2 v \equiv 0$$

Conditions on the Operators B_i

$$(A\,u - f,\, B_0\,(v))_h + < [\![u]\!],\, B_1(v) >_h + < [\![\boldsymbol{\sigma}]\!],\, B_2(v) >_h^0 = 0 \quad \forall v$$

The above equation gives back the original three equations on u (that is $A\,u = f$, $[\![u]\!] = 0$, and $[\![\boldsymbol{\sigma}]\!] = 0$) **if** (and, essentially, only if)

- $\forall K \in \mathscr{T}_h$ and $\forall \varphi \in C_0^\infty(K)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_0 v = \varphi \text{ in } K, \quad B_0 v = 0 \text{ in } \mathscr{T}_h \setminus K, \quad B_1 v \equiv 0, \quad B_2 v \equiv 0$$

- $\forall e \in \mathcal{E}_h$ and $\forall \boldsymbol{\psi} \in C_0^\infty(e)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_1 v = \boldsymbol{\psi} \text{ on } e, \quad B_1 v = 0 \text{ on } \mathcal{E}_h \setminus e \quad B_2 v \equiv 0$$

- $\forall e \in \mathcal{E}_h^\circ$ and $\forall \chi \in C_0^\infty(e)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_2 v = \chi \text{ on } e, \quad B_2 v = 0 \text{ on } \mathcal{E}_h^\circ \setminus e$$

Choosing B_0 and Using the Crucial Formula

- Choosing $B_0 v \equiv v$ and using the crucial formula, we can write:

$$(Au, B_0(v))_h \equiv \sum_{K \in \mathscr{T}_h} \int_K -\mathrm{div}(\kappa \nabla u)\, v \,\mathrm{d}x$$

Choosing B_0 and Using the Crucial Formula

- Choosing $B_0 v \equiv v$ and using the crucial formula, we can write:

$$(Au, B_0(v))_h \equiv \sum_{K \in \mathscr{T}_h} \int_K -\mathrm{div}(\kappa \nabla u)\, v \,\mathrm{d}x$$

$$= \sum_{K \in \mathscr{T}_h} \{ \int_K \kappa \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\partial K} \boldsymbol{\sigma} \cdot \mathbf{n}_K \; v \,\mathrm{d}s \}$$

Choosing B_0 and Using the Crucial Formula

- Choosing $B_0 v \equiv v$ and using the crucial formula, we can write:

$$
\begin{aligned}
(Au, B_0(v))_h &\equiv \sum_{K\in\mathcal{T}_h} \int_K -\mathrm{div}(\kappa\nabla u)\, v\, \mathrm{d}x \\
&= \sum_{K\in\mathcal{T}_h} \{ \int_K \kappa\nabla u \cdot \nabla v\, \mathrm{d}x + \int_{\partial K} \boldsymbol{\sigma}\cdot\mathbf{n}_K\ v\, \mathrm{d}s \} \\
&= \sum_{K\in\mathcal{T}_h} \int_K \kappa\nabla u \cdot \nabla v\, \mathrm{d}x + \sum_e \int_e [\![v]\!]\cdot\{\boldsymbol{\sigma}\}\mathrm{d}s + \sum_{e'} \int_{e'} \{v\}[\![\boldsymbol{\sigma}]\!]\, \mathrm{d}s
\end{aligned}
$$

Choosing B_0 and Using the Crucial Formula

- Choosing $B_0 v \equiv v$ and using the crucial formula, we can write:

$$
\begin{aligned}
(Au, B_0(v))_h &\equiv \sum_{K\in\mathscr{T}_h} \int_K -\mathrm{div}(\kappa\nabla u)\, v\,\mathrm{d}x \\
&= \sum_{K\in\mathscr{T}_h} \{ \int_K \kappa\nabla u\cdot\nabla v\,\mathrm{d}x + \int_{\partial K} \boldsymbol{\sigma}\cdot\mathbf{n}_K\; v\,\mathrm{d}s \} \\
&= \sum_{K\in\mathscr{T}_h} \int_K \kappa\nabla u\cdot\nabla v\,\mathrm{d}x + \sum_e \int_e [\![v]\!]\cdot\{\boldsymbol{\sigma}\}\mathrm{d}s + \sum_{e'} \int_{e'} \{v\}[\![\boldsymbol{\sigma}]\!]\,\mathrm{d}s \\
&= (\kappa\nabla u, \nabla v)_h + < \{\boldsymbol{\sigma}\}, [\![v]\!] >_h + < [\![\boldsymbol{\sigma}]\!], \{v\} >_h^0
\end{aligned}
$$

The New Bilinear Form

$$(A\,u - f,\, B_0\,(v))_h + < [\![u]\!],\, B_1(v) >_h + < [\![\boldsymbol{\sigma}]\!],\, B_2(v) >_h^0 = 0 \quad \forall v$$

The New Bilinear Form

$$(A\,u - f,\, B_0\,(v))_h + < [\![u]\!],\, B_1(v) >_h + < [\![\boldsymbol{\sigma}]\!],\, B_2(v) >_h^0 = 0 \quad \forall v$$

We had

$$\begin{aligned}(Au, B_0(v))_h &= (\kappa\nabla u,\, \nabla v)_h + < \{\boldsymbol{\sigma}\}, [\![v]\!] >_h + < [\![\boldsymbol{\sigma}]\!], \{v\} >_h^0 \\ &\equiv (\kappa\nabla u,\, \nabla\, v)_h - < \{\kappa\nabla u\}, [\![v]\!] >_h + < [\![\boldsymbol{\sigma}]\!], \{v\} >_h^0\end{aligned}$$

The New Bilinear Form

$$(A\,u - f,\ B_0\,(v))_h + < [\![u]\!],\ B_1(v) >_h + < [\![\boldsymbol{\sigma}]\!],\ B_2(v) >_h^0 = 0 \quad \forall v$$

We had

$$\begin{aligned}(Au, B_0(v))_h &= (\kappa\nabla u,\ \nabla v)_h + < \{\boldsymbol{\sigma}\}, [\![v]\!] >_h + < [\![\boldsymbol{\sigma}]\!], \{v\} >_h^0 \\ &\equiv (\kappa\nabla u,\ \nabla\, v)_h - < \{\kappa\nabla u\}, [\![v]\!] >_h + < [\![\boldsymbol{\sigma}]\!], \{v\} >_h^0\end{aligned}$$

- Choosing $B_2(v) \equiv -\{v\}$ gives

$$\begin{aligned}&(A\,u - f,\ B_0\,(v))_h + < [\![u]\!],\ B_1(v) >_h + < [\![\boldsymbol{\sigma}]\!],\ B_2(v) >_h^0 \\ &\equiv (\kappa\nabla u,\ \nabla\, v)_h - < \{\kappa\nabla u\}, [\![v]\!] >_h - (f, v)_h + < [\![u]\!],\ B_1(v) >_h = 0\end{aligned}$$

and we have just to choose B_1

Choices for B_1: Symmetrizing and Stabilizing

Our equations are

$$(\kappa\nabla u, \nabla v)_h - < \{\kappa\nabla u\}, [\![v]\!] >_h + < [\![u]\!], B_1(v) >_h = (f, v)_h$$

You are not allowed to take $B_1 \equiv 0$. You can take, for instance

- $B_1(v)_{|e} := -\{\kappa\nabla_h v\} + \left(\frac{c}{|e|}[\![v]\!]\right)$ (stabilized) IP

or

- $B_1(v)_{|e} := +\{\kappa\nabla_h v\} + \left(\frac{c}{|e|}[\![v]\!]\right)$ (stabilized) BO (NIPG)

or

- $B_1(v)_{|e} := \frac{c}{|e|}[\![v]\!]$ Wheeler-Sun (IIP)

Other Possibilities

On the other hand, the choice of B_2 can also be revisited. For instance taking

$$B_2 v_{|e} = -\{v\} - \gamma|e|[\![\nabla_h v]\!] \quad \text{and} \quad B_1(v)_{|e} := \frac{c(\kappa)}{|e|}[\![v]\!] - \{\kappa\nabla_h v\}$$

(Douglas-Dupont, Hansbo et als, etc.) your final bilinear form will be

$$(\kappa\nabla u, \nabla v)_h - < \{\kappa\nabla u\}, [\![v]\!] >_h - < \{\kappa\nabla_h v\}, [\![u]\!] >_h$$
$$+ \sum_{e\in\mathcal{E}_h} \frac{c(\kappa)}{|e|} \int_e [\![u]\!]\cdot[\![v]\!]\mathrm{d}s + \sum_{e\in\mathcal{E}_h^\circ} \gamma|e| \int_e [\![\kappa\nabla u]\!]\cdot[\![\nabla_h v]\!]\mathrm{d}s$$

where the red part is needed for consistency, and the others to symmetrize (if you choose to have it) and to stabilize

Error Estimates

$V_h := \{v \in L^2(\Omega) : \; v_{|K} \in P_k(K) \; \forall K \in \mathscr{T}_h\} \subset H^2(\mathscr{T}_h)$

The final bilinear form $a(u, v)$ should satisfy:

1) $a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$ 2) $\alpha |\!|\!| v_h |\!|\!|^2 \leq a(v_h, v_h) \quad \forall v_h \in V_h$

3) $a(w, v) \leq C |\!|\!| w |\!|\!| \, |\!|\!| v |\!|\!| \quad \forall w, v \in H^2(\mathscr{T}_h)$ 4) $|\!|\!| u - u_I |\!|\!| \leq C\, h^k \, \|u\|_{k+1}$

where

$$|\!|\!| v |\!|\!|^2 := \|\nabla_h v\|_0^2 + \sum_e |e|^{-1} \|[\![v]\!]\|_{0,e}^2 + \sum_K h_K^2 |v|_{2,K}^2$$

and u_h (discrete solution), and u_I (interpolant of u) are in V_h. Then:

$$\alpha |\!|\!| u_h - u_I |\!|\!|^2 \leq a(u_h - u_I, u_h - u_I) = a(u - u_I, u_h - u_I)$$
$$\leq C \, |\!|\!| u - u_I |\!|\!| \, |\!|\!| u_h - u_I |\!|\!| \leq C\, h^k |\!|\!| u_h - u_I |\!|\!|.$$

Advection-Diffusion-Reaction Problems

Let $f \in L^2(\Omega)$, $g \in H^{3/2}(\Gamma)$. Consider the problem

$$\begin{aligned} \mathrm{div}(-\varepsilon \nabla u + \boldsymbol{\beta} u) + \gamma u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma = \partial\Omega, \end{aligned}$$

Introducing the flux $\boldsymbol{\sigma}(u) = -\varepsilon \nabla u + \boldsymbol{\beta} u$ we can write

$$\begin{aligned} \mathrm{div}\, \boldsymbol{\sigma}(u) + \gamma u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma = \partial\Omega, \end{aligned}$$

For simplicity of exposition we shall assume ε, $\boldsymbol{\beta}$, γ constants:
$\varepsilon > 0$, $\gamma \geq 0$. Unique solution $u \in H^2(\Omega)$.

$\Gamma = \Gamma^+ \cup \Gamma^-$, $\Gamma^- =$ inflow $(\boldsymbol{\beta} \cdot \mathbf{n} < 0)$, $\Gamma^+ =$ outflow $(\boldsymbol{\beta} \cdot \mathbf{n} \geq 0)$

The Residuals

If we allow the solution to be a-priori discontinuous, we have to enforce the following equations:

$$
\begin{aligned}
R_0(u) &:= \operatorname{div} \boldsymbol{\sigma}(u) + \gamma u - f && = 0 && \text{in each } K \in \mathscr{T}_h, \\
R_1(u) &:= [\![u]\!] && = 0 && \text{on each } e \in \mathcal{E}_h^\circ, \\
R_2(u) &:= [\![\boldsymbol{\sigma}(u)]\!] && = 0 && \text{on each } e \in \mathcal{E}_h^\circ, \\
R_1^D(u) &:= u - g && = 0 && \text{on each } e \in \Gamma
\end{aligned}
$$

Hence we need four operators $B_0,\ B_1,\ B_2,\ B_1^D$ such that:

$$
\begin{aligned}
(R_0(u), B_0(v))_h \quad & + < R_1(u), B_1(v) >_h^0 + < R_2(u), B_2(v) >_h^0 \\
& + < R_1^D(u), B_1^D(v) >_\Gamma = 0 \qquad \forall v \in H^2(\mathscr{T}_h)
\end{aligned}
$$

Choices of the operators

Taking $B_0 v = v$ we have:

$$(\operatorname{div} \boldsymbol{\sigma}(u) + \gamma u - f, v)_h + < [\![u]\!], B_1(v) >_h^0 + < [\![\boldsymbol{\sigma}(u)]\!], B_2(v) >_h^0$$
$$+ < u - g, B_1^D(v) >_\Gamma = 0 \qquad \forall v \in H^2(\mathscr{T}_h)$$

Choices of the operators

Taking $B_0 v = v$ we have:

$$(\operatorname{div}\boldsymbol{\sigma}(u) + \gamma u - f, v)_h \quad + < [\![u]\!], B_1(v) >_h^0 + < [\![\boldsymbol{\sigma}(u)]\!], B_2(v) >_h^0$$
$$+ < u - g, B_1^D(v) >_\Gamma = 0 \qquad \forall v \in H^2(\mathscr{T}_h)$$

Integrating by parts and using the crucial formula:

$$\int_\Omega \operatorname{div}_h \boldsymbol{\sigma}(u) v = -\int_\Omega \boldsymbol{\sigma}(u) \cdot \boldsymbol{\nabla}_h v + \sum_{K \in \mathscr{T}_h} \int_{\partial K} \boldsymbol{\sigma}(u) \cdot \mathbf{n} v$$
$$= -(\boldsymbol{\sigma}(u), \boldsymbol{\nabla} v)_h + < [\![\boldsymbol{\sigma}(u)]\!], \{v\} >_h^0 + < \{\boldsymbol{\sigma}(u)\}, [\![v]\!] >_h$$

Choices of the operators

Taking $B_0 v = v$ we have:

$$(\operatorname{div} \boldsymbol{\sigma}(u) + \gamma u - f, v)_h \quad + < [\![u]\!], B_1(v) >_h^0 + < [\![\boldsymbol{\sigma}(u)]\!], B_2(v) >_h^0$$
$$+ < u - g, B_1^D(v) >_\Gamma = 0 \qquad \forall v \in H^2(\mathscr{T}_h)$$

Integrating by parts and using the crucial formula:

$$\int_\Omega \operatorname{div}_h \boldsymbol{\sigma}(u) v = -\int_\Omega \boldsymbol{\sigma}(u) \cdot \boldsymbol{\nabla}_h v + \sum_{K \in \mathscr{T}_h} \int_{\partial K} \boldsymbol{\sigma}(u) \cdot \mathbf{n} v$$
$$= -(\boldsymbol{\sigma}(u), \boldsymbol{\nabla} v)_h + < [\![\boldsymbol{\sigma}(u)]\!], \{v\} >_h^0 + < \{\boldsymbol{\sigma}(u)\}, [\![v]\!] >_h$$

Substituting in the equation we obtain:

$$(\gamma u, v) - (\boldsymbol{\sigma}(u), \boldsymbol{\nabla} v)_h + < [\![u]\!], B_1(v) >_h^0 + < \{\boldsymbol{\sigma}(u)\}, [\![v]\!] >_h$$
$$+ < [\![\boldsymbol{\sigma}(u)]\!], B_2(v) + \{v\} >_h^0 + < u, B_1^D(v) >_\Gamma$$
$$= (f, v) + < g, B_1^D(v) >_\Gamma$$

Guidelines for choosing the operators

Our bilinear form is:

$$a_h(u,v): \quad = (\gamma u, v) - (\boldsymbol{\sigma}(u), \nabla v)_h + < [\![u]\!], B_1(v) >_h^0 + < \{\boldsymbol{\sigma}(u)\}, [\![v]\!] >_h$$
$$+ < [\![\boldsymbol{\sigma}(u)]\!], B_2(v) + \{v\} >_h^0 + < u, B_1^D(v) >_\Gamma$$

We shall need stability (in the finite element space) in a suitable norm:

$$a_h(v,v) \geq \alpha |||v|||^2 \qquad \forall v \in V_h^k.$$

$$(k \geq 1 \longrightarrow V_h^k = \{v \in L^2(\Omega): \; v_{|K} \in P_k(K) \quad \forall K \in \mathscr{T}_h\})$$

$$|||v|||^2 = \gamma \|v\|_{0,\Omega}^2 + \varepsilon |v|_{1,h}^2 + \sum_{e \in \mathcal{E}_h} \frac{\varepsilon}{|e|} \|[\![v]\!]\|_{0,e}^2 + \sum_{e \in \mathcal{E}_h} |\boldsymbol{\beta} \cdot \mathbf{n}| \|[\![v]\!]\|_{0,e}^2$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma\|v\|^2_{0,\Omega} + \varepsilon|v|^2_{1,h} - \int_\Omega (\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h v)v + \cdots$$

$$\int_\Omega -(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h v)v = -\frac{1}{2}\int_\Omega \boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h(v^2) = -\sum_{K\in\mathscr{T}_h}\int_{\partial K}\frac{\boldsymbol{\beta}\cdot\mathbf{n}}{2}v^2$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma\|v\|^2_{0,\Omega} + \varepsilon|v|^2_{1,h} - \int_\Omega(\boldsymbol{\beta}\cdot\nabla_h v)v + \cdots$$

$$\int_\Omega -(\boldsymbol{\beta}\cdot\nabla_h v)v = -\frac{1}{2}\int_\Omega \boldsymbol{\beta}\cdot\nabla_h(v^2) = -\sum_{K\in\mathscr{T}_h}\int_{\partial K}\frac{\boldsymbol{\beta}\cdot\mathbf{n}}{2}v^2$$

$$= -\frac{1}{2} < \{\boldsymbol{\beta}\}, [\![v^2]\!] >_h - \frac{1}{2} < [\![\boldsymbol{\beta}]\!], \{v^2\} >_h^0$$

but $[\![\boldsymbol{\beta}]\!] = 0$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma \|v\|_{0,\Omega}^2 + \varepsilon |v|_{1,h}^2 - \int_\Omega (\boldsymbol{\beta} \cdot \nabla_h v) v + \cdots$$

$$\int_\Omega -(\boldsymbol{\beta} \cdot \nabla_h v) v = -\frac{1}{2} \int_\Omega \boldsymbol{\beta} \cdot \nabla_h (v^2) = -\sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\boldsymbol{\beta} \cdot \mathbf{n}}{2} v^2$$

$$= -\frac{1}{2} < \{\boldsymbol{\beta}\}, [\![v^2]\!] >_h^0 - \frac{1}{2} < \boldsymbol{\beta} \cdot \mathbf{n}, v^2 >_\Gamma$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma \|v\|^2_{0,\Omega} + \varepsilon |v|^2_{1,h} - \int_\Omega (\boldsymbol{\beta} \cdot \nabla_h v) v + \cdots$$

$$\int_\Omega -(\boldsymbol{\beta} \cdot \nabla_h v) v = -\frac{1}{2} \int_\Omega \boldsymbol{\beta} \cdot \nabla_h (v^2) = -\sum_{K \in \mathscr{T}_h} \int_{\partial K} \frac{\boldsymbol{\beta} \cdot \mathbf{n}}{2} v^2$$

$$= -\frac{1}{2} < \{\boldsymbol{\beta}\}, [\![v^2]\!] >^0_h - \frac{1}{2} < \boldsymbol{\beta} \cdot \mathbf{n}, v^2 >_\Gamma$$

$$<\{\boldsymbol{\sigma}(v)\}, [\![v]\!]>_h = - <\{\varepsilon \nabla_h v\}, [\![v]\!]>_h + <\{\boldsymbol{\beta} v\}, [\![v]\!]>_h$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma\|v\|^2_{0,\Omega} + \varepsilon|v|^2_{1,h} - \int_\Omega (\boldsymbol{\beta}\cdot\nabla_h v)v + \cdots$$

$$\int_\Omega -(\boldsymbol{\beta}\cdot\nabla_h v)v = -\frac{1}{2}\int_\Omega \boldsymbol{\beta}\cdot\nabla_h(v^2) = -\sum_{K\in\mathscr{T}_h}\int_{\partial K}\frac{\boldsymbol{\beta}\cdot\mathbf{n}}{2}v^2$$

$$= -\frac{1}{2} < \{\boldsymbol{\beta}\}, [\![v^2]\!] >^0_h - \frac{1}{2} < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma$$

$$< \{\boldsymbol{\sigma}(v)\}, [\![v]\!] >_h = - < \{\varepsilon\nabla_h v\}, [\![v]\!] >_h + < \{\boldsymbol{\beta} v\}, [\![v]\!] >^0_h + < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma \|v\|^2_{0,\Omega} + \varepsilon |v|^2_{1,h} - \int_\Omega (\boldsymbol{\beta}\cdot\nabla_h v)v + \cdots$$

$$\int_\Omega -(\boldsymbol{\beta}\cdot\nabla_h v)v = -\frac{1}{2}\int_\Omega \boldsymbol{\beta}\cdot\nabla_h(v^2) = -\sum_{K\in\mathscr{T}_h}\int_{\partial K}\frac{\boldsymbol{\beta}\cdot\mathbf{n}}{2}v^2$$

$$= -\frac{1}{2} < \{\boldsymbol{\beta}\}, [\![v^2]\!] >^0_h - \frac{1}{2} < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma$$

$$<\{\boldsymbol{\sigma}(v)\}, [\![v]\!]>_h = -<\{\varepsilon\nabla_h v\}, [\![v]\!]>_h + <\{\boldsymbol{\beta}v\}, [\![v]\!]>^0_h + <\boldsymbol{\beta}\cdot\mathbf{n}, v^2>_\Gamma$$

$$< \{\boldsymbol{\beta}v\}, [\![v]\!] >^0_h = \frac{1}{2} < \{\boldsymbol{\beta}\}, [\![v^2]\!] >^0_h$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma\|v\|^2_{0,\Omega} + \varepsilon|v|^2_{1,h} - \int_\Omega (\boldsymbol{\beta}\cdot\nabla_h v)v + \cdots$$

$$\int_\Omega -(\boldsymbol{\beta}\cdot\nabla_h v)v = -\frac{1}{2}\int_\Omega \boldsymbol{\beta}\cdot\nabla_h(v^2) = -\sum_{K\in\mathscr{T}_h}\int_{\partial K}\frac{\boldsymbol{\beta}\cdot\mathbf{n}}{2}v^2$$

$$= -\frac{1}{2} < \{\boldsymbol{\beta}\}, [\![v^2]\!] >^0_h - \frac{1}{2} < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma$$

$$<\{\boldsymbol{\sigma}(v)\}, [\![v]\!]>_h = - <\{\varepsilon\nabla_h v\}, [\![v]\!]>_h + \frac{1}{2} <\{\boldsymbol{\beta}\}, [\![v^2]\!]>^0_h + <\boldsymbol{\beta}\cdot\mathbf{n}, v^2>_\Gamma$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma \|v\|^2_{0,\Omega} + \varepsilon |v|^2_{1,h} - \int_\Omega (\boldsymbol{\beta} \cdot \boldsymbol{\nabla}_h v) v + \cdots$$

$$\int_\Omega -(\boldsymbol{\beta} \cdot \boldsymbol{\nabla}_h v) v = -\frac{1}{2} \int_\Omega \boldsymbol{\beta} \cdot \boldsymbol{\nabla}_h (v^2) = -\sum_{K \in \mathscr{T}_h} \int_{\partial K} \frac{\boldsymbol{\beta} \cdot \mathbf{n}}{2} v^2$$

$$= -\frac{1}{2} < \{\boldsymbol{\beta}\}, [\![v^2]\!] >^0_h - \frac{1}{2} < \boldsymbol{\beta} \cdot \mathbf{n}, v^2 >_\Gamma$$

$$< \{\boldsymbol{\sigma}(v)\}, [\![v]\!] >_h = - < \{\varepsilon \boldsymbol{\nabla}_h v\}, [\![v]\!] >_h + \frac{1}{2} < \{\boldsymbol{\beta}\}, [\![v^2]\!] >^0_h + < \boldsymbol{\beta} \cdot \mathbf{n}, v^2 >_\Gamma$$

$$< [\![\boldsymbol{\sigma}(v)]\!], B_2(v) + \{v\} >^0_h = - < [\![\varepsilon \boldsymbol{\nabla}_h v]\!], B_2(v) + \{v\} >^0_h$$

$$+ < [\![\boldsymbol{\beta} v]\!], B_2(v) + \{v\} >^0_h$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma\|v\|^2_{0,\Omega} + \varepsilon|v|^2_{1,h} - \int_\Omega (\boldsymbol{\beta}\cdot\nabla_h v)v + \cdots$$

$$\int_\Omega -(\boldsymbol{\beta}\cdot\nabla_h v)v = -\frac{1}{2}\int_\Omega \boldsymbol{\beta}\cdot\nabla_h(v^2) = -\sum_{K\in\mathscr{T}_h}\int_{\partial K}\frac{\boldsymbol{\beta}\cdot\mathbf{n}}{2}v^2$$

$$= -\frac{1}{2} < \{\boldsymbol{\beta}\}, [\![v^2]\!] >^0_h - \frac{1}{2} < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma$$

$$< \{\boldsymbol{\sigma}(v)\}, [\![v]\!] >_h = - < \{\varepsilon\nabla_h v\}, [\![v]\!] >_h + \frac{1}{2} < \{\boldsymbol{\beta}\}, [\![v^2]\!] >^0_h + < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma$$

$$< [\![\boldsymbol{\sigma}(v)]\!], B_2(v) + \{v\} >^0_h = - < [\![\varepsilon\nabla_h v]\!], B_2(v) + \{v\} >^0_h$$

$$+ < [\![\boldsymbol{\beta} v]\!], B_2(v) + \{v\} >^0_h$$

$$< [\![u]\!], B_1(v) >^0_h, \qquad < u, B_1^D(v) >_\Gamma$$

Guidelines for choosing the operators. (Towards stability)

$$B_2(v) = -\{v\} + Q_2(v), \qquad B_1(v) = \frac{c\varepsilon}{|e|}[\![v]\!] + Q_1(v)$$

$$B_1^D(v) = \frac{c\varepsilon}{|e|}v - \boldsymbol{\beta}\cdot\mathbf{n}v + Q_1^D(v) \text{ on } \Gamma^-, \quad B_1^D(v) = \frac{c\varepsilon}{|e|}v + Q_1^D(v) \text{ on } \Gamma^+$$

with $Q_1(v),\ Q_2(v),\ Q_1^D(v)$ to be chosen such that

$$< [\![v]\!], Q_1(v) >_h^0 + < [\![\boldsymbol{\beta} v]\!], Q_2(v) >_h^0 \geq \sum_{e\in\mathcal{E}_h^\circ} |\boldsymbol{\beta}\cdot\mathbf{n}| \|[\![v]\!]\|_{0,e}^2$$

$Q_1^D(v)$ not depending on $\boldsymbol{\beta}$.

In other words, we can upwind either with B_1 or with B_2

First choice - minimal choice

Set $S_e = \frac{c\varepsilon}{|e|}$, $Q_2(v) = 0$, $\quad Q_1(v) = \frac{\mathbf{n}^+}{2}[\![\beta v]\!] \quad Q_1^D(v) = 0$, that is

$$B_2(v) = -\{v\} \qquad B_1(v) = S_e[\![v]\!] + \frac{n^+}{2}[\![\beta v]\!]$$

$$B_1^D(v) = S_e v - \boldsymbol{\beta}\cdot\mathbf{n} v \text{ on } \Gamma^-, \quad B_1^D(v) = S_e v \text{ on } \Gamma^+$$

$$\begin{cases} \text{Find } u \in H^2(\mathscr{T}_h) \text{ such that } \forall v \in H^2(\mathscr{T}_h) \\ \int_\Omega (\gamma uv - \boldsymbol{\sigma}(u)\cdot\boldsymbol{\nabla}_h v) + \sum_{e\in\mathcal{E}_h} S_e \int_e [\![u]\!]\cdot[\![v]\!] - \sum_{e\in\mathcal{E}_h}\int_e \{\varepsilon\boldsymbol{\nabla}_h u\}\cdot[\![v]\!] \\ + \sum_{e\in\mathcal{E}_h^\circ}\int_e (\boldsymbol{\beta} u)_{upw}[\![v]\!] + \int_{\Gamma^+}\boldsymbol{\beta}\cdot\mathbf{n}uv \\ = \int_\Omega fv + \sum_{e\in\Gamma} S_e \int_e g\,v - \sum_{e\in\Gamma^-}\int_e \boldsymbol{\beta}\cdot\mathbf{n}gv. \end{cases}$$

For the diffusive part this choice corresponds to the IIP scheme.

Choice by Houston-Schwab-Süli (SINUM 2002)

Set

$$Q_2(v) = 0, \quad Q_1(v) = \{\varepsilon \nabla_h v\} + \frac{n^+}{2}[\![\beta v]\!], \quad Q_1^D(v) = \varepsilon \nabla_h v \cdot \mathbf{n}$$

Choice by Houston-Schwab-Süli (SINUM 2002)

Set

$$Q_2(v) = 0, \quad Q_1(v) = \{\varepsilon \nabla_h v\} + \frac{n^+}{2}[\![\beta v]\!], \quad Q_1^D(v) = \varepsilon \nabla_h v \cdot \mathbf{n}$$

$$\Longrightarrow \quad B_2(v) = -\{v\} \qquad B_1(v) = S_e[\![v]\!] + \{\varepsilon \nabla_h v\} + \frac{n^+}{2}[\![\beta v]\!]$$

$$B_1^D(v) = S_e v - \boldsymbol{\beta} \cdot \mathbf{n} v + \varepsilon \nabla_h v \cdot \mathbf{n} \text{ on } \Gamma^-,$$

$$B_1^D(v) = S_e v + \varepsilon \nabla_h v \cdot \mathbf{n} \text{ on } \Gamma^+$$

Choice by Houston-Schwab-Süli (SINUM 2002)

$$
\begin{cases}
\text{Find} \quad u \in H^2(\mathscr{T}_h) \quad \text{such that} \quad \forall\, v \in H^2(\mathscr{T}_h) \\
\displaystyle\int_\Omega (\gamma uv - \boldsymbol{\sigma}(u)\cdot \boldsymbol{\nabla}_h v) + \sum_{e\in\mathcal{E}_h} S_e \int_e [\![u]\!]\cdot[\![v]\!] + \sum_{e\in\mathcal{E}_h^\circ} \int_e (\boldsymbol{\beta} u)_{upw}\cdot [\![v]\!] \\
\displaystyle + \sum_{e\in\mathcal{E}_h} \int_e ([\![u]\!]\cdot\{\varepsilon\boldsymbol{\nabla}_h v\} - \{\varepsilon\boldsymbol{\nabla}_h u\}\cdot[\![v]\!]) + \sum_{e\in\Gamma^+}\int_e \boldsymbol{\beta}\cdot\mathbf{n}uv \\
\displaystyle = \int_\Omega fv + \sum_{e\in\Gamma}\int_e g\,(S_e v + \varepsilon\boldsymbol{\nabla}_h v\cdot\mathbf{n} - \boldsymbol{\beta}\cdot\mathbf{n}\,v)\,.
\end{cases}
$$

NIPG for the diffusive part (Wheeler-Rivière-Girault)

Choice by Hughes-Scovazzi-Bochev-Buffa (CMAME 2006)

$$Q_1(v) = \theta(\varepsilon \nabla_h v)_{upw} \equiv \theta(\{\varepsilon \nabla_h v\} + \frac{\mathbf{n}^+}{2}[\![\varepsilon \nabla_h v]\!]),$$

$$Q_2(v) = \frac{\mathbf{n}^+}{2} \cdot [\![v]\!], \qquad Q_1^D(v) = \theta \varepsilon \nabla_h v \cdot \mathbf{n}$$

Choice by Hughes-Scovazzi-Bochev-Buffa (CMAME 2006)

$$Q_1(v) = \theta(\varepsilon \nabla_h v)_{upw} \equiv \theta(\{\varepsilon \nabla_h v\} + \frac{\mathbf{n}^+}{2} [\![\varepsilon \nabla_h v]\!]),$$

$$Q_2(v) = \frac{\mathbf{n}^+}{2} \cdot [\![v]\!], \qquad Q_1^D(v) = \theta \varepsilon \nabla_h v \cdot \mathbf{n}$$

$$\Longrightarrow \quad B_2(v) = -(v)_{dw} \equiv -\{v\} + \frac{\mathbf{n}^+}{2} \cdot [\![v]\!]$$

$$B_1(v) = S_e [\![v]\!] + \theta(\varepsilon \nabla_h v)_{upw}$$

$$B_1^D(v) = S_e v - \boldsymbol{\beta} \cdot \mathbf{n} v + \theta \varepsilon \nabla_h v \cdot \mathbf{n} \text{ on } \Gamma^-,$$

$$B_1^D(v) = S_e v + \theta \varepsilon \nabla_h v \cdot \mathbf{n} \text{ on } \Gamma^+$$

Choice by Hughes-Scovazzi-Bochev-Buffa (CMAME 2006)

$$\begin{cases} \text{Find } u \in H^2(\mathscr{T}_h) \text{ such that } \forall v \in H^2(\mathscr{T}_h) \\ \displaystyle\int_\Omega (\gamma uv - \boldsymbol{\sigma}(u)\cdot\nabla_h v) + \sum_{e\in\mathcal{E}_h} S_e \int_e [\![u]\!]\cdot[\![v]\!] \\ \displaystyle + \sum_{e\in\mathcal{E}_h^\circ}\int_e (\theta[\![u]\!]\cdot(\varepsilon\nabla_h v)_{upw} - (\varepsilon\nabla_h u)_{upw}\cdot[\![v]\!]) + \sum_{e\in\mathcal{E}_h^\circ}\int_e (\boldsymbol{\beta} u)_{upw}\cdot[\![v]\!] \\ \displaystyle + \sum_{e\in\Gamma}\int_e \theta u(\varepsilon\nabla_h v\cdot\mathbf{n}) - (\varepsilon\nabla_h u\cdot\mathbf{n})v + \sum_{e\in\Gamma^+}\int_e \boldsymbol{\beta}\cdot\mathbf{n}uv \\ \displaystyle = \int_\Omega fv + \sum_{e\in\Gamma}\int_e g(S_e v + \theta\varepsilon\nabla v\cdot\mathbf{n}) - \sum_{e\in\Gamma^-}\int_e g\boldsymbol{\beta}\cdot\mathbf{n}v. \end{cases}$$

$\theta = -1 \longrightarrow$ symmetric, $\theta = 1 \longrightarrow$ nonsymmetric, $\theta = 0 \longrightarrow$ neutral

Analyzed by Hughes-Buffa-Sangalli (to appear in SINUM)

Another choice

Let us first introduce a new average. Definition of weighted average on an internal edge. For $\alpha^1,\ \alpha^2$ real numbers, with $\alpha^1 + \alpha^2 = 1$:

$$\{\varphi\}_\alpha = \alpha^1\varphi^1 + \alpha^2\varphi^2$$

Another choice

Let us first introduce a new average. Definition of weighted average on an internal edge. For $\alpha^1,\ \alpha^2$ real numbers, with $\alpha^1 + \alpha^2 = 1$:

$$\{\varphi\}_\alpha = \alpha^1\varphi^1 + \alpha^2\varphi^2$$

$$\alpha^1 = \alpha^2 = 1/2 \longrightarrow \text{ arithmetic average}$$

$$\alpha^i = (sign(\boldsymbol{\beta} \cdot \mathbf{n}^i) + 1)/2 \longrightarrow \text{ classical upwind}$$

Another choice

Let us first introduce a new average. Definition of weighted average on an internal edge. For $\alpha^1,\ \alpha^2$ real numbers, with $\alpha^1 + \alpha^2 = 1$:

$$\{\varphi\}_\alpha = \alpha^1\varphi^1 + \alpha^2\varphi^2$$

$$\alpha^1 = \alpha^2 = 1/2 \longrightarrow \text{ arithmetic average}$$

$$\alpha^i = (sign(\boldsymbol{\beta}\cdot\mathbf{n}^i) + 1)/2 \longrightarrow \text{ classical upwind}$$

Indeed:

$$\{\varphi\}_\alpha = \{\varphi\} + \frac{[\![\alpha]\!]}{2}[\![\varphi]\!]$$

Another choice

Let us first introduce a new average. Definition of weighted average on an internal edge. For $\alpha^1,\ \alpha^2$ real numbers, with $\alpha^1+\alpha^2=1$:

$$\{\varphi\}_\alpha = \alpha^1\varphi^1 + \alpha^2\varphi^2$$

$$\alpha^1 = \alpha^2 = 1/2 \longrightarrow \text{ arithmetic average}$$

$$\alpha^i = (sign(\boldsymbol{\beta}\cdot\mathbf{n}^i)+1)/2 \longrightarrow \text{ classical upwind}$$

Indeed:

$$\{\varphi\}_\alpha = \{\varphi\} + \frac{[\![\alpha]\!]}{2}[\![\varphi]\!]$$

Ex: If K_1 upwind element $(\boldsymbol{\beta}\cdot\mathbf{n}^1 > 0)$ then $\alpha=(1,0)$

$$\{\varphi\}_\alpha \equiv (\varphi)_{upw} = \varphi^1 \qquad \{\varphi\}_{1-\alpha} \equiv (\varphi)_{dw} = \varphi^2$$

Another choice

$$Q_1(v) = \theta(\{\boldsymbol{\sigma}(v)\}_\alpha - \{\boldsymbol{\beta} v\}), \qquad Q_2(v) = \frac{[\![\alpha]\!]}{2} \cdot [\![v]\!], \quad Q_1^D(v) = -\theta\varepsilon\boldsymbol{\nabla}_h v \cdot$$

Another choice

$$Q_1(v) = \theta(\{\boldsymbol{\sigma}(v)\}_\alpha - \{\boldsymbol{\beta} v\}), \qquad Q_2(v) = \frac{[\![\alpha]\!]}{2} \cdot [\![v]\!], \quad Q_1^D(v) = -\theta\varepsilon\boldsymbol{\nabla}_h v \cdot$$

$$B_1 v = S_e[\![v]\!] + \theta(\{\boldsymbol{\sigma}(v)\}_\alpha - \{\boldsymbol{\beta} v\}),$$
$$B_2 v = -\{v\}_{1-\alpha} \equiv -\{v\} + \frac{[\![\alpha]\!]}{2}[\![v]\!],$$
$$B_1^D v = S_e v - \boldsymbol{\beta} \cdot \mathbf{n} v - \theta\varepsilon\boldsymbol{\nabla}_h v \cdot \mathbf{n} \text{ on } \Gamma^-,$$
$$B_1^D v = S_e v - \theta\varepsilon\boldsymbol{\nabla}_h v \cdot \mathbf{n} \text{ on } \Gamma^+ .$$

Another choice

$$
\begin{cases}
\text{Find} \quad u \in H^2(\mathscr{T}_h) \quad \text{such that} \quad \forall\, v \in H^2(Th) \\
\int_\Omega (\gamma uv - \boldsymbol{\sigma}(u) \cdot \nabla_h v) + \sum_{e \in \mathcal{E}_h} S_e \int_e [\![u]\!] \cdot [\![v]\!] \\
+ \sum_{e \in \mathcal{E}_h^\circ} \int_e \{\boldsymbol{\sigma}(u)\}_\alpha \cdot [\![v]\!] + \sum_{e \in \mathcal{E}_h^\circ} \int_e \theta\, [\![u]\!] \cdot (\{\boldsymbol{\sigma}(v)\}_\alpha - \{\boldsymbol{\beta} v\}) \\
- \sum_{e \in \Gamma} \int_e (\varepsilon \nabla_h u \cdot \mathbf{n}\, v + \theta u\, \varepsilon \nabla_h v \cdot \mathbf{n}) + \sum_{e \in \Gamma^+} \int_e \boldsymbol{\beta} \cdot \mathbf{n} uv \\
= \int_\Omega fv + \sum_{e \in \Gamma} \int_e g(S_e v - \theta \varepsilon \nabla_h v \cdot \mathbf{n}) - \sum_{e \in \Gamma^-} \int_e \boldsymbol{\beta} \cdot \mathbf{n} gv
\end{cases}
$$

Another choice

$$\begin{cases} \text{Find} \quad u \in H^2(\mathscr{T}_h) \quad \text{such that} \quad \forall\, v \in H^2(Th) \\ \int_\Omega (\gamma uv - \boldsymbol{\sigma}(u)\cdot \boldsymbol{\nabla}_h v) + \sum_{e\in\mathcal{E}_h} S_e \int_e [\![u]\!]\cdot[\![v]\!] \\ + \sum_{e\in\mathcal{E}_h^\circ} \int_e \{\boldsymbol{\sigma}(u)\}_\alpha \cdot [\![v]\!] + \sum_{e\in\mathcal{E}_h^\circ} \int_e \theta\, [\![u]\!]\cdot(\{\boldsymbol{\sigma}(v)\}_\alpha - \{\boldsymbol{\beta} v\}) \\ - \sum_{e\in\Gamma} \int_e (\varepsilon \boldsymbol{\nabla}_h u \cdot \mathbf{n}\, v + \theta u\, \varepsilon \boldsymbol{\nabla}_h v \cdot \mathbf{n}) + \sum_{e\in\Gamma^+} \int_e \boldsymbol{\beta}\cdot\mathbf{n} uv \\ = \int_\Omega fv + \sum_{e\in\Gamma} \int_e g(S_e v - \theta\varepsilon \boldsymbol{\nabla}_h v\cdot\mathbf{n}) - \sum_{e\in\Gamma^-} \int_e \boldsymbol{\beta}\cdot\mathbf{n} g v \end{cases}$$

$\theta = 1 \longrightarrow$ symmetric, $\theta = 0 \longrightarrow$ neutral

For $\theta = -1 \longrightarrow$ nonsymmetric, but stability in a weaker norm $\Longrightarrow$ suboptimal order of convergence.

Error estimates

$$k \geq 1 \longrightarrow V_h^k = \{\mathbf{v} \in L^2(\Omega) : \ \mathbf{v}_{|K} \in P_k(K) \quad \forall K \in \mathscr{T}_h\}$$

In all the cases the discrete problem reads:

$$\begin{cases} \text{Find } u_h \in V_h^k \text{ such that :} \\ a_h(u_h, v) = F(v) \qquad \forall v \in V_h^k. \end{cases}$$

We have consistency (by construction), and stability in the norm:

$$|||v|||^2 = \gamma \|v\|_{0,\Omega}^2 + \varepsilon |v|_{1,h}^2 + \sum_{e \in \mathcal{E}_h} \frac{\varepsilon}{|e|} \|[\![v]\!]\|_{0,e}^2 + \sum_{e \in \mathcal{E}_h} |\boldsymbol{\beta} \cdot \mathbf{n}| \|[\![v]\!]\|_{0,e}^2$$

The following estimate holds:

$$|||u - u_h||| \leq C \begin{cases} h^{k+1/2} & \text{if advection dominates,} \\ h^k & \text{if diffusion dominates.} \end{cases}$$

Introducing a term of SUPG type

It is often desirable to have direct estimates in the norm

$$|||v|||^2_{SUPG} = |||v|||^2 + \sum_{K \in \mathscr{T}_h} C_{0,K} \frac{h_K}{|\boldsymbol{\beta}|} ||\boldsymbol{\beta} \cdot \nabla v||^2$$

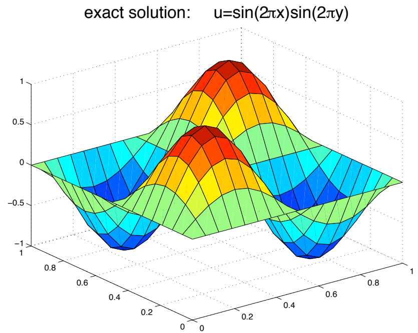
For this we only have to choose B_0 as

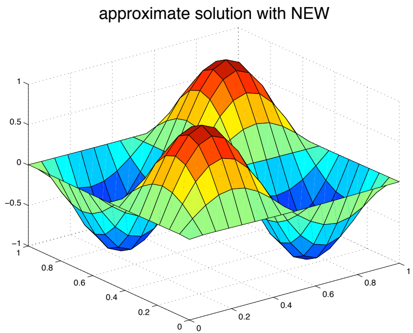
$$B_0(v) = v + \sum_{K \in \mathscr{T}_h} C_{0,K} \frac{h_K}{|\boldsymbol{\beta}|} \boldsymbol{\beta} \cdot \nabla v$$

The same estimates hold

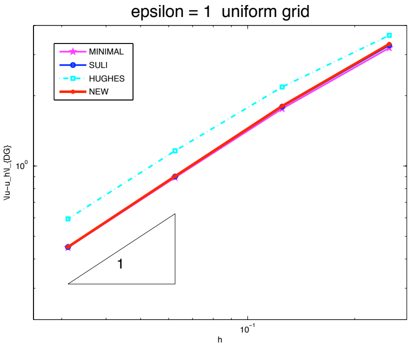
$$|||u - u_h|||_{SUPG} \leq C \begin{cases} h^{k+1/2} & \text{if advection dominates,} \\ h^k & \text{if diffusion dominates.} \end{cases}$$

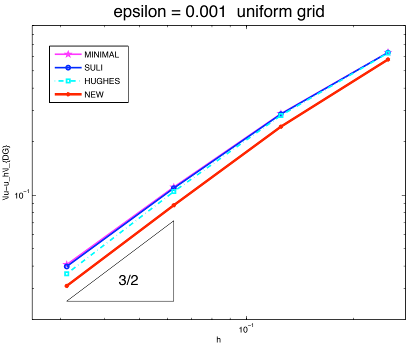
Approximation to a Smooth Solution



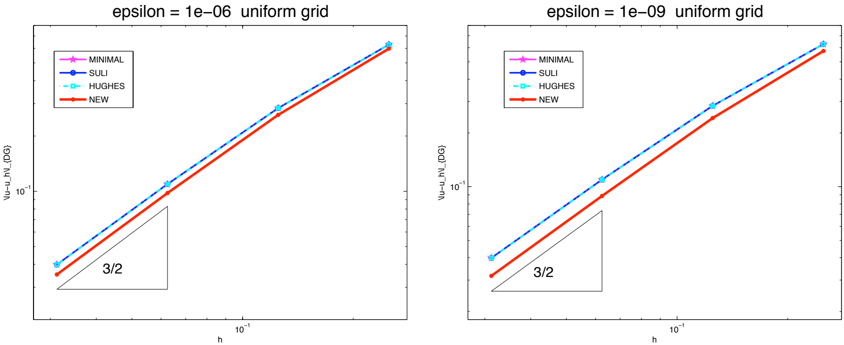


Convergence Diagrams: Smooth Solution

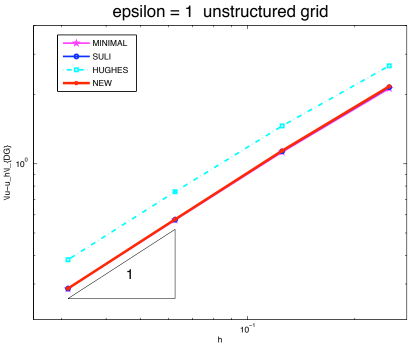


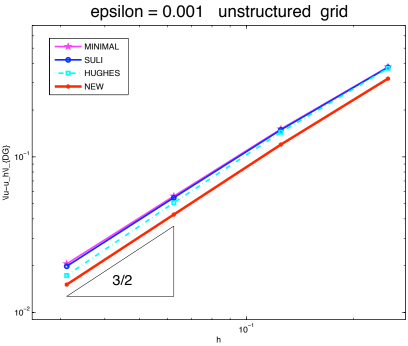


Convergence Diagrams: Smooth Solution

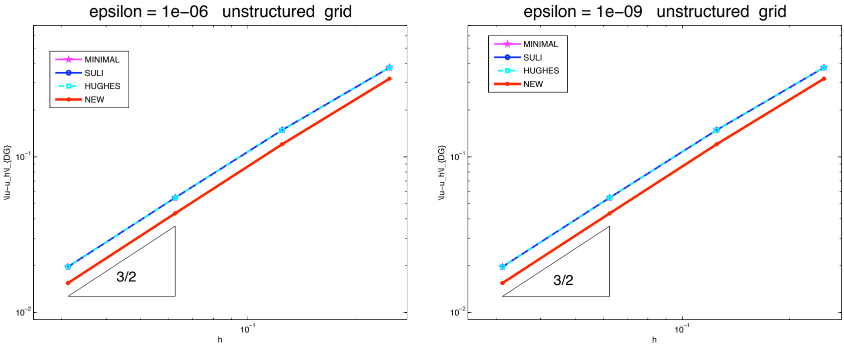


Convergence Diagrams:Unstructured Meshes

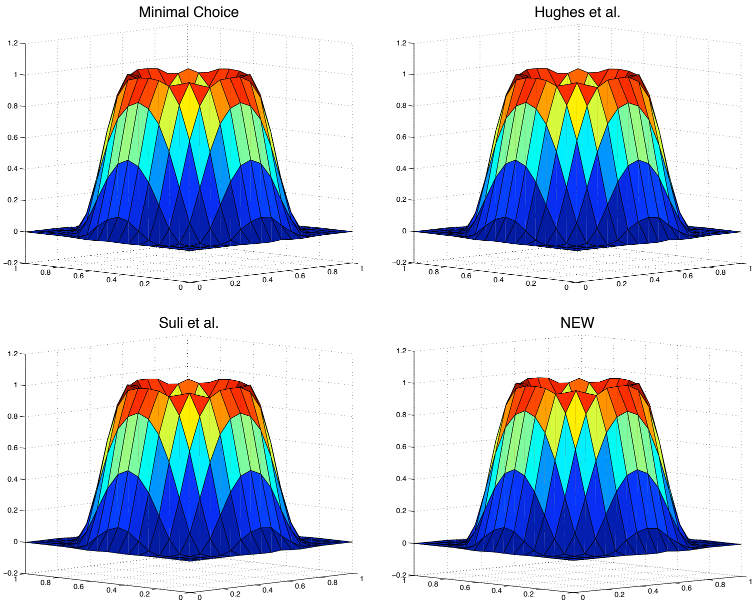




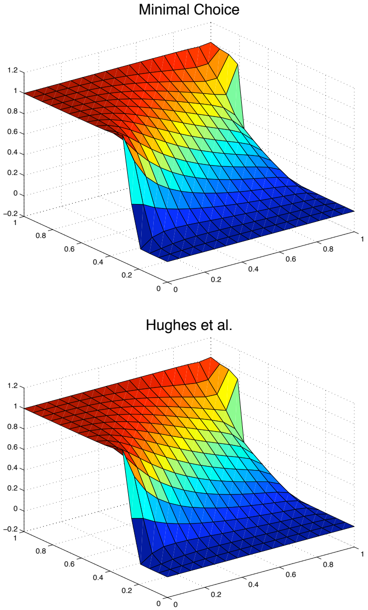
Convergence Diagrams: Smooth Solution



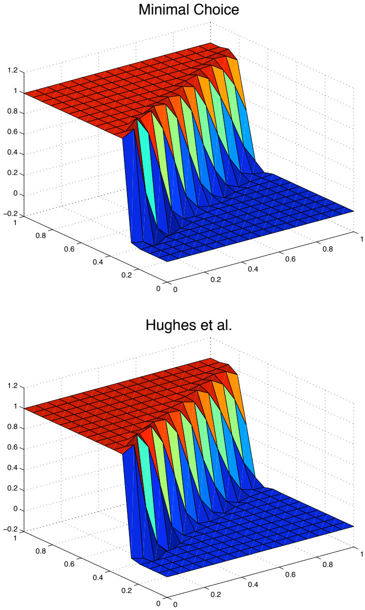
Rotating Flow $\varepsilon = 1e - 09$



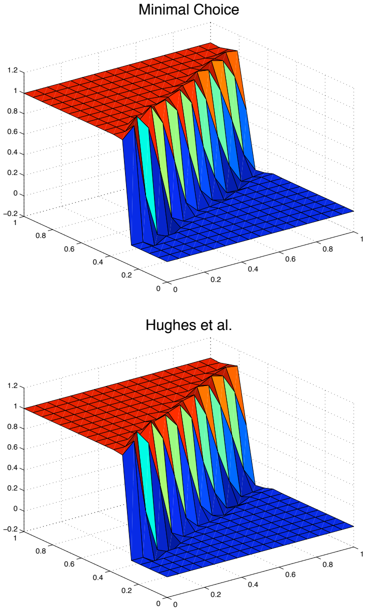
Internal Layer $\varepsilon = 0.1$



Internal Layer $\varepsilon = 0.001$



Internal Layer $\varepsilon = 1e - 06$



Internal Layer $\varepsilon = 1e - 09$