Convergence and Optimality of an AFEM for General Elliptic Operators

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Multiscale Problems: Modeling, Adaptive Discretization, Stabilization, and Solvers.
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Outline

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Main Results

$$\begin{aligned} \mathcal{L}u := -\operatorname{div} \mathbf{A}\nabla u + \mathbf{b}\cdot\nabla u + c\,u = f & \qquad \text{in } \Omega \\ u = 0 & \qquad \text{on } \partial\Omega \end{aligned}$$

- Ω polyhedral bounded domain in $\mathbb{R}^d$ (d=2,3).
- $\mathbf{A} : \Omega \mapsto \mathbb{R}^{d\times d}$ is piecewise Lipschitz in $\mathcal{T}_0$ and symmetric positive definite.
- $\mathbf{b} \in [L^\infty(\Omega)]^d$ (in general $\operatorname{div}\mathbf{b} \neq 0$ in Ω).
- $c \in L^\infty(\Omega)$ ($c \geq 0$ in Ω).
- $f \in L^2(\Omega)$.

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- **Theorem** [Convergence of AFEM] There exists $0 < \alpha < 1$ such that

$$|||u - u_{k+1}|||_\Omega^2 + \operatorname{osc}\left(u_{k+1}, \mathcal{T}_{k+1}\right)^2 \leq \alpha^2 \left\{ |||u - u_k|||_\Omega^2 + \operatorname{osc}\left(u_k, \mathcal{T}_k\right)^2 \right\}.$$

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- **Theorem** [Optimality of AFEM] Let $(u,\, f,\, \mathbf{A},\, \mathbf{b},\, c)$ satisfy PDE and for all $\epsilon > 0$

$$\exists \mathcal{T} \supset \mathcal{T}_0 \;:\; \inf_{v_\mathcal{T}\in\mathbb{V}_\mathcal{T}} \left\{ \|u - v_\mathcal{T}\|_\Omega + \operatorname{osc}\left(v_\mathcal{T}, \mathcal{T}\right) \right\} \le \epsilon \quad \& \quad \#\mathcal{T} - \#\mathcal{T}_0 \preccurlyeq \epsilon^{-1/s}.$$

  Then

$$\|u - u_k\|_\Omega + \operatorname{osc}\left(u_k, \mathcal{T}_k\right) \preccurlyeq \left\{\#\mathcal{T}_k - \#\mathcal{T}_0\right\}^{-s}.$$

Morin-Nochetto-Siebert Algorithm

1. $\mathcal{T}_0,\ u_{-1} = 0,\ k = 0.$
2. $u_k = \text{SOLVE}(\mathcal{T}_k, u_{k-1}).$
3. $\{\eta_k, \text{osc}_k\} = \text{ESTIMATE}(\mathcal{T}_k, u_k).$
4. $\mathcal{M}_k = \text{MARK}(\eta_k, \text{osc}_k, \theta_{\text{est}}, \theta_{\text{osc}}, \mathcal{T}_k).$
5. $\mathcal{T}_{k+1} = \text{REFINE}(\mathcal{T}_k, \mathcal{M}_k).$
6. $k \leftarrow k + 1.$
7. go to 2.

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- **Procedure** $\text{MARK}(\eta_k, \text{osc}_k, \theta_{\text{est}}, \theta_{\text{osc}})$.

Select $\mathcal{M}_k \subset \mathcal{T}_k$ such that

$$\eta_k(\mathcal{M}_k)^2 \geq \theta_{\text{est}} \eta_k^2,$$

if necessary enlarge $\mathcal{M}_k$ *to satisfy*

$$\text{osc}\,(u_k, \mathcal{M}_k)^2 \geq \theta_{\text{osc}}\, \text{osc}\,(u_k, \mathcal{T}_k)^2 .$$

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Is MNS optimal?

- η_k has large indicators in a small area $\Rightarrow$ Dörfler's Property holds for a small number of elements.
- $\text{osc}\,(u_k, \mathcal{T}_k) \ll \eta_k$.
- $\text{osc}\,(u_k, \mathcal{T}_k)$ is uniformly distributed in $\mathcal{T}_k \Rightarrow$ A large percentage of triangles must be marked.

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$\Rightarrow$**MNS is suboptimal!!**

MNS Algorithm - Counterexample

$$
\begin{aligned}
-\operatorname{div} \mathbf{A}\nabla u &= f \qquad \text{in } \Omega, \\
u &= u \qquad \text{on } \partial\Omega.
\end{aligned}
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- $\Omega = (-1,1)^2$.
- $\mathbf{A} = \sum_{i=1}^{4} a_i \mathbf{I} \chi_i$ (Checkerboard pattern).

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- $\mathbf{A} = \sum_{i=1}^4 a_i \mathbf{I}\chi_i$ (Checkerboard pattern).
- (u, f):

$$u(\mathbf{x}) = u_K(\mathbf{x}) + u_S(\mathbf{x}) \qquad f(\mathbf{x}) = f_K(\mathbf{x}) + f_S(\mathbf{x})$$

  - Kellogg's example

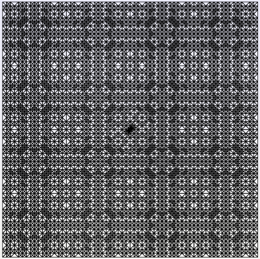
$$u_K(r,\theta) = r^{0.1}\phi(\theta) \quad \Rightarrow \quad f_K = 0$$

  - Smooth function

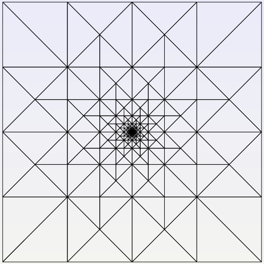
$$u_S(\mathbf{x}) = 10^{-6}\sum_{i=1}^{4} a_i^{-1}\chi_i \sin^2(10\pi x)\sin^2(10\pi y) \quad \Rightarrow \quad f_S \text{ smooth function}$$

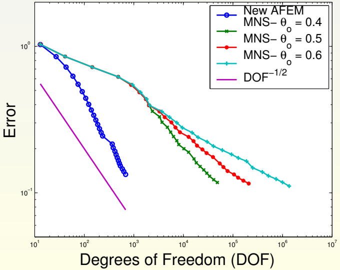
MNS Algorithm - Counterexample

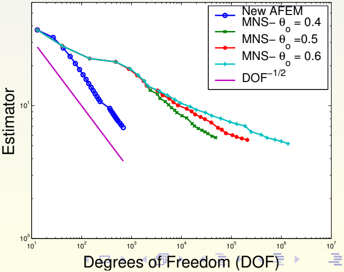
MNS Algorithm (iter 15, DOFs 8077)



New Algorithm (iter 15, DOFs 368)







Stevenson Algorithm (Simplified)

1. $\mathcal{T}_0,\ u_{-1} = 0,\ \delta_0 = \|f\|_{H^{-1}},$
   $k = 0$, tol > 0.
2. do $\delta_k = \delta_k/2$,
   $[\mathcal{T}_{k,*}, f_k] = \text{RHS}(\mathcal{T}_k, f, \delta_k)$
   $\mathcal{T}_k = \text{MAKECONFORM}(\mathcal{T}_{k,*})$
   $u_k = \text{SOLVE}(\mathcal{T}_k, u_{k-1})$
   $\eta_k = \text{ESTIMATE}(\mathcal{T}_k, u_k)$
   if $C_1\eta_k <$ tol STOP
   until $\delta_k < \omega\eta_k$
3. $\mathcal{M}_k = \text{MARK}(\eta_k, \theta_{\text{est}}, \mathcal{T}_k)$
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- RHS is not specified for $f \in H^{-1}$.
- The algorithm does not work with general elliptic operators $\mathcal{L}$.
- RHS reduces the oscillation of f relative to the current error indicator η_k.
- It has to solve the linear system (most expensive task) at all steps of the inner loop.

Preliminary Results

$\mathcal{T}_0 \to \{\mathcal{T}_k\}_{k\in\mathbb{N}}$.

$\mathbb{V} := H_0^1(\Omega)$, $\qquad \|\cdot\|_{\mathbb{V}}$.

$\mathbb{V}_k$, Continuous piecewise polynomials of degree one over $\mathcal{T}_k$.

$\mathcal{B}(u,v) := \langle \mathbf{A}\nabla u, \nabla v\rangle + \langle \mathbf{b}\cdot\nabla u + c\, u, v\rangle$.

$|\!|\!| v |\!|\!|_\Omega := \mathcal{B}(v,v)^{1/2}$.

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- **Continuous Problem**

$$u \in \mathbb{V}: \qquad \mathcal{B}(u,v) = \langle f, v\rangle \qquad \forall v \in \mathbb{V}.$$

- **Discrete Problem**

$$u_k \in \mathbb{V}_k: \qquad \mathcal{B}(u_k,v) = \langle f, v\rangle \qquad \forall v \in \mathbb{V}_k.$$

Preliminary Results

- **Orthogonality** (Case $\mathbf{b} = 0$).

$$\mathbb{V}_l \subset \mathbb{V}_m \subset \mathbb{V}_n, \qquad \mathbb{V}_\infty = \mathbb{V},\ u_\infty = u,$$
$$|\!|\!| u_n - u_m |\!|\!|_\Omega^2 = |\!|\!| u_n - u_l |\!|\!|_\Omega^2 - |\!|\!| u_m - u_l |\!|\!|_\Omega^2 .$$

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- **A-posteriori Error Estimate and Oscillation** ($v \in \mathbb{V}_k$)

$$R(v) := (f + \operatorname{div} \mathbf{A}\nabla v - \mathbf{b} \cdot \nabla v - cv)_{|T},$$
$$J(v) := ([\![\mathbf{A}\nabla v]\!] \cdot \boldsymbol{\nu})_{|\partial T},$$
$$\eta_k(T)^2 := \|hR(u_k)\|_{L^2(T)}^2 + \|h^{\frac{1}{2}} J(u_k)\|_{L^2(\partial T)}^2 ,$$
$$\operatorname{osc}(v, T)^2 := \left\| h_T^2 (R(v) - \overline{R}(v)) \right\|_{L^2(T)}^2 .$$

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- For any subset $\mathcal{M} \in \mathcal{T}_k$ we use the convention

$$\eta_k(\mathcal{M})^2 = \sum_{T\in\mathcal{M}} \eta_k(T)^2, \qquad \operatorname{osc}(v, \mathcal{M})^2 = \sum_{T\in\mathcal{M}} \operatorname{osc}(v, T)^2 .$$

- For $\mathcal{M} = \mathcal{T}_k$ we use the notation: $\eta_k^2 = \sum_{T\in\mathcal{M}} \eta_k(T)^2$.

Notation

- $\mathcal{T}_{k,*} \supset \mathcal{T}_k$ nonconforming refinement.
- Marked elements set:
  $\mathcal{M}_{k,*}$: elements of $\mathcal{T}_k$ that give rise to $\mathcal{T}_{k,*}$ after their
  not conforming refinement.
  $\mathcal{M}_{k,\mathrm{est}} := \mathsf{MARK}(\eta_k, \theta_{\mathrm{est}}, \mathcal{T}_k)$.
  $\mathcal{M}^i_{k,\mathrm{osc}} := \mathsf{MARK}(\mathrm{osc}^i_k, \theta_{\mathrm{osc}}, \mathcal{T}^i_k)$.
  $\overline{\mathcal{M}}$: discrete vicinity of $\mathcal{M}$ (all T's in $\mathcal{M}$ and theirs stars).

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- $\mathbf{D} := (\mathbf{A}, \mathbf{b}, c)$

$$\mathrm{osc}\,(\mathbf{D}, T)^2 := h_T^2 \max_j \max_{x,y\in T} |\partial_i \{a_{ij}(x) - a_{ij}(y)\}^2| + \ldots$$

$$\mathrm{osc}\,(\mathbf{D}, \mathcal{T}_k)^2 := \max_{T\in\mathcal{T}_k} \mathrm{osc}\,(\mathbf{D}, T)^2 .$$

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- $e_k := \left\| u - u_k \right\|_\Omega \qquad \epsilon_k := \left\| u_{k+1} - u_k \right\|_\Omega.$

A Posteriori Error Estimates

- **Upper bound**

$$\text{Global}: \quad \|u - u_k\|_\Omega^2 \le C_1 \eta_k^2$$

$$\text{Localized}: \quad \|u_{k,*} - u_k\|_\Omega^2 \le C_1 \eta_k(\overline{\mathcal{M}}_{k,*})^2$$

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- **Lower bound**

$$\text{Global}: \quad C_2 \eta_k^2 \le \|u - u_k\|_\Omega^2 + \operatorname{osc}(u_k, \mathcal{T}_k)^2$$

$$\text{Localized}: \quad C_2 \eta_k(\mathcal{M}_{k,*})^2 \le \|u_{k,*} - u_k\|_\Omega^2 + \operatorname{osc}(u_k, \mathcal{T}_k)^2$$

$+$ Interior Node Property

A Posteriori Error Estimates

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OPTIMALITY $\Leftarrow$ Localized : $\|u_{k,*} - u_k\|_\Omega^2 \le C_1 \eta_k(\overline{\mathcal{M}}_{k,*})^2$

- **Lower bound**

Global : $C_2 \eta_k^2 \le \|u - u_k\|_\Omega^2 + \operatorname{osc}(u_k, \mathcal{T}_k)^2$

Localized : $C_2 \eta_k(\mathcal{M}_{k,*})^2 \le \|u_{k,*} - u_k\|_\Omega^2 + \operatorname{osc}(u_k, \mathcal{T}_k)^2$

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$$\text{Global}: \quad C_2 \eta_k^2 \le \|u - u_k\|_\Omega^2 + \operatorname{osc}(u_k, \mathcal{T}_k)^2$$

CONVERGENCE $\Leftarrow$ Localized : $C_2 \eta_k(\mathcal{M}_{k,*})^2 \le \|u_{k,*} - u_k\|_\Omega^2 + \operatorname{osc}(u_k, \mathcal{T}_k)^2$

$+$ Interior Node Property

Oscillation

We consider $\mathcal{T}_k \subset \mathcal{T}_{k,*}, \quad \mathbb{V}_k \subset \mathbb{V}_{k,*}$

- Local: $\forall T \in \mathcal{T}_{k,*}$

$$\operatorname{osc}(u_{k,*}, T)^2 \leq 2 \operatorname{osc}(u_k, T)^2 + 2 \operatorname{osc}(\mathbf{D}, T)^2 \, \| u_{k,*} - u_k \|_\Omega^2 .$$

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- Global:

$$\operatorname{osc}(u_{k,*}, \mathcal{T}_{k,*})^2 \leq 2\operatorname{osc}(u_k, \mathcal{T}_{k,*})^2 + 2\operatorname{osc}(\mathbf{D}, \mathcal{T}_{k,*})^2 \, \|u_{k,*} - u_k\|_\Omega^2 .$$

By monotonicity,

$$\operatorname{osc}(u_{k,*}, \mathcal{T}_{k,*})^2 \leq 2\operatorname{osc}(u_k, \mathcal{T}_k)^2 + 2\operatorname{osc}(\mathbf{D}, \mathcal{T}_k)^2 \, \|u_{k,*} - u_k\|_\Omega^2 .$$

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2. $u_k = \text{SOLVE}(\mathcal{T}_k, u_{k-1}).$

- SOLVE: solves the discrete linear system.

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- ESTIMATE: computes local a posteriori error indicators and oscillation.

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2. $u_k = \mathsf{SOLVE}(\mathcal{T}_k, u_{k-1}).$
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4. $\mathcal{M}_{k,\mathrm{est}} = \mathsf{MARK}(\eta_k, \theta_{\mathrm{est}}, \mathcal{T}_k).$

- MARK: marks elements according to relative size of error indicators (Dörfler Marking).

New Algorithm

1. $\mathcal{T}_0,\ u_{-1} = 0,\ k = 0.$
2. $u_k = \text{SOLVE}(\mathcal{T}_k, u_{k-1}).$
3. $\{\eta_k, \text{osc}_k\} = \text{ESTIMATE}(\mathcal{T}_k, u_k).$
4. $\mathcal{M}_{k,\text{est}} = \text{MARK}(\eta_k, \theta_{\text{est}}, \mathcal{T}_k).$
5. if $\text{osc}\left(u_k, \mathcal{T}_k\right)^2 > \sigma_k = \delta\eta_k^2$
   $\{\mathcal{T}_k^+, \mathcal{M}_k^+\} = \text{REDUCE-OSC}(\mathcal{T}_k, \mathcal{M}_{k,\text{est}}, \text{osc}\left(u_k, \mathcal{T}_k\right), u_k, \sigma_k).$

- REDUCE-OSC: reduces the oscillation only relative to the estimator η_k .It generates a refinement $\mathcal{T}_k^+$ satisfying

$$\text{osc}\left(u_k, \mathcal{T}_k^+\right)^2 < \delta\eta_k^2.$$

New Algorithm

1. $\mathcal{T}_0,\ u_{-1} = 0,\ k = 0.$
2. $u_k = \text{SOLVE}(\mathcal{T}_k, u_{k-1}).$
3. $\{\eta_k, \text{osc}_k\} = \text{ESTIMATE}(\mathcal{T}_k, u_k).$
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5. if $\text{osc}\,(u_k, \mathcal{T}_k)^2 > \sigma_k = \delta\eta_k^2$
   $\{\mathcal{T}_k^+, \mathcal{M}_k^+\} = \text{REDUCE-OSC}(\mathcal{T}_k, \mathcal{M}_{k,\text{est}}, \text{osc}\,(u_k, \mathcal{T}_k)\,, u_k, \sigma_k).$
6. $\mathcal{T}_{k+1} = \text{REFINE}(\mathcal{T}_k^+, \mathcal{M}_{k,\text{est}}^+).$

- REFINE: refines the mesh $\mathcal{T}_k^+$ locally according to marked elements $\mathcal{M}_{k,\text{est}}^+$.

New Algorithm

1. $\mathcal{T}_0,\ u_{-1} = 0,\ k = 0.$
2. $u_k = \mathsf{SOLVE}(\mathcal{T}_k, u_{k-1}).$
3. $\{\eta_k, \mathsf{osc}_k\} = \mathsf{ESTIMATE}(\mathcal{T}_k, u_k).$
4. $\mathcal{M}_{k,\mathrm{est}} = \mathsf{MARK}(\eta_k, \theta_{\mathrm{est}}, \mathcal{T}_k).$
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   $\{\mathcal{T}_k^+, \mathcal{M}_k^+\} = \mathsf{REDUCE\text{-}OSC}(\mathcal{T}_k, \mathcal{M}_{k,\mathrm{est}}, \mathrm{osc}\,(u_k, \mathcal{T}_k)\,, u_k, \sigma_k).$
6. $\mathcal{T}_{k+1} = \mathsf{REFINE}(\mathcal{T}_k^+, \mathcal{M}_{k,\mathrm{est}}^+).$
7. $k \leftarrow k + 1.$

New Algorithm

1. $\mathcal{T}_0,\ u_{-1} = 0,\ k = 0.$
2. $u_k = \text{SOLVE}(\mathcal{T}_k, u_{k-1}).$
3. $\{\eta_k, \text{osc}_k\} = \text{ESTIMATE}(\mathcal{T}_k, u_k).$
4. $\mathcal{M}_{k,\mathrm{est}} = \text{MARK}(\eta_k, \theta_{\mathrm{est}}, \mathcal{T}_k).$
5. if $\text{osc}\,(u_k, \mathcal{T}_k)^2 > \sigma_k = \delta\eta_k^2$
   $\{\mathcal{T}_k^+, \mathcal{M}_k^+\} = \text{REDUCE-OSC}(\mathcal{T}_k, \mathcal{M}_{k,\mathrm{est}}, \text{osc}\,(u_k, \mathcal{T}_k)\,, u_k, \sigma_k).$
6. $\mathcal{T}_{k+1} = \text{REFINE}(\mathcal{T}_k^+, \mathcal{M}_{k,\mathrm{est}}^+).$
7. $k \leftarrow k + 1.$
8. go to 2.

New Algorithm - REDUCE-OSC

$\{\mathcal{T}^+, \mathcal{M}^+\} = \text{REDUCE-OSC}(\mathcal{T}, \mathcal{M}, \text{osc}(v, \mathcal{T}), v, \sigma)$

New Algorithm - REDUCE-OSC

$\{\mathcal{T}^+, \mathcal{M}^+\} =$ REDUCE-OSC$(\mathcal{T}, \mathcal{M}, \text{osc}(v, \mathcal{T}), v, \sigma)$

1. $\text{osc}^0 = \text{osc}(v, \mathcal{T})$, $\mathcal{T}^0 = \mathcal{T}$, $\mathcal{M}^+ = \mathcal{M}$, $i = 0$.

New Algorithm - REDUCE-OSC

$\{\mathcal{T}^+, \mathcal{M}^+\}$ = REDUCE-OSC$(\mathcal{T}, \mathcal{M}, \mathrm{osc}\,(v, \mathcal{T})\,, v, \sigma)$

1. $\mathrm{osc}^0 = \mathrm{osc}\,(v, \mathcal{T})$, $\mathcal{T}^0 = \mathcal{T}$, $\mathcal{M}^+ = \mathcal{M}$, $i = 0$.
2. do
   $\mathcal{M}^i_{\mathrm{osc}} = \mathrm{MARK}(\mathrm{osc}^i, \mathcal{T}^i, \theta_{\mathrm{osc}})$.
   $\mathcal{M}^+ = \mathrm{UPDATE}(\mathcal{M}^+, \mathcal{M}^i_{\mathrm{osc}})$.
   $\mathcal{T}^{i+1} = \mathrm{REFINE}(\mathcal{T}^i, \mathcal{M}^i_{\mathrm{osc}})$.
   $\mathrm{osc}^{i+1} = \mathrm{ESTIMATE\text{-}OSC}(\mathcal{T}^i, v)$
   $i \leftarrow i + 1$.
   while$(\mathrm{osc}^i > \sigma^{1/2})$.

New Algorithm - REDUCE-OSC

$\{\mathcal{T}^+, \mathcal{M}^+\} = \text{REDUCE-OSC}(\mathcal{T}, \mathcal{M}, \text{osc}(v, \mathcal{T}), v, \sigma)$

1. $\text{osc}^0 = \text{osc}(v, \mathcal{T})$, $\mathcal{T}^0 = \mathcal{T}$, $\mathcal{M}^+ = \mathcal{M}$, $i = 0$.
2. do

   $\mathcal{M}^i_{\text{osc}} = \text{MARK}(\text{osc}^i, \mathcal{T}^i, \theta_{\text{osc}})$.
   $\mathcal{M}^+ = \text{UPDATE}(\mathcal{M}^+, \mathcal{M}^i_{\text{osc}})$.
   $\mathcal{T}^{i+1} = \text{REFINE}(\mathcal{T}^i, \mathcal{M}^i_{\text{osc}})$.
   $\text{osc}^{i+1} = \text{ESTIMATE-OSC}(\mathcal{T}^i, v)$
   $i \leftarrow i + 1$.

   while($\text{osc}^i > \sigma^{1/2}$).

- MARK: standard Dörfler marking of oscillation,

$$\sum_{T \in \mathcal{M}^i_{\text{osc}}} \text{osc}(v, T)^2 \geq \theta_{\text{osc}} \, \text{osc}\left(v, \mathcal{T}^i\right)^2$$

New Algorithm - REDUCE-OSC

$\{\mathcal{T}^+, \mathcal{M}^+\}$ = REDUCE-OSC$(\mathcal{T}, \mathcal{M}, \text{osc}(v, \mathcal{T}), v, \sigma)$

1. $\text{osc}^0 = \text{osc}(v, \mathcal{T})$, $\mathcal{T}^0 = \mathcal{T}$, $\mathcal{M}^+ = \mathcal{M}$, $i = 0$.
2. do

   $\mathcal{M}^i_{\text{osc}} = \text{MARK}(\text{osc}^i, \mathcal{T}^i, \theta_{\text{osc}})$.
   $\mathcal{M}^+ = \text{UPDATE}(\mathcal{M}^+, \mathcal{M}^i_{\text{osc}})$.
   $\mathcal{T}^{i+1} = \text{REFINE}(\mathcal{T}^i, \mathcal{M}^i_{\text{osc}})$.
   $\text{osc}^{i+1} = \text{ESTIMATE-OSC}(\mathcal{T}^i, v)$
   $i \leftarrow i + 1$.

   while$(\text{osc}^i > \sigma^{1/2})$.

- UPDATE: deletes the triangles in $\mathcal{M}^+$ that belong to $\mathcal{M}^+ \cap \mathcal{M}^i_{\text{osc}}$.

New Algorithm - REDUCE-OSC

$\{\mathcal{T}^+, \mathcal{M}^+\}$ = REDUCE-OSC$(\mathcal{T}, \mathcal{M}, \mathrm{osc}\,(v, \mathcal{T})\,, v, \sigma)$

1. $\mathrm{osc}^0 = \mathrm{osc}\,(v, \mathcal{T})$, $\mathcal{T}^0 = \mathcal{T}$, $\mathcal{M}^+ = \mathcal{M}$, $i = 0$.
2. do
   $\mathcal{M}^i_{\mathrm{osc}}$ = MARK$(\mathrm{osc}^i, \mathcal{T}^i, \theta_{\mathrm{osc}})$.
   $\mathcal{M}^+$ = UPDATE$(\mathcal{M}^+, \mathcal{M}^i_{\mathrm{osc}})$.
   $\mathcal{T}^{i+1}$ = REFINE$(\mathcal{T}^i, \mathcal{M}^i_{\mathrm{osc}})$.
   osc^{i+1} = ESTIMATE-OSC$(\mathcal{T}^i, v)$
   $i \leftarrow i + 1$.
   while$(\mathrm{osc}^i > \sigma^{1/2})$.

- REFINE: conforming local refinement of $\mathcal{T}^i$ according to $\mathcal{M}^i_{\mathrm{osc}}$.

New Algorithm - REDUCE-OSC

$\{\mathcal{T}^+, \mathcal{M}^+\}$ = REDUCE-OSC$(\mathcal{T}, \mathcal{M}, \operatorname{osc}(v, \mathcal{T}), v, \sigma)$

1. $\operatorname{osc}^0 = \operatorname{osc}(v, \mathcal{T})$, $\mathcal{T}^0 = \mathcal{T}$, $\mathcal{M}^+ = \mathcal{M}$, $i = 0$.
2. do
   - $\mathcal{M}^i_{\text{osc}} = \text{MARK}(\operatorname{osc}^i, \mathcal{T}^i, \theta_{\text{osc}})$.
   - $\mathcal{M}^+ = \text{UPDATE}(\mathcal{M}^+, \mathcal{M}^i_{\text{osc}})$.
   - $\mathcal{T}^{i+1} = \text{REFINE}(\mathcal{T}^i, \mathcal{M}^i_{\text{osc}})$.
   - osc^{i+1} = ESTIMATE-OSC$(\mathcal{T}^i, v)$
   - $i \leftarrow i + 1$.

   while$(\operatorname{osc}^i > \sigma^{1/2})$.

- ESTIMATE-OSC: computes oscillation indicators (v is fixed throughout).

New Algorithm - REDUCE-OSC

$\{\mathcal{T}^+, \mathcal{M}^+\} = \text{REDUCE-OSC}(\mathcal{T}, \mathcal{M}, \text{osc}(v, \mathcal{T}), v, \sigma)$

1. $\text{osc}^0 = \text{osc}(v, \mathcal{T})$, $\mathcal{T}^0 = \mathcal{T}$, $\mathcal{M}^+ = \mathcal{M}$, $i = 0$.
2. do
   $\mathcal{M}^i_{\text{osc}} = \text{MARK}(\text{osc}^i, \mathcal{T}^i, \theta_{\text{osc}})$.
   $\mathcal{M}^+ = \text{UPDATE}(\mathcal{M}^+, \mathcal{M}^i_{\text{osc}})$.
   $\mathcal{T}^{i+1} = \text{REFINE}(\mathcal{T}^i, \mathcal{M}^i_{\text{osc}})$.
   $\text{osc}^{i+1} = \text{ESTIMATE-OSC}(\mathcal{T}^i, v)$
   $i \leftarrow i + 1$.
   while$(\text{osc}^i > \sigma^{1/2})$.
3. $\mathcal{T}^+ = \mathcal{T}^i$.

Convergence (Case $\mathbf{b} = 0$)

Theorem[Convergence] Let $\{u_k\}_k$ be the sequence generated by AFEM. If $\delta = \delta(C_1, C_2, \theta_{\text{est}})$ is adequately chosen and

$$\operatorname{osc}(\mathbf{D}, \mathcal{T}_0)^2 \leq \frac{\theta_{\text{est}} C_2}{8 C_1},$$

then there exists $0 < \alpha < 1$ depending on $\mathcal{T}_0$ and θ_{est} such that

$$|\!|\!| u - u_{k+1} |\!|\!|_{\Omega}^2 + \operatorname{osc}(u_{k+1}, \mathcal{T}_{k+1})^2 \leq \alpha^2 \left\{ |\!|\!| u - u_k |\!|\!|_{\Omega}^2 + \operatorname{osc}(u_k, \mathcal{T}_k)^2 \right\}.$$

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Proof: Outline:

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Proof: Outline:

- Estimate e_{k+1}^2 in terms of e_k^2, $\operatorname{osc}(u_k, \mathcal{T}_k)^2$.

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Proof: Outline:

- Estimate e_{k+1}^2 in terms of e_k^2, $\operatorname{osc}(u_k, \mathcal{T}_k)^2$.
- Estimate $\operatorname{osc}(u_{k+1}, \mathcal{T}_{k+1})^2$ in terms of e_k^2, $\operatorname{osc}(u_k, \mathcal{T}_k)^2$.

Convergence (Case $\mathbf{b} = 0$)

Theorem[Convergence] Let $\{u_k\}_k$ be the sequence generated by AFEM. If $\delta = \delta(C_1, C_2, \theta_{\rm est})$ is adequately chosen and

$$\operatorname{osc}(\mathbf{D}, \mathcal{T}_0)^2 \le \frac{\theta_{\rm est} C_2}{8 C_1},$$

then there exists $0 < \alpha < 1$ depending on $\mathcal{T}_0$ and $\theta_{\rm est}$ such that

$$|\!|\!| u - u_{k+1} |\!|\!|_\Omega^2 + \operatorname{osc}(u_{k+1}, \mathcal{T}_{k+1})^2 \le \alpha^2 \left\{ |\!|\!| u - u_k |\!|\!|_\Omega^2 + \operatorname{osc}(u_k, \mathcal{T}_k)^2 \right\}.$$

Proof: Outline:

- Estimate e_{k+1}^2 in terms of e_k^2, $\operatorname{osc}(u_k, \mathcal{T}_k)^2$.
- Estimate $\operatorname{osc}(u_{k+1}, \mathcal{T}_{k+1})^2$ in terms of e_k^2, $\operatorname{osc}(u_k, \mathcal{T}_k)^2$.
- Sum and determine $0 < \alpha < 1$.

Optimality of AFEM

Theorem [Optimality of AFEM] Let $(u, \mathbf{A}, \mathbf{b}, c, f)$ satisfy PDE and for all $\epsilon > 0$

$$\exists \mathcal{T} \supset \mathcal{T}_0 \;:\; \inf_{v_\mathcal{T} \in \mathbb{V}_\mathcal{T}} \left\{ |\!|\!| u - v_\mathcal{T} |\!|\!|_\Omega + \operatorname{osc}\left(v_\mathcal{T}, \mathcal{T}\right) \right\} \le \epsilon \quad \& \quad \#\mathcal{T} - \#\mathcal{T}_0 \preccurlyeq \epsilon^{-1/s}.$$

Then

$$|\!|\!| u - u_k |\!|\!|_\Omega + \operatorname{osc}\left(u_k, \mathcal{T}_k\right) \preccurlyeq \left\{ \#\mathcal{T}_k - \#\mathcal{T}_0 \right\}^{-s}.$$

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Then

$$|||u - u_k|||_\Omega + \operatorname{osc}(u_k, \mathcal{T}_k) \preccurlyeq \left\{ \#\mathcal{T}_k - \#\mathcal{T}_0 \right\}^{-s}.$$

$$\Leftrightarrow \quad \#\mathcal{T}_k - \#\mathcal{T}_0 \preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \left\{ |||u - u_k|||_\Omega + \operatorname{osc}(u_k, \mathcal{T}_k) \right\}^{-1/s}$$

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Definition. For $\epsilon > 0$, let $\mathcal{P}_\epsilon$ be the set of all nonconforming partitions $\mathcal{T}$ of $\mathcal{T}_0$ such that

$$\mathcal{P}_\epsilon = \left\{ \mathcal{T} \supset \mathcal{T}_0 \; : \; \inf_{v \in \mathbb{V}_{\mathcal{T}}} \left\{ |\!|\!| u - v_{\mathcal{T}} |\!|\!|_{\Omega} + \operatorname{osc}(v_{\mathcal{T}}, \mathcal{T}) \right\} < \epsilon \right\},$$

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Theorem [Optimality of AFEM] Let $(u,\ \mathbf{A},\ \mathbf{b},\ c,\ f)$ satisfy PDE and for all $\epsilon > 0$

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Then

$$|\!|\!| u - u_k |\!|\!|_\Omega + \operatorname{osc}(u_k, \mathcal{T}_k) \preccurlyeq \left\{ \#\mathcal{T}_k - \#\mathcal{T}_0 \right\}^{-s}.$$

$$\Leftrightarrow \quad \#\mathcal{T}_k - \#\mathcal{T}_0 \preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \left\{ |\!|\!| u - u_k |\!|\!|_\Omega + \operatorname{osc}(u_k, \mathcal{T}_k) \right\}^{-1/s}$$

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and define

$$|u, f, \mathbf{D}|_s := \sup_{\epsilon > 0} \left\{ \epsilon \inf_{\mathcal{T} \in \mathcal{P}_\epsilon} (\#\mathcal{T} - \#\mathcal{T}_0)^s \right\}$$

Optimality of AFEM

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \Big\{ \| u - u_k \|_\Omega + \operatorname{osc}\left(u_k, \mathcal{T}_k\right) \Big\}^{-1/s}$$

Proof: Outline

Optimality of AFEM

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \Big\{ \, |\!|\!| u - u_k |\!|\!|_\Omega + \text{osc}\,(u_k, \mathcal{T}_k) \, \Big\}^{-1/s}$$

Proof: Outline

- **Theorem 0**. (Binev, Dahmen, DeVore).

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq \sum_{k=0}^{n-1} \left(\#\mathcal{M}_{k,\text{est}} + \sum_{i=0}^{n_k-1} \#\mathcal{M}^i_{k,\text{osc}} \right)$$

Optimality of AFEM

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \Big\{ \| u - u_k \|_\Omega + \operatorname{osc}(u_k, \mathcal{T}_k) \Big\}^{-1/s}$$

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- **Theorem 1**. (Upper bound of DOFs added by η_k)

$$\#\mathcal{M}_{k,\mathrm{est}} \preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \Big\{ \| u - u_k \|_\Omega + \operatorname{osc}(u_k, \mathcal{T}_k) \Big\}^{-1/s}$$

Optimality of AFEM

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \left\{ \||u - u_k\||_\Omega + \operatorname{osc}(u_k, \mathcal{T}_k) \right\}^{-1/s}$$

Proof: Outline

- **Theorem 0**. (Binev, Dahmen, DeVore).

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq \sum_{k=0}^{n-1} \left(\#\mathcal{M}_{k,\mathrm{est}} + \sum_{i=0}^{n_k - 1} \#\mathcal{M}^i_{k,\mathrm{osc}} \right)$$

- **Theorem 1**. (Upper bound of DOFs added by η_k)

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- **Theorem 2**. (Upper bound of DOFs added by $\operatorname{osc}(u_k, \mathcal{T}_k)$)

$$\#\mathcal{M}^i_{k,\mathrm{osc}} \preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \operatorname{osc}(u_k, \mathcal{T}^i_k)^{-1/s}$$

Optimality of AFEM

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq |u,f,\mathbf{D}|_s^{1/s} \Big\{ \, \| u - u_k \|_\Omega + \operatorname{osc}(u_k, \mathcal{T}_k) \, \Big\}^{-1/s}$$

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- **Theorem 0**. (Binev, Dahmen, DeVore).

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- **Theorem 1**. (Upper bound of DOFs added by η_k)

$$\#\mathcal{M}_{k,\mathrm{est}} \preccurlyeq |u,f,\mathbf{D}|_s^{1/s} \Big\{ \, \| u - u_k \|_\Omega + \operatorname{osc}(u_k, \mathcal{T}_k) \, \Big\}^{-1/s}$$

- **Theorem 2**. (Upper bound of DOFs added by $\operatorname{osc}(u_k, \mathcal{T}_k)$)

$$\#\mathcal{M}_{k,\mathrm{osc}}^i \preccurlyeq |u,f,\mathbf{D}|_s^{1/s} \operatorname{osc}(u_k, \mathcal{T}_k^i)^{-1/s}$$

- **Theorem**[Optimality].

Thms 0,1,2 + Geometric Convergence $\Rightarrow$ **Optimality**

Theorem 1 (Upper bound for DOFs added by η_k)

Lemma (Optimal marking for η_k)

Total error reduction $\mathcal{T}_k \to \mathcal{T}_{k,*}$ $\Rightarrow$ A Dörfler's property for $(\eta_k, \mathcal{M}_{k,*})$:

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- Link between nonlinear approximation theory and AFEM.

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- Link between nonlinear approximation theory and AFEM.
  - If best approximation satisfies a total error reduction, then it also satisfies a Dörfler's property for $(\eta_k, \mathcal{M}_{k,*})$.
  - The solution of AFEM safisfies an optimal Dörfler's property for $(\eta_k, \mathcal{M}_{k,\mathrm{est}})$.

$$\Rightarrow \qquad \#\mathcal{M}_{k,\mathrm{est}} \leq \#\mathcal{M}_{k,*}.$$

Theorem 1 - Optimal marking for η_k

Lemma (Optimal marking for η_k) Let $\mathcal{T}_{k,*} \supset \mathcal{T}_k$ be such that

$$|\!|\!| u - u_{k,*} |\!|\!|_\Omega^2 + \operatorname{osc}\left(u_{k,*}, \mathcal{T}_{k,*}\right)^2 \le \lambda_{\mathrm{est}} \left\{ |\!|\!| u - u_k |\!|\!|_\Omega^2 + \operatorname{osc}\left(u_k, \mathcal{T}_k\right)^2 \right\}$$

for $\lambda_{\mathrm{est}} = \lambda_{\mathrm{est}}(\theta_{\mathrm{est}}) < \frac{1}{2}$, then

$$\eta_k(\overline{\mathcal{M}}_{k,*})^2 \ge \theta_{\mathrm{est}} \eta_k^2.$$

Proof: Localized upper bound

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$$\Rightarrow \qquad \eta_k(\overline{\mathcal{M}}_{k,*})^2 \ge \frac{(1 - 2\lambda_{\mathrm{est}})C_2}{1 + (1 + 2\operatorname{osc}(\mathbf{D}, \mathcal{T}_0)^2)C_1} \eta_k^2 \ge \theta_{\mathrm{est}} \eta_k^2$$

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$$\theta_{\mathrm{est}} \in (0, \theta_{\mathrm{est}}^*), \qquad \theta_{\mathrm{est}}^* := \frac{C_2}{1 + C_1(1 + 2\operatorname{osc}(\mathbf{D}, \mathcal{T}_0)^2)}.$$

Theorem 1 - Proof I

$$\#\mathcal{T}_{k,\mathrm{est}} - \#\mathcal{T}_k \preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \left\{ \|u - u_k\|_\Omega + \operatorname{osc}\left(u_k, \mathcal{T}_k\right) \right\}^{-1/s}$$

Proof:

Theorem 1 - Proof I

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Proof:

$$\epsilon^2 = \lambda_{\mathrm{est}}\Lambda^{-1}\left(|||u-u_k|||_\Omega^2 + \mathrm{osc}\left(u_k,\mathcal{T}_k\right)\right)^2$$

$$\mathcal{T}_\epsilon \supset \mathcal{T}_0 \quad : \quad |||u-u_\epsilon|||_\Omega^2 + \mathrm{osc}\left(u_\epsilon,\mathcal{T}_\epsilon\right)^2 < \epsilon^2$$

$$\#\mathcal{T}_\epsilon - \#\mathcal{T}_0 \preccurlyeq \epsilon^{-1/s}|u,f,\mathbf{D}|_s^{1/s}$$

$$\mathcal{T}_{k,*} = \mathcal{T}_\epsilon \cup \mathcal{T}_k$$

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$$\mathcal{T}_{k,*} = \mathcal{T}_\epsilon \cup \mathcal{T}_k \qquad \Rightarrow \qquad \#\mathcal{T}_{k,*} - \#\mathcal{T}_k \le \#\mathcal{T}_\epsilon - \#\mathcal{T}_0$$

- $\mathcal{T}_k$, $\mathcal{T}_\epsilon$ are refinements of $\mathcal{T}_0$.
- $\mathcal{T}_{k,*}$ is generated from $\mathcal{T}_k$ adding some elements of $\mathcal{T}_\epsilon$, anyway less that the number of triangles that must be added to go from $\mathcal{T}_0$ to $\mathcal{T}_\epsilon$.

Theorem 1 - Proof II

- There is a total error reduction between $\mathcal{T}_{k,*}$ and $\mathcal{T}_k$.
- From **Lemma**(Optimal marking for η_k) we have

$$\#\mathcal{M}_{k,\mathrm{est}} \leq \#\overline{\mathcal{M}}_{k,*} \leq C\mathcal{M}_{k,*}$$

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$$\begin{aligned}\Rightarrow \#\mathcal{M}_{k,\text{est}} &\preccurlyeq \#\mathcal{M}_{k,*} \\ &\preccurlyeq \#\mathcal{T}_{k,*} - \#\mathcal{T}_k \preccurlyeq \#\mathcal{T}_\epsilon - \#\mathcal{T}_0 \\ &\preccurlyeq \lambda_{\text{est}}^{-1/s} |u, f, \mathbf{D}|_s^{1/s} \Big\{ \| u - u_k \|_\Omega + \operatorname{osc}(u_k, \mathcal{T}_k) \Big\}^{-1/s}\end{aligned}$$

- λ_{est} gives the discrepance between θ_{est} and θ^*_{est}.

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Theorem 2 (Upper bound for DOFs added by osc)

Lemma (Optimal marking for $\operatorname{osc}(u_k, \mathcal{T}_k)$)

"Oscillation reduction" $\Rightarrow$ A Dörfler's property for $(\operatorname{osc}(u_k, \mathcal{T}_k), \mathcal{M}_{k,*})$

Theorem 2 (Upper bound for DOFs added by osc)

Lemma (Optimal marking for $\operatorname{osc}(u_k, \mathcal{T}_k)$) Let $\mathcal{T}_0$ satisfy

$$\operatorname{osc}(\mathbf{D}, \mathcal{T}_0) \leq \frac{\delta}{24 C_1.}$$

If $v \in \mathbb{V}_{\mathcal{T}}$ and $\mathcal{T}_{k,*} \supset \mathcal{T} \supset \mathcal{T}_k$ are such that

$$\operatorname{osc}(v, \mathcal{T}_{k,*}) \leq \lambda_{\text{osc}} \operatorname{osc}(v, \mathcal{T}_k) \qquad |\!|\!| u - v |\!|\!|_{\Omega} \leq C_1^{1/2} \eta_k \qquad \delta^{1/2} \eta_k \leq \operatorname{osc}(u_k, \mathcal{T}_k)$$

with $\lambda_{\text{osc}} > 0$ suitably small, then

$$\operatorname{osc}(u_k, \mathcal{M}_{k,*})^2 \geq \theta_{\text{osc}} \operatorname{osc}(u_k, \mathcal{T}_k)^2$$

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Proof: Use relation of oscillation between different meshes.

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Proof: Use relation of oscillation between different meshes.

$$\theta_{\text{osc}} \in (0, \theta^*_{\text{osc}}), \qquad \theta^*_{\text{osc}} = \frac{1 - 24\ C_1\ \delta^{-1} \operatorname{osc}(\mathbf{D}, \mathcal{T}_0)^2}{4}$$

Theorem 2 - Proof

$$\#\mathcal{T}^i_{k,\mathrm{osc}} - \#\mathcal{T}^i_k \preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \operatorname{osc}\left(u_k, \mathcal{T}^i_k\right)^{-1/s}$$

Proof:

$$\epsilon_i = \lambda_{\mathrm{osc}} \Lambda^{-1} \operatorname{osc}(u_k, \mathcal{T}_k)$$

$$\mathcal{T}_{\epsilon_i} \supset \mathcal{T}_0 \quad : \quad |||u - u_{\epsilon_i}|||^2_\Omega + \operatorname{osc}(u_{\epsilon_i}, \mathcal{T}_{\epsilon_i}) \leq \epsilon_i$$
$$\#\mathcal{T}_{\epsilon_i} - \#\mathcal{T}_0 \preccurlyeq \epsilon_i^{-1/s} |u, f, \mathbf{D}|_s^{1/s}$$

$$\mathcal{T}^i_{k,*} = \mathcal{T}_{\epsilon_i} \cup \mathcal{T}^i_k \qquad \Rightarrow \qquad \#\mathcal{T}^i_{k,*} - \#\mathcal{T}^i_k \leq \#\mathcal{T}_{\epsilon_i} - \#\mathcal{T}_0$$

- Take $u_{\epsilon_i} = v$ in **Lemma** (Optimal marking for $\operatorname{osc}(u_k, \mathcal{T}_k)$), verify the hypothesis and conclude as in **Theorem 1**.

Optimality of AFEM - Proof I

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq \sum_{k=0}^{n-1} \left(\#\mathcal{M}_{k,\mathrm{est}} + \sum_{i=0}^{n_k-1} \#\mathcal{M}^i_{k,\mathrm{osc}} \right)$$

Optimality of AFEM - Proof I

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq \sum_{k=0}^{n-1} \left(\#\mathcal{M}_{k,\mathrm{est}} + \sum_{i=0}^{n_k-1} \#\mathcal{M}^i_{k,\mathrm{osc}} \right)$$

$$\sum_{i=0}^{n_k-1} \#\mathcal{M}^i_{k,\mathrm{osc}} \preccurlyeq \sum_{i=0}^{n_k-1} \mathrm{osc}\left(u_k, \mathcal{T}^i_k\right)^{-1/s} |u, f, \mathbf{D}|_s^{1/s}$$

Optimality of AFEM - Proof I

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$$\preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \operatorname{osc}\left(u_k, \mathcal{T}^{n_k-1}_k\right)^{-1/s} \sum_{i=0}^{n_k-1} \rho^{i/s}$$

$$\operatorname{osc}\left(u_k, \mathcal{T}^{n_k-1}_k\right) \le \rho^{n_k-1-i} \operatorname{osc}\left(u_k, \mathcal{T}^i_k\right) \qquad 0 \le i \le n_k - 1$$

Optimality of AFEM - Proof I

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq \sum_{k=0}^{n-1} \left(\#\mathcal{M}_{k,\mathrm{est}} + \sum_{i=0}^{n_k-1} \#\mathcal{M}^i_{k,\mathrm{osc}} \right)$$

$$\sum_{i=0}^{n_k-1} \#\mathcal{M}^i_{k,\mathrm{osc}} \preccurlyeq \sum_{i=0}^{n_k-1} \mathrm{osc}\left(u_k, \mathcal{T}^i_k\right)^{-1/s} |u, f, \mathbf{D}|_s^{1/s}$$

$$\preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \, \mathrm{osc}\left(u_k, \mathcal{T}_k^{n_k-1}\right)^{-1/s} \sum_{i=0}^{n_k-1} \rho^{i/s}$$

$$\mathrm{osc}\left(u_k, \mathcal{T}_k^{n_k-1}\right) \geq \delta^{1/2}\eta_k \geq \begin{cases} \dfrac{\delta^{1/2}}{C_1^{1/2}} \, \|u - u_k\|_\Omega \\ \delta^{1/2} \, \mathrm{osc}\left(u_k, \mathcal{T}_k\right) \end{cases}$$

Optimality of AFEM - Proof I

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq \sum_{k=0}^{n-1} \left(\#\mathcal{M}_{k,\mathrm{est}} + \sum_{i=0}^{n_k-1} \#\mathcal{M}^i_{k,\mathrm{osc}} \right)$$

$$\begin{aligned}
\sum_{i=0}^{n_k-1} \#\mathcal{M}^i_{k,\mathrm{osc}} &\preccurlyeq \sum_{i=0}^{n_k-1} \mathrm{osc}\left(u_k, \mathcal{T}^i_k\right)^{-1/s} |u, f, \mathbf{D}|_s^{1/s} \\
&\preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \, \mathrm{osc}\left(u_k, \mathcal{T}^{n_k-1}_k\right)^{-1/s} \sum_{i=0}^{n_k-1} \rho^{i/s} \\
&\preccurlyeq \left(|\!|\!| u - u_k |\!|\!|_\Omega + \mathrm{osc}\left(u_k, \mathcal{T}_k\right)\right)^{-1/s} |u, f, \mathbf{D}|_s^{1/s}
\end{aligned}$$

Optimality of AFEM - Proof I

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq \sum_{k=0}^{n-1} \left(\#\mathcal{M}_{k,\text{est}} + \sum_{i=0}^{n_k-1} \#\mathcal{M}^i_{k,\text{osc}} \right)$$

$$\begin{aligned}
\sum_{i=0}^{n_k-1} \#\mathcal{M}^i_{k,\text{osc}} &\preccurlyeq \sum_{i=0}^{n_k-1} \text{osc}\left(u_k, \mathcal{T}_k^i\right)^{-1/s} |u, f, \mathbf{D}|_s^{1/s} \\
&\preccurlyeq |u, f, \mathbf{D}|_s^{1/s} \, \text{osc}\left(u_k, \mathcal{T}_k^{n_k-1}\right)^{-1/s} \sum_{i=0}^{n_k-1} \rho^{i/s} \\
&\preccurlyeq \left(|\!|\!| u - u_k |\!|\!|_\Omega + \text{osc}\left(u_k, \mathcal{T}_k\right)\right)^{-1/s} |u, f, \mathbf{D}|_s^{1/s}
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Optimality of AFEM - Proof II

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$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq \sum_{k=0}^{n-1} \left(|\!|\!| u - u_k |\!|\!|_\Omega^2 + \operatorname{osc}\left(u_k, \mathcal{T}_k\right)^2 \right)^{-1/2s} |u, f, \mathbf{D}|_s^{1/s}$$

Optimality of AFEM - Proof II

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq \sum_{k=0}^{n-1} \left(\#\mathcal{M}_{k,\mathrm{est}} + \sum_{i=0}^{n_k-1} \#\mathcal{M}^i_{k,\mathrm{osc}} \right)$$

$$\begin{aligned}\#\mathcal{T}_n - \#\mathcal{T}_0 &\preccurlyeq \sum_{k=0}^{n-1} \left(\| u - u_k \|_\Omega^2 + \mathrm{osc}\,(u_k, \mathcal{T}_k)^2 \right)^{-1/2s} |u, f, \mathbf{D}|_s^{1/s} \\ &\leq |u, f, \mathbf{D}|_s^{1/s} \left(\| u - u_n \|_\Omega^2 + \mathrm{osc}\,(u_n, \mathcal{T}_n)^2 \right)^{-1/2s} \sum_{k=1}^{n} \alpha^{k/s}\end{aligned}$$

$$\| u - u_n \|_\Omega^2 + \mathrm{osc}\,(u_n, \mathcal{T}_n)^2 \leq \alpha^{2(n-k)} \left(\| u - u_k \|_\Omega^2 + \mathrm{osc}\,(u_k, \mathcal{T}_k)^2 \right), \quad 0 \leq k \leq n-1$$

Optimality of AFEM - Proof II

$$\#\mathcal{T}_n - \#\mathcal{T}_0 \preccurlyeq \sum_{k=0}^{n-1} \left(\#\mathcal{M}_{k,\mathrm{est}} + \sum_{i=0}^{n_k - 1} \#\mathcal{M}^i_{k,\mathrm{osc}} \right)$$

$$\begin{aligned}
\#\mathcal{T}_n - \#\mathcal{T}_0 &\preccurlyeq \sum_{k=0}^{n-1} \left(|\!|\!| u - u_k |\!|\!|_\Omega^2 + \operatorname{osc}\left(u_k, \mathcal{T}_k\right)^2 \right)^{-1/2s} |u, f, \mathbf{D}|_s^{1/s} \\
&\leq |u, f, \mathbf{D}|_s^{1/s} \left(|\!|\!| u - u_n |\!|\!|_\Omega^2 + \operatorname{osc}\left(u_n, \mathcal{T}_n\right)^2 \right)^{-1/2s} \sum_{k=1}^{n} \alpha^{k/s} \\
&\leq \left(|\!|\!| u - u_n |\!|\!|_\Omega^2 + \operatorname{osc}\left(u_n, \mathcal{T}_n\right)^2 \right)^{-1/2s} |u, f, \mathbf{D}|_s^{1/s}
\end{aligned}$$

Conclusions and Open problems

- We modify the MNS algorithm with a new procedure for oscillation reduction relative to error estimator, and prove its convergence and optimality.
- Keys for Convergence: orthogonality, discrete lower bound.
- Keys for Optimality: localized upper bound, optimal marking Lemma (error/oscillation reduction implies a Dörfler's property).
- Characterization of the approximation class

$$\mathcal{A}_s := \{(u, f, \mathbf{D}) : |u, f, \mathbf{D}|_s < \infty\}.$$

- Extension to the Laplace-Beltrami operator and AFEMs in $H(\mathrm{div})$ and $H(\mathrm{curl})$.
- Extension to a posteriori error estimators which do not dominate the oscillation (local problems).