Discontinuous Galerkin Approximations of the Maxwell Eigenproblem

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Motivation

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- DG approximations of problems with associated compact inverse operators: difficulties due to the use of non-conforming approximation spaces
- DG approximations of problems with associated non compact inverse operators: difficulties due to the use of non-conforming approximation spaces *plus* lack of compactness

Model Problem

The Maxwell Eigenproblem

Find $\mathbf{u} \in H(\mathrm{curl};\Omega)$, $\mathbf{u} \neq \mathbf{0}$, and $k \in \mathbb{C}$ s.t.

$$\begin{aligned} \nabla \times \nabla \times \mathbf{u} &= k^2 \mathbf{u} && \text{in } \Omega \subset \mathbb{R}^3 \\ \mathbf{n} \times \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega \end{aligned}$$

We assume, for simplicity, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$ and "topologically trivial" domain; $k = \omega\sqrt{\varepsilon_0 \mu_0}$

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We assume, for simplicity, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$ and "topologically trivial" domain; $k = \omega\sqrt{\varepsilon_0\mu_0}$

- $\nabla H_0^1(\Omega)$ is an infinite dimensional eigenspace associated with the essential spectrum $\sigma_{ess} = \{0\}$

Model Problem

Functional space and norm

Set $\mathbf{V} = H_0(\mathrm{curl};\Omega)$ with $|\mathbf{v}|_{\mathbf{V}} = \|\nabla \times \mathbf{v}\|_{0,\Omega}$ and

$$\|\mathbf{v}\|_{\mathbf{V}}^2 = \|\mathbf{v}\|_{0,\Omega}^2 + |\mathbf{v}|_{\mathbf{V}}^2$$

Variational Formulation

Find $(\mathbf{0} \neq \mathbf{u}, k) \in \mathbf{V} \times \mathbb{C}$, s.t.

$$a(\mathbf{u}, \mathbf{v}) := (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) = k^2(\mathbf{u}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}$$

Spectral Properties

- $k^2 = 0$ is an eigenvalue with *infinite dimensional* associated eigenspace
- $k^2 = 0$ is an isolated eigenvalue and all the other eigenvalues are real and strictly positive and form a sequence accumulating only at $+\infty$
- all the eigenspaces associated with eigenvalues $\neq 0$ are finite dimensional
- eigenfunctions associated with different eigenvalues are L^2–orthogonal and $\mathbf{V}$–orthogonal

Spectrally Correct Approximation

Following [Descloux-Nassif-Rappaz, 1978]:

i) isolation of the discrete essential spectrum, i.e., all the discrete eigenvalues approaching $\sigma_{ess} = \{0\}$ are separated from the other ones

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iii) completeness of the spectrum, i.e., all continuous eigenvalues smaller than an arbitrarily large fixed value are approximated when the mesh is sufficiently fine

iv) non-pollution and completeness of the eigenspaces, i.e., there are no spurious eigenfunctions and the eigenspace approximations associated with eigenvalues which are not approaching $\sigma_{ess} = \{0\}$ have the right dimension

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Lack of spectral correctness: we expect *spurious* solutions for the associated parabolic or hyperbolic evolution problems

Some References

Conforming Discretizations

- Boffi, 2000
- Caorsi-Fernandes-Raffetto, 2000
- Demkovicz-Monk, 2001
- Buffa, 2005

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DG for Maxwell in Frequency-Domain

- Houston, Monk, Perugia, Schneebeli, Schötzau, 2002-05
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DG for the Maxwell Eigenproblem

- Hesthaven-Warburton, 2004
- Warburton-Embree, 2005
- Creuzé-Nicaise, 2006

Features of DG Methods

Non conforming methods based on completely discontinuous polynomial approximation spaces

- Flexibility in the mesh design
  - non-matching grids (*hanging nodes*)
  - non-uniform approximation degrees
- Freedom in the choice of basis functions
  - simpler than Nédélec's elements, especially for high orders
- Capability to reproduce discontinuities of solutions (e.g., due to coefficients, transport terms...)
- Drawback: high number of degrees of freedom

DG Discretizations

- $\mathcal{T}_h$ shape-regular tetrahedral mesh, $\mathcal{F}_h$ set of all faces
- $\mathbf{V}_h := \{\mathbf{v} \in L^2(\Omega)^3 : \mathbf{v}|_K \in \mathcal{P}^\ell(K)^3 \ \ \forall K \in \mathcal{T}_h\}$

$$\mathbf{V}_h \not\subset \mathbf{V}$$

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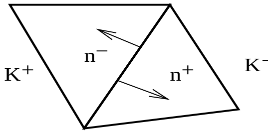
- Averages and jumps:



  - $\{\!\{\mathbf{v}\}\!\} := (\mathbf{v}^+ + \mathbf{v}^-)/2$
  - $[\![\mathbf{v}]\!]_T := \mathbf{n}^+ \times \mathbf{v}^+ + \mathbf{n}^- \times \mathbf{v}^-$

DG Discretizations

- $|\mathbf{v}|^2_{\mathbf{V}(h)} = \sum_{K \in \mathcal{T}_h} \|\nabla \times \mathbf{v}\|^2_{0,K} + \sum_{f \in \mathcal{F}_h} \mathtt{h}^{-1} \|[\![\mathbf{v}]\!]_T\|^2_{0,f}$

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- $a_h(\cdot,\cdot)$ bilinear from obtained by discretizing $a(\cdot,\cdot)$ (curl-curl operator) by any (either *symmetric* or *unsymmetric*) DG method

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- $a_h(\cdot,\cdot)$ bilinear from obtained by discretizing $a(\cdot,\cdot)$ (curl-curl operator) by any (either *symmetric* or *unsymmetric*) DG method

DG Method

Find $(0 \neq \mathbf{u}_h, k_h) \in \mathbf{V}_h \times \mathbb{C}$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = k_h^2(\mathbf{u}_h, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h$$

Example: Interior Penalty Family

$$\nabla \times \nabla \times \mathbf{u} = k^2 \mathbf{u} \quad \text{in } \Omega \qquad \mathbf{u} \times \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega$$

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Integration by parts (element-by-element)

$$\int_K \nabla \times \nabla \times \mathbf{u} \cdot \overline{\mathbf{v}} = \int_K \nabla \times \mathbf{u} \cdot \nabla \times \overline{\mathbf{v}} + \int_{\partial K} \mathbf{n}_K \times (\nabla \times \mathbf{u}) \cdot \overline{\mathbf{v}}$$

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Key formula

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \mathbf{n}_K \times (\nabla \times \mathbf{u}) \cdot \overline{\mathbf{v}}$$

$$= - \sum_{f \in \mathcal{F}_h} \int_f [\![\overline{\mathbf{v}}]\!]_T \cdot \{\!\{\nabla_h \times \mathbf{u}\}\!\} + \sum_{f \in \mathcal{F}_h^{\mathcal{I}}} \{\!\{\overline{\mathbf{v}}\}\!\} \cdot [\![\nabla_h \times \mathbf{u}]\!]_T$$

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$\mathbf{u}$ analytical solution $\Rightarrow$ $[\![\nabla_h \times \mathbf{u}]\!]_T = 0$

Example: Interior Penalty Family

IP Bilinear Forms

$$a_h(\mathbf{u}, \mathbf{v}) := \sum_{K \in \mathcal{T}_h} \int_K \nabla \times \mathbf{u} \cdot \nabla \times \overline{\mathbf{v}} - \sum_{f \in \mathcal{F}_h} \int_f [\![\overline{\mathbf{v}}]\!]_T \cdot \{\!\{\nabla_h \times \mathbf{u}\}\!\}$$

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$$-\, k\sum_{f\in\mathcal{F}_h}\int_f [\![\mathbf{u}]\!]_T\cdot\{\!\{\nabla_h\times\overline{\mathbf{v}}\}\!\} + \sum_{f\in\mathcal{F}_h}\int_f \alpha\,\mathrm{h}^{-1}\,[\![\mathbf{u}]\!]_T\cdot[\![\overline{\mathbf{v}}]\!]_T$$

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α stability parameter independent of the mesh size

$k = 1$ SIP (Douglas, Wheeler, Arnold)
$k = -1$ NIP (Baumann-Oden, Rivière-Wheeler-Girault)
$k = 0$ IIP (Dawson-Sun-Wheeler)

A Second Example: LDG

$$\nabla \times \nabla \times \mathbf{u} = k^2 \mathbf{u} \quad \text{in } \Omega \qquad \mathbf{u} \times \mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega$$

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- Mixed form: $\mathbf{s} = \nabla \times \mathbf{u} \qquad \nabla \times \mathbf{s} = k^2 \mathbf{u}$

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- DG spaces: $\mathbf{\Sigma}_h = \mathbf{V}_h = \mathcal{P}_\ell(\mathcal{T}_h)^3$
- LDG method:

$$\int_K \mathbf{s}_h \cdot \mathbf{t} = \int_K \mathbf{u}_h \cdot \nabla \times \mathbf{t} - \int_{\partial K} \widehat{\mathbf{u}}_h \cdot \mathbf{n}_K \times \mathbf{t}$$

$$\int_K \mathbf{s}_h \cdot \nabla \times \mathbf{v} - \int_{\partial K} \widehat{\mathbf{s}}_h \cdot \mathbf{n}_K \times \mathbf{v} = k^2 \int_K \mathbf{u}_h \cdot \mathbf{v} \, dx$$

$\widehat{\mathbf{u}}_h$ and $\widehat{\mathbf{s}}_h$: numerical fluxes

A Second Example: LDG

Numerical fluxes

$$\begin{cases} \widehat{\mathbf{s}} = \{\!\{\mathbf{s}\}\!\} - \alpha\, \mathrm{h}^{-1} [\![\mathbf{u}]\!]_T + \mathrm{b}[\![\mathbf{s}]\!]_T \\ \widehat{\mathbf{u}} = \{\!\{\mathbf{u}\}\!\} + \mathrm{b}[\![\mathbf{u}]\!]_T \end{cases}$$

α stability parameter; b independent of h

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Elimination of the auxiliary variable $\mathbf{s}_h$:

$$\mathbf{s}_h = \nabla_h \times \mathbf{u}_h - \mathcal{L}([\![\mathbf{u}_h]\!]_T)$$

($\mathcal{L}$ lifts functions on faces into functions in $\mathbf{\Sigma}_h$)

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LDG bilinear form

$$a_h(\mathbf{u},\mathbf{v}) := \int_\Omega \left[\nabla_h \times \mathbf{u} - \mathcal{L}([\![\mathbf{u}]\!]_T)\right] \cdot \left[\nabla_h \times \mathbf{v} - \mathcal{L}([\![\mathbf{v}]\!]_T)\right] + \int_{\mathcal{F}_h} \alpha\, \mathrm{h}^{-1} [\![\mathbf{u}]\!]_T \cdot [\![\mathbf{v}]\!]_T$$

Some Results on DG

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Positive Definite Source Problem

Find $\mathbf{u}_s \in \mathbf{V}$ s.t.

$$b(\mathbf{u}_s, \mathbf{v}) := (\nabla \times \mathbf{u}_s, \nabla \times \mathbf{v}) + (\mathbf{u}_s, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}$$

$\mathbf{u}_s \in H^r(\mathrm{curl}; \Omega)$, $r > 1/2$
[Amrouche-Bernardi-Dauge-Giralut, 1998]

Some Results on DG

DG Method for the Positive Definite Source Problem

Find $\mathbf{u}_h \in \mathbf{V}_h$ s.t.

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Some Results on DG

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- Quasi-optimality [Perugia-Schötzau, 2003]:

$$\|\mathbf{u}_s - \mathbf{u}_h\|_{\mathbf{V}(h)} \leq C h^{\min\{\ell, r\}} \|\mathbf{u}_s\|_{H^r(\mathrm{curl};\Omega)}$$

- For symmetric DG methods (adjoint consistent): optimality also in L^2–norm

DG for Compact Operators (Overview)

The Laplace Eigenproblem

Find $u \in H^1(\Omega)$, $u \neq 0$, and $\lambda \in \mathbb{C}$ s.t.

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega \subset \mathbb{R}^2 \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

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Set $V = H_0^1(\Omega)$ with $\|v\|_V = \|\nabla v\|_{L^2(\Omega)}$

Variational Formulation

Find $(0 \neq u, \lambda) \in V \times \mathbb{C}$ s.t.

$$a(u, v) := (\nabla u, \nabla v) = \lambda(u, v) \qquad \forall v \in V$$

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The Laplace Eigenproblem

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$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega \subset \mathbb{R}^2 \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Set $V = H^1_0(\Omega)$ with $\|v\|_V = \|\nabla v\|_{L^2(\Omega)}$

Variational Formulation

Find $(0 \neq u, \lambda) \in V \times \mathbb{C}$ s.t.

$$a(u,v) := (\nabla u, \nabla v) = \lambda (u,v) \qquad \forall v \in V$$

DG Method

Find $(0 \neq u_h, \lambda_h) \in V_h \times \mathbb{C}$ such that

$$a_h(u_h, v_h) = \lambda_h (u_h, v_h) \qquad \forall v_h \in V_h$$

Symmetric DG Methods, $\ell = 1$

[Antonietti-Buffa-Perugia, 2005]

$$\Omega = (0,\pi) \times (0,\pi)$$
$$\lambda^{mn} = m^2 + n^2 \qquad m, n \in \mathbb{N} \setminus \{0\}$$
$$u^{mn}(x,y) = \sin(mx)\sin(ny)$$

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Convergence rates SIP and LDG methods
(mesh of 1024 el. to mesh of 4106 el.)

ev	2	5	8	10	13	17
SIP	1.9982	1.9999	1.9998	1.9993	1.9994	1.9988
LDG	2.0000	2.0005	2.0007	2.0012	2.0010	2.0020

- On unstructured grids: the 2 sequences converging to an eigenvalue of multiplicity 2 are not identical

Unsymmetric DG Methods, $\ell = 1$

Convergence rates NIP method
(mesh of 1024 el. to mesh of 4106 el.)

ev	2	5	8	10	13	17
$\alpha = 10$	1.9755	1.9789	1.9792	1.9820	1.9816	1.9858
$\alpha = 1$	2.0347	2.0327	2.0326	2.0313	2.0315	2.0295

- On unstructured grids: complex eigenvalues

Symmetric DG Methods, $\ell > 1$

Convergence rates SIP method
(mesh of 1024 el. to mesh of 4106 el.)

ev	2	5	8	10	13	17
$\ell = 2$	3.9844	3.9815	3.9814	3.9765	3.9769	3.9696
$\ell = 3$	5.7614	5.9804	5.9868	5.9798	5.9804	5.9653

Unsymmetric DG Methods, $\ell > 1$

Convergence rates NIP method
(mesh of 1024 el. to mesh of 4106 el.)

ev	2	5	8	10	13	17
$\ell = 2$	2.0478	2.0373	2.0367	2.0190	2.0190	2.0217
$\ell = 3$	4.0397	4.0416	4.0417	4.0448	4.0445	4.0488

Summary

Optimal rates: 2ℓ

	SIP	LDG	NIP	IIP
$\ell = 1$	2	2	2	2
$\ell = 2$	4	4	2	2
$\ell = 3$	6	6	4	4
$\ell = 4$	8	8	4	4
$\ell = 5$	10	10	6	6

Computed rates for symm. methods: 2ℓ

Computed rates for unsymm. methods: ℓ, for even ℓ

$\ell + 1$, for odd ℓ

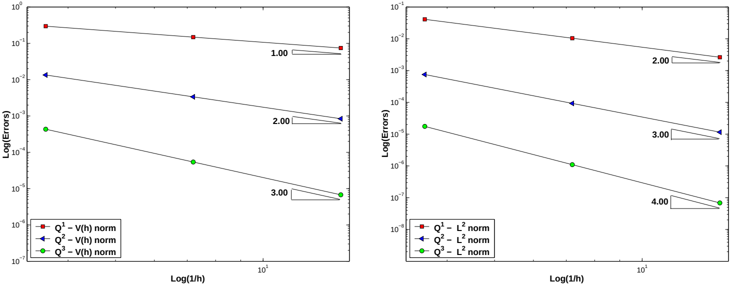
(see also [Harriman-Houston-Senior-Süli, 2003])

(see also convergence in L^2–norm)

Eigenfunctions

Optimal and computed rates: ℓ in $V(h)$–norm

NIP 1st eigenfunction: errors $(\ell = 1, 2, 3)$



(a) $V(h)$–norm.   (b) L^2–norm.

Assumptions

Poincaré Inequality

$$\|v\|_{L^2(\Omega)} \leq C\|v\|_{V(h)} \quad \forall v \in V_h + H_0^1(\Omega)$$

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Convergence for the Source Problem

Let u_s be s.t. $-\Delta u_s = f$ in Ω, $u_s = 0$ on $\partial\Omega$, with $f \in L^2(\Omega)$, and let u_h its the DG approximation; whenever $u_s \in H^{1+t}(\Omega)$, $1/2 < t \leq \ell$,

$$\|u_s - u_h\|_{V(h)} \leq C\, h^t \|u_s\|_{L^2(\Omega)}$$

Results

Spectral Correctness and Convergence Rates

- Non-pollution and completeness of spectrum and eigenspaces
- Optimal eigenfunction approximation
- Optimal eigenvalue approximation for symmetric DG methods, suboptimal eigenvalue approximation for unsymmetric DG methods

DG for Non Compact Operators

[Buffa-Perugia, to appear]

Set $\mathbf{W} := H_0(\mathrm{curl};\Omega) \cap \{\nabla H_0^1(\Omega)\}^\perp$

Standard Assumptions

- Approximation property of $\mathbf{V}_h$:

$$\lim_{h\to 0} \inf_{\mathbf{v}_h\in\mathbf{V}_h} \|\mathbf{v}-\mathbf{v}_h\|_{\mathbf{V}(h)} = 0 \qquad \forall \mathbf{v}\in\mathbf{W}$$

- Coercivity in seminorm and continuity:

$$\mathrm{Re}\,[a_h(\mathbf{v},\mathbf{v})] \geq \alpha\,|\mathbf{v}|^2_{\mathbf{V}(h)} \qquad \forall \mathbf{v}\in\mathbf{V}_h$$
$$|a_h(\mathbf{u},\mathbf{v})| \leq \gamma\|\mathbf{u}\|_{\mathbf{V}(h)}\|\mathbf{v}\|_{\mathbf{V}(h)} \qquad \forall \mathbf{u},\mathbf{v}\in\mathbf{V}_h$$

- Convergence for the *positive definite* source problem

Additional Assumptions

Discrete kernel and its $\mathbf{V}(h)$–orthogonal complement:

$$K_h = \{\mathbf{v} \in \mathbf{V}_h : a_h(\mathbf{v}, \mathbf{w}) = 0 \; \forall \mathbf{w} \in \mathbf{V}_h\}$$

$$K_h^{\perp} = \{\mathbf{v} \in \mathbf{V}_h : (\mathbf{v}, \mathbf{w})_{\mathbf{V}(h)} = 0 \; \forall \mathbf{w} \in K_h\}.$$

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Discrete Friedrichs Inequality (DFI)

$$\|\mathbf{v}\|_{0,\Omega}^2 \leq C \, \mathrm{Re} \, [a_h(\mathbf{v}, \mathbf{v})] \quad \forall \mathbf{v} \in K_h^{\perp}$$

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Gap Property (GAP)

For h small enough, for any $\mathbf{w}_h \in K_h^\perp$, $\exists$ $\mathbf{w} \in \{\nabla H_0^1(\Omega)\}^\perp$ s.t.

$$\|\mathbf{w} - \mathbf{w}_h\|_{0,\Omega} \le \eta_h \|\mathbf{w}_h\|_{\mathbf{V}(h)}$$

with $\eta_h \to 0$ as $h \to 0$

Results

Define the solution operators:

$$T : L^2(\Omega)^3 \to \mathbf{V} \qquad b(T\mathbf{f}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$
$$T_h : L^2(\Omega)^3 \to \mathbf{V}_h \qquad b_h(T_h\mathbf{f}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)$$

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(DFI) is equivalent to

Isolation of the Discrete Essential Spectrum

If $1 \neq \lambda_h \in \sigma(T_h)$, then

$$\mathrm{Re}\,[\lambda_h] \leq \beta < 1$$

Results

(GAP) implies convergence of $T_h \to T$, as $h \to 0$, in mesh-dependent norm, which implies

Non-Pollution of the Spectrum

Let $0 \neq z \in \rho(T)$; then, for h small enough,

$$\|(z - T_h)\mathbf{f}\|_{\mathbf{V}(h)} \geq C\|\mathbf{f}\|_{\mathbf{V}(h)}$$

In words: if z is in the resolvent set of T, then, for h small enough, it is also in the resolvent set of T_h (no spurious eigenvalues)

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- Completeness of the spectrum
- Non-pollution and completeness of the eigenspaces
- Eigenvalue and eigenfunction convergence rates

Convergence Rates

Eigenvalue Approximation

Let $\lambda \neq 1$ be an eigenvalue of T with multiplicity m; for h small enough, there exist m discrete eigenvalues $\lambda_{i,h}$ s.t.

$$\sup_{1\leq i\leq m} |\lambda - \lambda_{i,h}| \leq Ch^t$$

$$\sup_{1\leq i\leq m} |\lambda - \lambda_{i,h}| \leq Ch^{2t} \quad \text{for symmetric methods}$$

$t = \min\{\ell, \sigma_\lambda\}$, with σ_λ s.t. $\mathbf{v} \in H^{\sigma_\lambda}(\mathrm{curl};\Omega)$ for all $\mathbf{v} \in E_\lambda$

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Distance between closed subspaces of $\mathbf{V} + \mathbf{V}_h$:

$$\delta(Y,Z) := \sup_{y\in Y,\ \|y\|_{\mathbf{V}(h)}=1} \inf_{z\in Z} \|y - z\|_{\mathbf{V}(h)}$$

$$\widehat{\delta}(Y,Z) := \max\{\delta(Y,Z), \delta(Z,Y)\}$$

Eigenfunction Approximation

For h small enough, $\quad \widehat{\delta}(E_\lambda, E_{\{\lambda_{i,h}\}}) \le Ch^t$

Example 1

[Buffa-Houston-Perugia, 2006]

$$\Omega = (0,\pi) \times (0,\pi)$$
$$\varepsilon = I,\ \mu = I$$
$$\lambda^{mn} = m^2 + n^2, \quad m, n \in \mathbb{N}$$

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[Buffa-Houston-Perugia, 2006]

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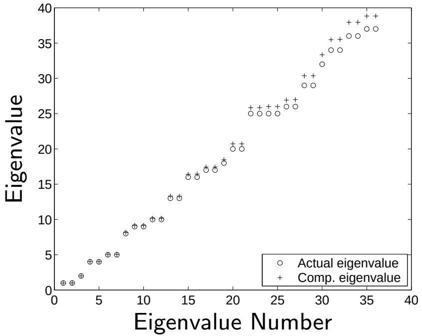
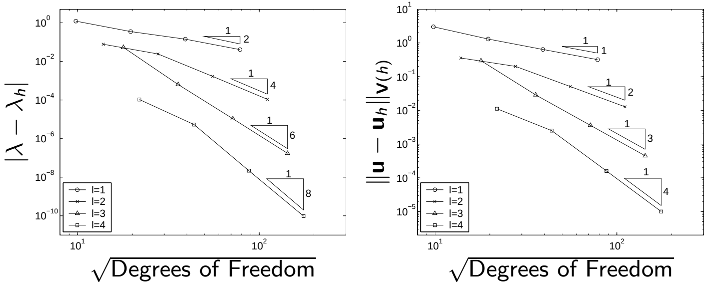
SIP, conforming triangular mesh, $\ell = 1$



Example 1

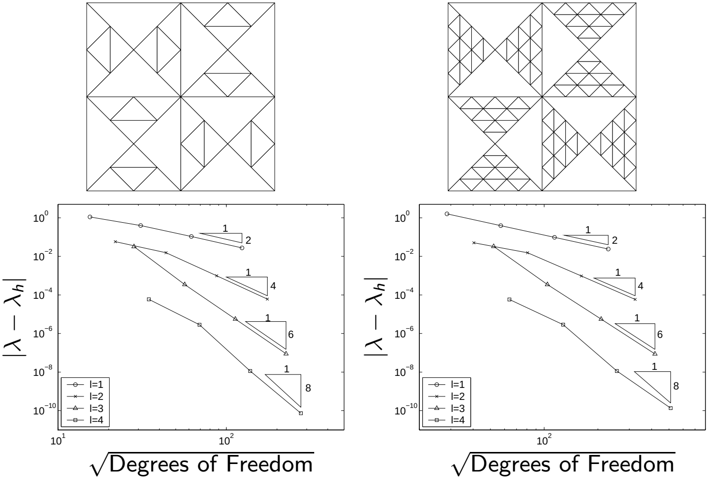
SIP, error 8th eigenv. and eigenfct. on conforming meshes



- Computed rates: 2ℓ for eigenvalues, ℓ for eigenfunctions

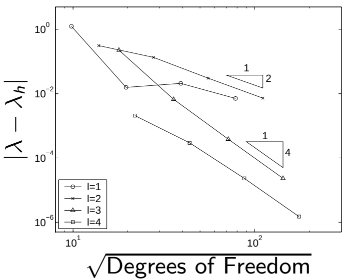
Example 1

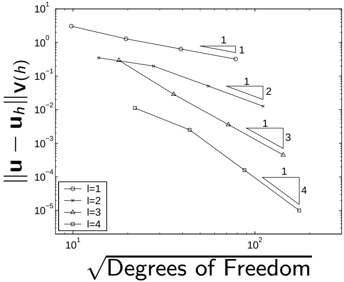
SIP, error 8th eigenv. on k–irregular meshes



Example 1

NIP, error 8th eigenv. and eigenfct. on conforming meshes

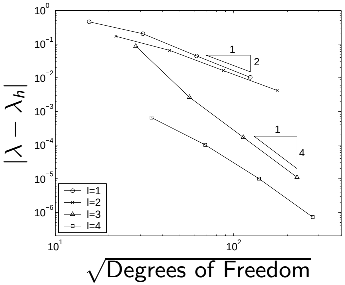


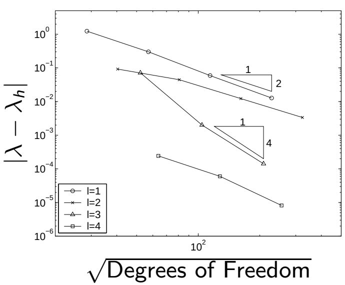


- Computed rates: ℓ if ℓ is even, $\ell + 1$ if ℓ is odd for eigenvalues
  ℓ for eigenfunctions

Example 1

NIP, error 8th eigenv. on 1–irregular and 3–irregular meshes

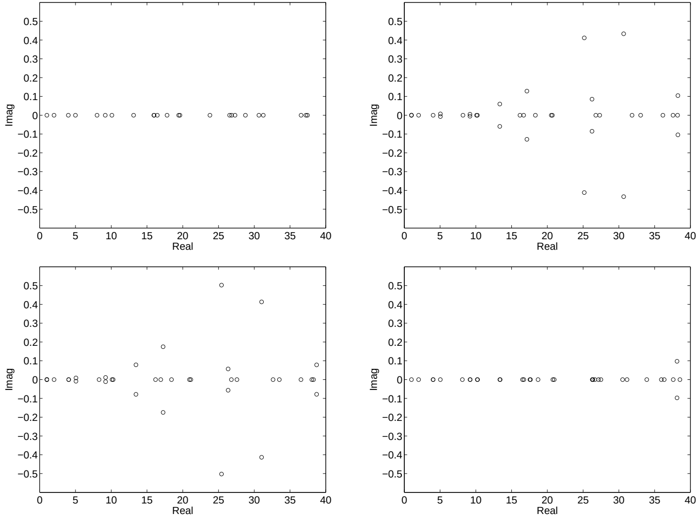




- Computed rates: same as for conforming meshes

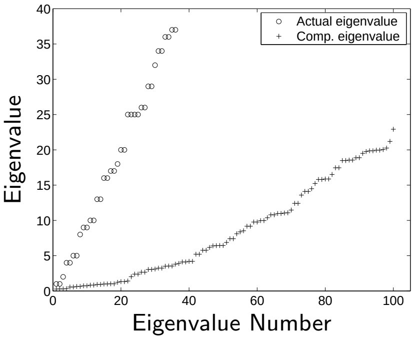
Example 1

NIP, firts 36 eigenvalues on (confroming) structured, 1–irregular, 3–irregular and unstructured meshes, resp.



Example 1

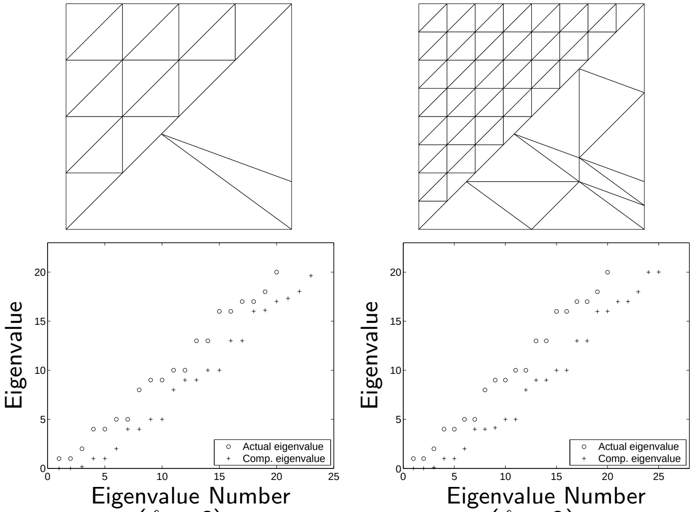
SIP, square mesh, $Q1$ elements



- Spurious modes as for the underlying H(curl)-conforming finite element approximation

Example 1

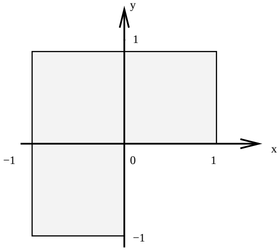
SIP, general non-conforming meshes

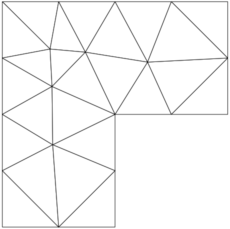


$(\ell = 6)$ $(\ell = 8)$

Example 2

$\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$
$\varepsilon = I, \ \mu = I$





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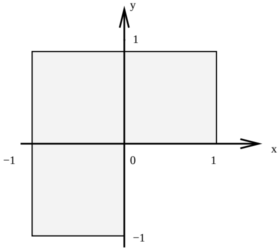
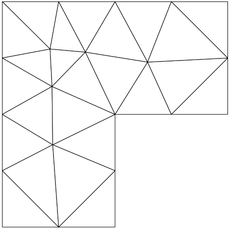


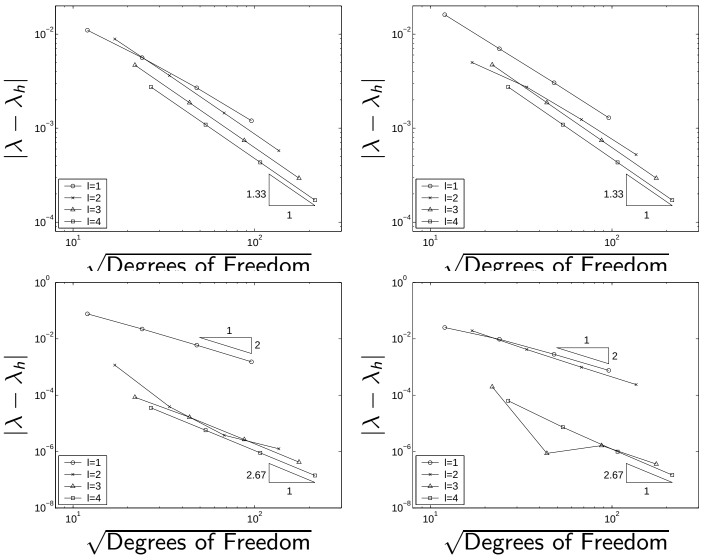


[M. Dauge's webpage]

- First 5 eigenvalues:
  1.4756, 3.5340, π^2, π^2, 11.3895
- First 5 eigenfunctions:
  strongly sing., $H^1(\Omega)^2$, analytic, analytic, strongly sing.

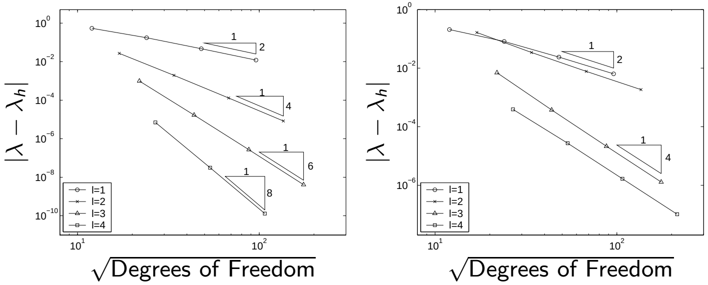
Example 2

SIP (right) and NIP (left), 1st and 2nd eigenvalues



Example 2

SIP (right) and NIP (left), 3rd eigenvalues



- 1st eigenv.: computed rate 1.33 for both SIP and NIP
- 2nd eigenv.: computed rate $\min\{2\ell, 2.67\}$ for SIP; for NIP with $\ell = 2$, inferior rate 2
- 3rd eigenv. (analytic eigenfct.): 2ℓ for SIP, ℓ (even) or $\ell + 1$ (odd) for NIP

Example 3

$$\Omega = (-1,1)^2$$
$$\varepsilon = \varepsilon_r I, \;\; \mu = I$$

$\varepsilon_r = 1$	$\varepsilon_r = 0.1$
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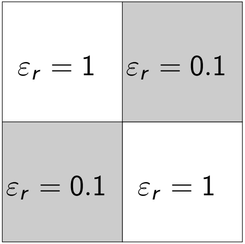
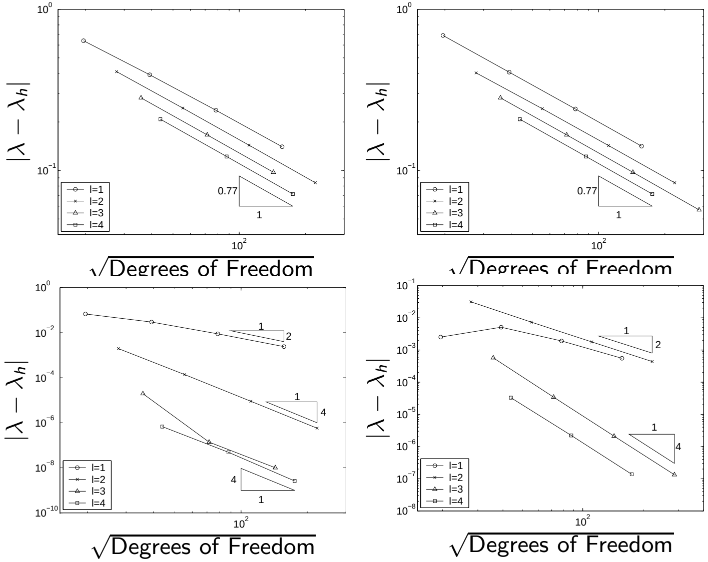


Strongest singularity: $r^{-0.6}$ as $r \to 0$ ($r =$ dist. form origin); the eigenfct. corresponding to the 2nd eigenv. contains such a singularity ([M. Dauge's webpage])

Example 3

SIP (right) and NIP (left), 2nd and 3rd eigenvalues



Remarks

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- K_h is approximating in $\nabla H_0^1(\Omega)$:

$$\lim_{h\to 0} \inf_{\mathbf{k}_h \in K_h} \|\mathbf{k}-\mathbf{k}_h\|_{\mathbf{V}(h)} = 0 \qquad \forall \mathbf{k} \in \nabla H_0^1(\Omega)$$

(approx. property of $\mathbf{V}_h$ required for the whole $\mathbf{V}$)

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- Locally divergence-free elements? [Baker-Jureidini-Karakashian, 1990]

Application: the Maxwell Source Problem

Indefinite Maxwell Problem

Find $\mathbf{u} \in H(\mathrm{curl};\Omega)$ s.t.

$$\begin{aligned} \nabla \times (\mu^{-1} \nabla \times \mathbf{u}) - k^2 \varepsilon \mathbf{u} &= \mathbf{f} \qquad \text{in } \Omega \subset \mathbb{R}^3 \\ \mathbf{n} \times \mathbf{u} &= \mathbf{0} \qquad \text{on } \partial\Omega \end{aligned}$$

- Assume k *not* a Maxwell eigenvalue

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DG for the Indefinite Maxwell Problem

Find $\mathbf{u}_h \in \mathbf{V}_h$ s.t.

$$a_h(\mathbf{u}_h, \mathbf{v}_h) - k^2 (\varepsilon \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h$$

- k *might be* a discrete eigenvalue...

Application: the Maxwell Source Problem

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$$b_h(z\mathbf{u}_h, \mathbf{v}_h) - b_h(T_h\mathbf{u}_h, \mathbf{v}_h) = b_h(zT_h\mathbf{g}_h, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h$$

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- Let $\mathbf{g}_h$ be s.t. $(\varepsilon \mathbf{g}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)$ $\forall \mathbf{v}_h \in \mathbf{V}_h$

$$b_h(\mathbf{u}_h, \mathbf{v}_h) - (1 + k^2)(\varepsilon \mathbf{u}_h, \mathbf{v}_h) = (\varepsilon \mathbf{g}_h, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h$$

$$b_h(z\mathbf{u}_h, \mathbf{v}_h) - b_h(T_h\mathbf{u}_h, \mathbf{v}_h) = b_h(zT_h\mathbf{g}_h, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h$$

$b_h(\cdot, \cdot)$ coercive $\Rightarrow (z - T_h)\mathbf{u}_h = zT_h\mathbf{g}_h$

Application: The Maxwell Source Problem

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Recall:

Non-Pollution of the Spectrum

Let $0 \neq z \in \rho(T)$; then, for h small enough,

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Then:

Well-Posedness and Convergence

- $\exists\,!$ of the solution $\mathbf{u}_h$, for h small enough
- continuous dependence on the datum $\mathbf{f}$
- well-posedness $\longrightarrow$ inf-sup condition
- inf-sup condition $\longrightarrow$ quasi-optimal error estimates

Concluding Remarks

- Asymptotic analysis of DG spectral approximations of second order operators with non-compact inverse (Maxwell, grad-div)
- Sufficient (and necessary) conditions for spectral correctness, provided that the considered DG method is well-posed and convergent for the corresponding positive definite source problem
- Optimality of eigenfunction approximation; optimality of eigenvalue approximation for symmetric methods (suboptimality for unsymmetric methods)
- Application to the indefinite Maxwell source problem with piecewise smooth coefficients
- Relations between our analysis and standard analyzes of conforming approximations