The fine-scale Green's Function

and the construction of variational multiscale methods

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INdAM Workshop on Multiscale Problems:
Modeling, Adaptive Discretization, Stabilization, Solvers
Cortona, September 18-22, 2006

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   - Typical multiscale approach
   - New approach: fine scales optimization
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   - H_0^1-optimality
   - P1 coarse scales
   - Higher-order coarse scales
   - L^2-optimality
3. The 2D advection-diffusion problem
   - The 2D setting
   - Numerical evaluation of g' (H_0^1, L^2, and nodal optimality)
   - H_0^1-optimality and SUPG

The abstract problem and framework

Given a Hilbert space V, with norm $\|\cdot\|_V$ and s.p. $(\cdot,\cdot)_V$, dual space V^*, and $\mathcal{L}: V \to V^*$, we consider the problem:

$$\begin{cases} \text{find } u \in V : \\ \mathcal{L}u = f. \end{cases}$$

We split the space V, where the exact solution is, into:

$$\bar{V} = \text{space of coarse scales,}$$
$$V' = \text{space of fine scales,}$$

and then consider:

$$\begin{cases} \text{find } \bar{u} \in \bar{V}, u' \in V' : \\ \mathcal{L}(\bar{u} + u') = f. \end{cases}$$

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Variational multiscale (VMS) formulation

The variational formulation of the problem is:

$$\text{find } \bar{u} + u' \in V : \qquad {}_{V^*}\langle \mathcal{L}(\bar{u} + u'), v \rangle_V = {}_{V^*}\langle f, v \rangle_V, \quad \forall v \in V.$$

Then, we split the problem as:

$$\begin{aligned} {}_{V^*}\langle \mathcal{L}\bar{u}, \bar{v} \rangle_V + {}_{V^*}\langle \mathcal{L}u', \bar{v} \rangle_V &= {}_{V^*}\langle f, \bar{v} \rangle_V, \qquad && \forall \bar{v} \in \bar{V}, \\ {}_{V^*}\langle \mathcal{L}\bar{u}, v' \rangle_V + {}_{V^*}\langle \mathcal{L}u', v' \rangle_V &= {}_{V^*}\langle f, v' \rangle_V, \qquad && \forall v' \in V'. \end{aligned}$$

$\rightsquigarrow u' = \mathcal{G}'(f - \mathcal{L}\bar{u}) \rightsquigarrow$

VMS formulation (for $\bar{u}$)

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- the fine-scale effect is determined by the fine scale Green's operator $\mathcal{G}' : V^* \to V'$ which gives $V' \ni u' = \mathcal{G}' r$ such that ${}_{V^*}\langle \mathcal{L}u', v'\rangle_V = {}_{V^*}\langle r, v'\rangle_V, \forall v' \in V'$,
- $\mathcal{G}'$ is not the classical Green's operator $\mathcal{G} \equiv \mathcal{L}^{-1} : V^* \to V$,
- in order to derive a VMS formulation, we need $\bar{V} \cap V' = \{0\}$, that is we need a direct sum of $\bar{V} \oplus V'$.

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Example of typical VMS methods

For 2D advection-diffusion problems, various choices for $\bar{V} \oplus V'$ have been proposed in literature:

- $P1 \oplus$ residual-free bubbles: [F. BREZZI AND A. RUSSO, '94], [T. J. R. HUGHES, '95], [F. BREZZI, L. P. FRANCA, T. J. R. HUGHES, AND A. RUSSO, '97], ...
- $P2 \oplus$ residual-free bubbles: [M. I. ASENSIO, A. RUSSO, AND G. SANGALLI, '04]
- $P1 \oplus$ (r.-f. bubbles$+ \ldots$): [F. BREZZI AND L.D. MARINI, '02][A.CANGIANI AND E. SÜLI, 05], [L. P. FRANCA, A. L. MADUREIRA AND F. VALENTIN, 05], ...

In all cases,

$$\bar{V} \oplus V' \subsetneq V \equiv \text{all-scale space.}$$

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New approach to VMS: error optimization

$$\text{Minimize } \Phi(u') \text{ subject to } \begin{cases} \bar{u} \in \bar{V}, \\ u' \in V, \\ \mathcal{L}(\bar{u} + u') = f \end{cases}$$

- we do not assume a-priori constraints on the fine scales,
- we minimize the fine scale u' (i.e., the numerical error $u - \bar{u} = u'$) w.r.t. a functional $\Phi(\cdot)$,
- then $\bar{u}$ and u' are uniquely determined, and the numerical solution $\bar{u}$ is optimal ($\Phi(u - \bar{u})$ is minimized) by design,
- here, we consider $\Phi(\cdot) = \|\cdot\|^2$ (for example, $\|\cdot\| = \|\cdot\|_{H_0^1}$ or $\|\cdot\| = \|\cdot\|_{L^2}$).
- other possibilities: $\Phi(\cdot)$ is not a quadratic form.

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$$\begin{cases} \bar{u} \in \bar{V}, \\ u' \in V, \\ \mathcal{L}(\bar{u} + u') = f \end{cases} \quad \Leftrightarrow u' \in u + \bar{V} \quad \rightsquigarrow \delta u' \in \bar{V}.$$

Then, under suitable condition on $\Phi(\cdot)$:

$$\Phi(u') = \min_{v' \in u + \bar{V}} \Phi(v') \quad \Leftrightarrow \quad D\Phi(u'; \delta u') = 0, \quad \forall \delta u' \in \bar{V}.$$

In case of $\Phi(\cdot) = \|\cdot\|^2$, then $D\Phi(u'; \delta u') = (u', \delta u') = 0$, for all $\delta u' \in \bar{V}$, that is, u' is orthogonal to $\bar{V}$, that is,

$$\mathcal{P}u' = 0,$$

where $\mathcal{P} : V \to \bar{V}$ is the orthogonal projector on $\bar{V}$.

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In case of $\Phi(\cdot) = \|\cdot\|^2$, then $D\Phi(u'; \delta u') = (u', \delta u') = 0$, for all $\delta u' \in \bar{V}$, that is, u' is orthogonal to $\bar{V}$, that is,

$$\mathcal{P} u' = 0,$$

where $\mathcal{P} : V \to \bar{V}$ is the orthogonal projector on $\bar{V}$.

$$\begin{cases} \bar{u} \in \bar{V}, \\ u' \in V, \\ \mathcal{L}(\bar{u} + u') = f \end{cases} \qquad \Leftrightarrow u' \in u + \bar{V} \qquad \rightsquigarrow \delta u' \in \bar{V}.$$

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Scales splitting + optimization

Find $\bar{u}$ and u' such that:

$$\begin{cases} \bar{u} \in \bar{V}, \\ u' \in V, \text{ with } \mathcal{P}u' = 0, \\ \mathcal{L}(\bar{u} + u') = f \end{cases}$$

- the fine scale space is implicitly defined by the optimality condition:

$$V' = \{v \in V : \mathcal{P}v = 0\};$$

- we have $\bar{V} \oplus V'$;
- when $\Phi(\cdot) = \|\cdot\|^2$,, i.e., $\mathcal{P}$ is an orthogonal projector, V' is the orthogonal complement of $\bar{V}$ (in V) and $\bar{u} = \mathcal{P}u$;
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The fine-scale problem reads: find u' such that $\mathcal{P}u' = 0$ and

$$_{V^*}\langle \mathcal{L}u', v' \rangle_V = {}_{V^*}\langle r, v' \rangle_V, \quad \forall v' \in V \text{ with } \mathcal{P}v' = 0,$$

where $r := f - \mathcal{L}\bar{u}$. It can be written in unconstrained form introducing a Lagrange multiplier: find $u' \in V$, and $\bar{\lambda} \in \bar{V}^*$ s.t.

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where $\mathcal{P}^T : \bar{V}^* \to V^*$. We want $\mathcal{G}'$ such that $u' = \mathcal{G}'r$.

Theorem

Let $\mathcal{G} \equiv \mathcal{L}^{-1}$ be the Green's operator. Then,

$$\mathcal{G}' = \mathcal{G} - \mathcal{G}\mathcal{P}^T(\mathcal{P}\mathcal{G}\mathcal{P}^T)^{-1}\mathcal{P}\mathcal{G}.$$

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where $r := f - \mathcal{L}\bar{u}$. From (1) we get

$$u' = \mathcal{G}(r - \mathcal{P}^T\bar{\lambda}); \qquad (3)$$

substituting in (2) gives $\mathcal{P}\mathcal{G}r - \mathcal{P}\mathcal{G}\mathcal{P}^T\bar{\lambda} = 0$, whence $\bar{\lambda} = (\mathcal{P}\mathcal{G}\mathcal{P}^T)^{-1}\mathcal{P}\mathcal{G}r$. Finally, using this in (3) yield

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1D advection-diffusion problem and H_0^1-optimality

$$\mathcal{L}u := -\kappa \frac{d^2}{dx^2}u + \beta \frac{d}{dx}u = f \quad \text{in } (0, L), \quad u(0) = u(L) = 0.$$

The Green's operator is represented by the Green's function:

$$u(y) = \int_0^L g(x, y) f(x) \, dx,$$

We set $V = H_0^1 \equiv H_0^1(0, L)$, $\bar{V}$ =finite elements;

$$\Phi(v) = \|v\|_{H_0^1}^2 := \int_0^L \left(\frac{d}{dx} v(x) \right)^2 \, dx,$$

$$\int_0^L \frac{d}{dx}(\mathcal{P}v - v) \frac{d}{dx} \bar{v} = 0, \quad \forall \bar{v} \in \bar{V}.$$

$$\int_0^L \mathcal{L}\bar{u}(x)\bar{v}(x)\,dx + \int_0^L \mathcal{L}u'(x)\bar{v}(x)\,dx = \int_0^L f(x)\bar{v}(x)\,dx, \quad \forall \bar{v} \in \bar{V}$$

$$\int_0^L \mathcal{L}u'(x)v'(x)\,dx = \int_0^L (f(x) - \mathcal{L}\bar{u}(x))\bar{v}(x)\,dx, \quad \forall v' \in V'.$$

$$u'(y) = \int_0^L g'(x,y)(f(x) - \mathcal{L}\bar{u}(x))\,dx = \int_0^L g'(x,y)r(x)\,dx,$$

$\rightsquigarrow$ VMS for 1D advection-diffusion equation:

$$\int_0^L \mathcal{L}\bar{u}(x)\bar{v}(x)\,dx + \int_0^L \int_0^L \mathcal{L}^*\bar{v}(y)g'(x,y)r(x)\,dxdy$$
$$= \int_0^L f(x)\bar{v}(x)\,dx, \quad \forall \bar{v} \in \bar{V}.$$

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P1 coarse scales

Consider a grid of nodes $0 = x_0 < x_1 < \ldots < x_{n_{el}-1} < x_{n_{el}} = L$, that subdivides $(0, L)$ into n_{el} elements (x_{i-1}, x_i) $(i = 1, \ldots, n_{el})$, and take piecewise affine $\bar{V} \subset H_0^1$, with $N := dim(\bar{V}) \equiv n_{el} - 1$.

Then the abstract formula gives:

$$g'(x,y) = g(x,y) - [g(x_1,y) \ldots g(x_N,y)] \begin{bmatrix} g(x_1,x_1) & \ldots & g(x_N,x_1) \\ \vdots & \ddots & \vdots \\ g(x_1,x_N) & \ldots & g(x_N,x_N) \end{bmatrix}^{-1} \begin{bmatrix} g(x,x_1) \\ \vdots \\ g(x,x_N) \end{bmatrix}$$

Structure of $g'(\cdot,\cdot)$ for 1D, linear element, H^1_0-optimality

- $g'(x,y) \neq 0$ only if x and y belong to the same element
- g' is the *element* Green's function g^{el} on each $(x_{i-1},x_i) \times (x_{i-1},x_i)$

The structure of g' for this case is known (RFB-FEM), indeed

$$V' = \bigoplus_{i=1,\dots,n_{el}} H^1_0(x_{i-1},x_i),$$

(u' is a bubble, $\bar{u}$ is nodally exact), whence

$$\mathcal{L}u' = f - \mathcal{L}\bar{u}, \text{ on } (x_{i-1},x_i), \quad u'(x_{i-1}) = u'(x_i) = 0.$$

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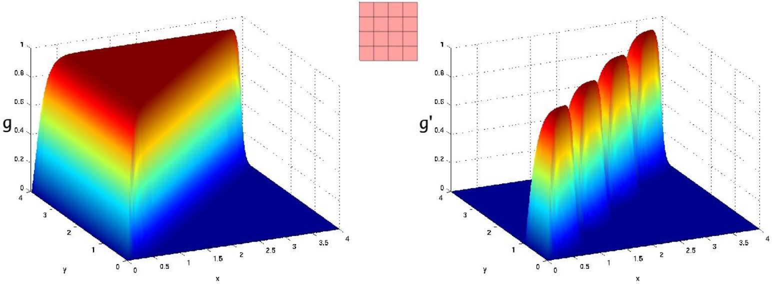
- g' is the *element* Green's function g^{el} on each $(x_{i-1},x_i)\times(x_{i-1},x_i)$

The structure of g' for this case is known (RFB-FEM), indeed

$$V' = \bigoplus_{i=1,\dots,n_{el}} H^1_0(x_{i-1},x_i),$$

(u' is a bubble, $\bar{u}$ is nodally exact), whence

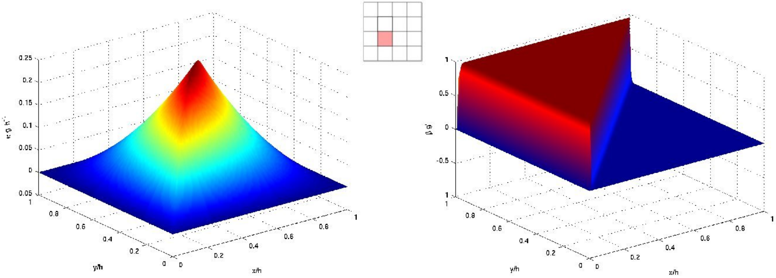
$$\mathcal{L}u' = f - \mathcal{L}\bar{u}, \text{ on } (x_{i-1},x_i), \quad u'(x_{i-1}) = u'(x_i) = 0.$$



Comparison between the Green's function g (left) and the fine scale Green's function g' (right) for linear elements, $\kappa = 10^{-1}$, $\beta = 1$, $L = 4$ and a grid of $n_{el} = 4$ uniform elements.

Effect of the fine scale on the coarse scale equation:

$$
\begin{aligned}
&\int_0^L \int_0^L \mathcal{L}^* \bar{v}(y) g'(x,y) r(x)\, dxdy \\
&= \sum_{i=1}^{n_{el}} \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} \mathcal{L}^* \bar{v}(y) g'(x,y) r(x)\, dxdy \\
&= \sum_{i=1}^{n_{el}} \frac{\int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} g'(x,y)\, dxdy}{|x_i - x_{i-1}|} \int_{x_{i-1}}^{x_i} r(x) \mathcal{L}^* \bar{v}(x)\, dx \\
&= \sum_{i=1}^{n_{el}} \tau_1 \int_{x_{i-1}}^{x_i} r(x) \mathcal{L}^* \bar{v}(x)\, dx \\
&= \sum_{i=1}^{n_{el}} \tau_1 \int_{x_{i-1}}^{x_i} \left(\beta \frac{d}{dx} \bar{u}(x) - f(x) \right) \left(\beta \frac{d}{dx} \bar{v}(x) \right) dx
\end{aligned}
$$



Fine scale Green's functions g' for linears in the diffusive ($\alpha = 10^{-2}$, left) and in the advective ($\alpha = 10^{2}$, right) regime; $\alpha := \frac{\beta h}{2\kappa}$ is the mesh Peclét number.

Higher-order coarse scales

$$\bar{V} = \left\{ \bar{v} \in H_0^1(0,L) \text{ such that } \bar{v}_{|(x_{i-1},x_i)} \in \mathbb{P}_k,\ 0 \le i \le n_{el} \right\};$$

unlike the linear case, $\bar{V}$ now contains bubbles, and then:

$$V' \underset{\neq}{\subset} \bigoplus_{i=1,\ldots,n_{el}} H_0^1(x_{i-1}, x_i).$$

Then

$$\int_{x_{i-1}}^{x_i} \mathcal{L}u'(x)v'(x)\,dx = \int_{x_{i-1}}^{x_i} (f(x) - \mathcal{L}\bar{u}(x))v'(x)\,dx, \quad \forall v' \in V'$$

$$\not\Downarrow$$

$$\mathcal{L}u' = f - \mathcal{L}\bar{u}, \text{ on } (x_{i-1}, x_i), \quad u'(x_{i-1}) = u'(x_i) = 0.$$

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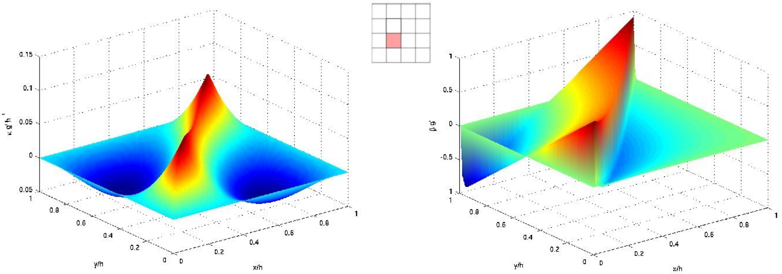
$$\mathcal{L}u' = f - \mathcal{L}\bar{u}, \text{ on } (x_{i-1}, x_i), \quad u'(x_{i-1}) = u'(x_i) = 0.$$

If $k=2$ then we obtain for $0 \leq x, y \leq h$

$$g'(x,y) = g^{el}(x,y) - \frac{\int_0^h g^{el}(s,y)ds \int_0^h g^{el}(x,t)dt}{\int_0^h \int_0^h g^{el}(s,t)dsdt} = I + II.$$

Term I is the element Green's function, and term II is:

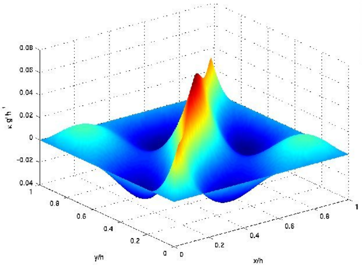
$$II = \frac{2\left(ye^{\frac{\beta h}{\kappa}} - he^{\frac{\beta y}{\kappa}} + h - y\right)\left(-x - he^{\frac{\beta h}{\kappa}} + e^{-\frac{\beta(-h+x)}{\kappa}}h + e^{\frac{\beta h}{\kappa}}x\right)}{h\left(e^{\frac{\beta h}{\kappa}} - 1\right)\left(he^{\frac{\beta h}{\kappa}}\beta - 2\,e^{\frac{\beta h}{\kappa}}\kappa + \beta h + 2\,\kappa\right)}.$$

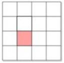


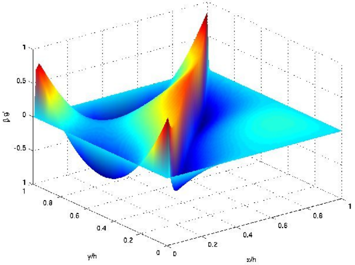
Fine scale Green's functions g' for quadratics in the diffusive ($\alpha = 10^{-2}$, left) and in the advective ($\alpha = 10^{2}$, right) regime; $\alpha := \frac{\beta h}{2\kappa}$ is the mesh Peclét number.

For $k = 3$, for $0 \le x \le h$ and $0 \le y \le h$ we have

$$g'(x,y) = g^{el}(x,y) - \begin{bmatrix} \int_0^h g^{el}(s,y)ds & \int_0^h sg^{el}(s,y)ds \end{bmatrix}$$
$$\times \begin{bmatrix} \int_0^h \int_0^h g^{el}(s,t)dsdt & \int_0^h \int_0^h sg^{el}(s,t)dsdt \\ \int_0^h \int_0^h tg^{el}(s,t)dsdt & \int_0^h \int_0^h stg^{el}(s,t)dsdt \end{bmatrix}^{-1}$$
$$\times \begin{bmatrix} \int_0^h g^{el}(x,t)dt \\ \int_0^h tg^{el}(x,t)dt \end{bmatrix}.$$







Fine scale Green's functions g' for cubics in the diffusive ($\alpha = 10^{-2}$, left) and in the advective ($\alpha = 10^{2}$, right) regime; $\alpha := \frac{\beta h}{2\kappa}$ is the mesh Peclét number.

The fine-scale effect on the coarse-scale equation is now:

$$\int_0^L \int_0^L \mathcal{L}^* \bar{v}(y) g'(x,y) r(x)\, dxdy$$
$$= \sum_{i=1}^{n_{el}} \int_{x_{i-1}}^{x_i} \int_{x_{i-1}}^{x_i} \mathcal{L}^* \bar{v}(y) g'(x,y) r(x)\, dxdy$$
$$= \sum_{i=1}^{n_{el}} \tau_k \int_{x_{i-1}}^{x_i} \left(\frac{d^{k-1}}{dx^{k-1}} r(x) \right) \left(\frac{d^{k-1}}{dx^{k-1}} \mathcal{L}^* \bar{v}(x) \right) dx.$$

- still local (at the element level)
- since $\mathcal{G}'\mathcal{P}^T = 0$ and $\mathcal{P}\mathcal{G}' = 0$, then g' is L^2-orthogonal to $\mathbb{P}_{k-2}$ in both x and y, on each $(x_{i-1}, x_i) \times (x_{i-1}, x_i)$.

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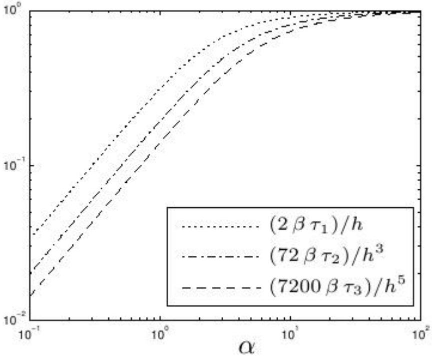
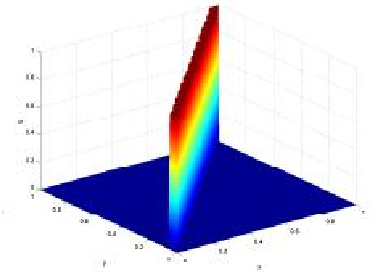
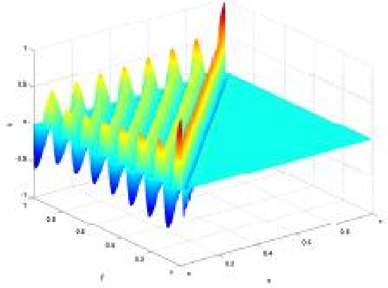


The τ_k are positive and of order h^{2k-1}/β and $\alpha h^{2k-1}/\beta = h^{2k}/\kappa$ in the advective and in the diffusive regimes, respectively.

H_0^1 vs. L^2-optimality in 1D: localization of g'





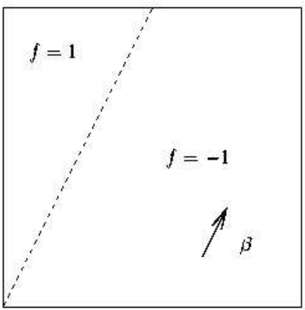
g' for the 1D problem and P1 coarse scales, $\kappa = 10^{-3}$, $\beta = 1$, $L = 1$, 16 elem.; $\mathcal{P} = H_0^1-$proj. (left) and $\mathcal{P} = L^2-$proj. (right): in the latter case, g' is global and unattenuated.

2D advection-diffusion model problem

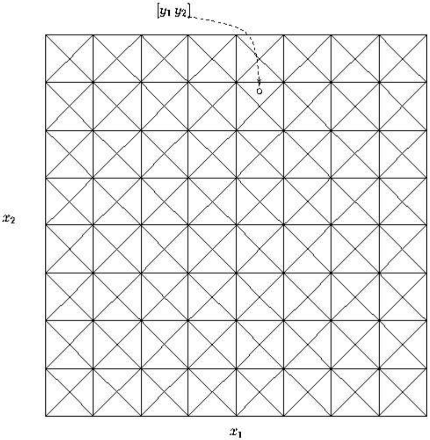
Consider:

$$\begin{cases} -\kappa \Delta u + \beta \cdot \nabla u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\kappa = 2 \cdot 10^{-3}$, $\|\beta\| = 1/\sqrt{2}$.

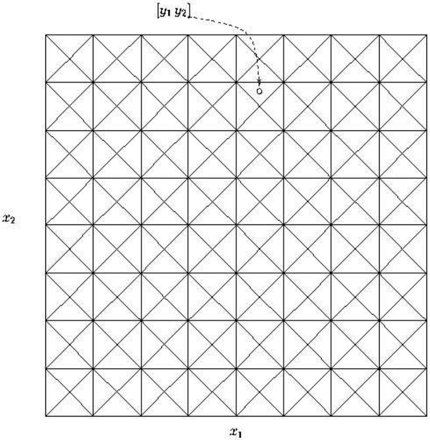


- domain for plotting $x \mapsto g'(x, y)$, with $y = 1/8 \cdot [4.5\ 5.75]$:



- g and g' are computed numerically.

- domain for plotting $x \mapsto g'(x, y)$, with $y = 1/8 \cdot [4.5\ 5.75]$:



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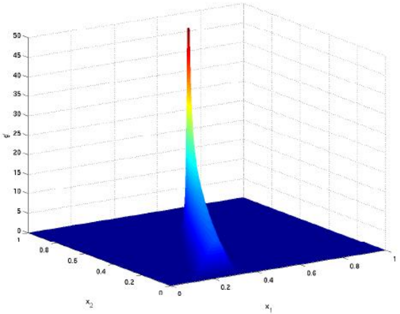
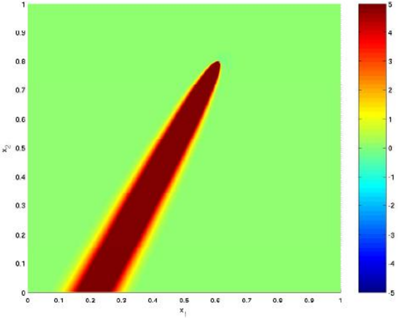
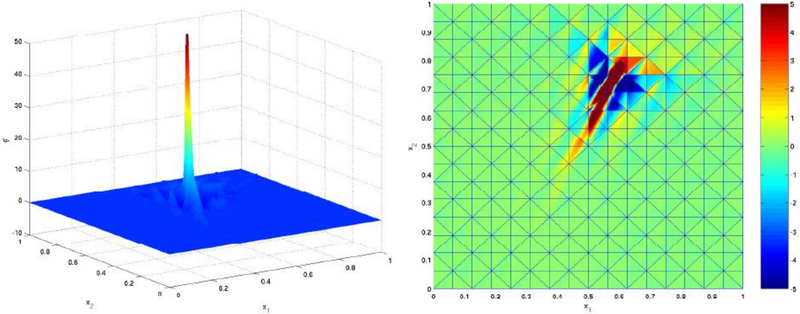
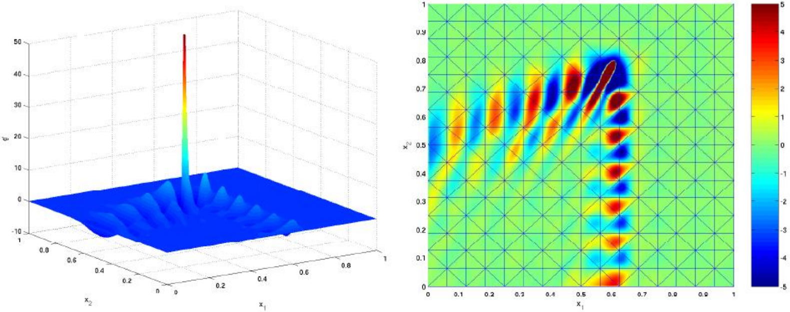




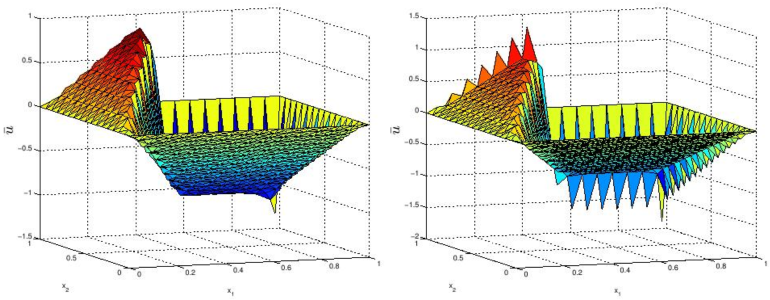
Plot of: $x \mapsto g(x, y)$



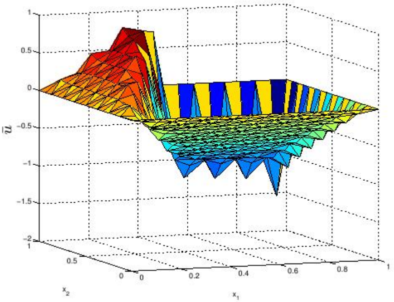
plot of $x \mapsto g'(x, y)$, with $\mathcal{P} \equiv H_0^1$-proj.



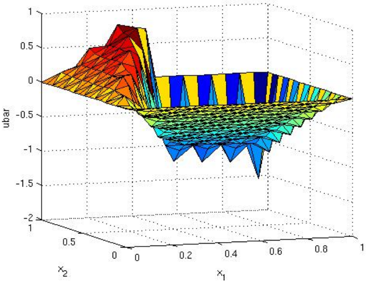
plot of $x \mapsto g'(x, y)$, with $\mathcal{P} \equiv L^2$-proj.



Coarse-scale component $\bar{u}$ for the model problem.
$\mathcal{P} = H_0^1$-proj. (left) and $\mathcal{P} = L^2$-proj. (right).

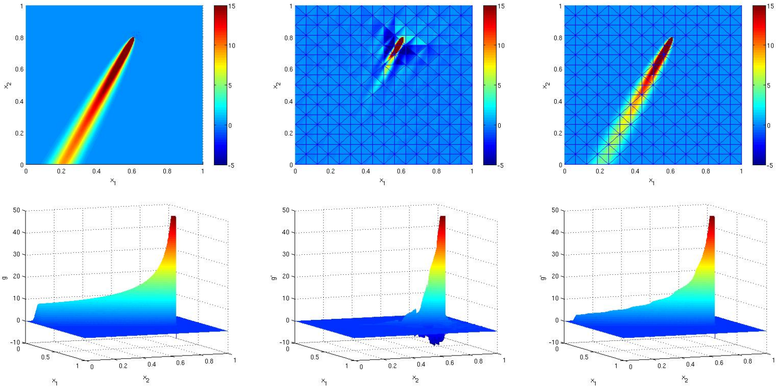


Coarse-scale component $\bar{u}$ for the model problem (different mesh). $\mathcal{P} = H_0^1$-proj. (left) and $\mathcal{P} = L^2$-proj. (right).



$\mathcal{P} = H_0^1$-proj. (left) and $\mathcal{P} =$"nodal"-proj. (right).

g $\quad$ g', H_0^1-proj. $\quad$ g',"nodal"-proj.



Comparison of $x \mapsto g(x, y)$ (left), $x \mapsto g'(x, y)$ for $\mathcal{P} = H_0^1$-proj. (middle) and $\mathcal{P} =$"nodal"-proj. (right).

H^1_0-optimal method and SUPG.

In the case $\mathcal{P} = H^1_0$-proj., because of $P\mathcal{G}' = 0$, we have:

$$\int_\Omega g'(x,y)\Delta\bar{v}(y)\,dy = 0, \quad \forall \bar{v} \in \bar{V}.$$

Therefore, the fine-scale effect on the coarse-scale eq. is:

$$\int_\Omega \int_\Omega (f(x) - \mathcal{L}\bar{u}(x))\, g'(x,y)\mathcal{L}^*\bar{v}(y)\,dxdy$$
$$= -\int_\Omega \int_\Omega (f(x) - \mathcal{L}\bar{u}(x))\, g'(x,y)\beta\cdot\nabla\bar{v}(y)\,dxdy.$$

- In 1D, where g' is fully localized, this is the classical SUPG stabilization [A. N. Brooks and T. J. R. Hughes, '82]
- In 2D, g' is not fully localized, and SUPG is obtained replacing g' by the element-wise constant τ.

Summary

- We have derived an expression for the fine-scale Green's function g' arising in VMS: the specification of a functional $\Phi(\cdot)$, and then of a projector $\mathcal{P}$ defining the decomposition into coarse and fine-scales, renders the problem well-posed.
- For the adv.-diff. 1D problem, we have explicitly calculated g': for higher order-elements, we have obtained a new higher-order residual-based stabilization.
- For the 2D problem, we have numerically computed g': it is found that the projector induced by the H_0^1-seminorm is associated to a g' with dominantly local support, whereas the projector induced by the L^2-norm is not; g' is only attenuated for the "nodal" interpolation.
- Further extension: non-linear fine-scale optimization.

References

F. Brezzi and A. Russo, *Choosing bubbles for advection-diffusion problems*, M^3AS, 4 (1994), pp. 571–587.

T. J. R. Hughes, *Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles and the origins of stabilized methods*, CMAME, 127 (1995), pp. 387–401.

T. J. R. Hughes and G. Sangalli, *Variational multiscale analysis: the fine-scale Green's function, projection, optimization, localization, and stabilized methods*, Tech. Report 05-46, ICES, November 2005.