Extending the theory for domain decomposition algorithms to less regular subdomains

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Joint projects with

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Limitations of the standard theory

We will consider finite element approximations of a selfadjoint elliptic problem on a region Ω (scalar elliptic or linear elasticity.) The domain Ω is subdivided into nonoverlapping subdomains Ω_i. In between the interface Γ.

We will consider tools for proofs of results on *iterative substructuring methods*, such as FETI-DP and BDDC.

We will also consider *two-level overlapping Schwarz methods* with a coarse space component borrowed from iterative substructuring methods, in particular BDDC. In addition, these preconditioners also have local components based on overlapping subregions. Related work in the past by Dryja, Sarkis, and W. (on special multigrid methods); cf. Numer. Math. 1996. That paper introduced quasi-monotonicity. More recent work by Sarkis et al. (These methods have some advantages.)

Assumptions in previous work

In the standard theory for iterative substructuring methods, we typically assume that:

The partition into subdomains Ω_i is such that each subdomain is the union of shape-regular coarse tetrahedral elements of a global conforming mesh $\mathcal{T}_\mathcal{H}$ and the number of such tetrahedra forming an individual subdomain is uniformly bounded.

(Also assume material properties constant in each subdomain.)

In the standard theory for two-level Schwarz methods, we often assume that a conventional coarse space is used, defined on a coarse triangulation, and that the coefficients do not vary a lot or that they are at least quasi-monotone. A major weakness concerns the variation of the coefficients.

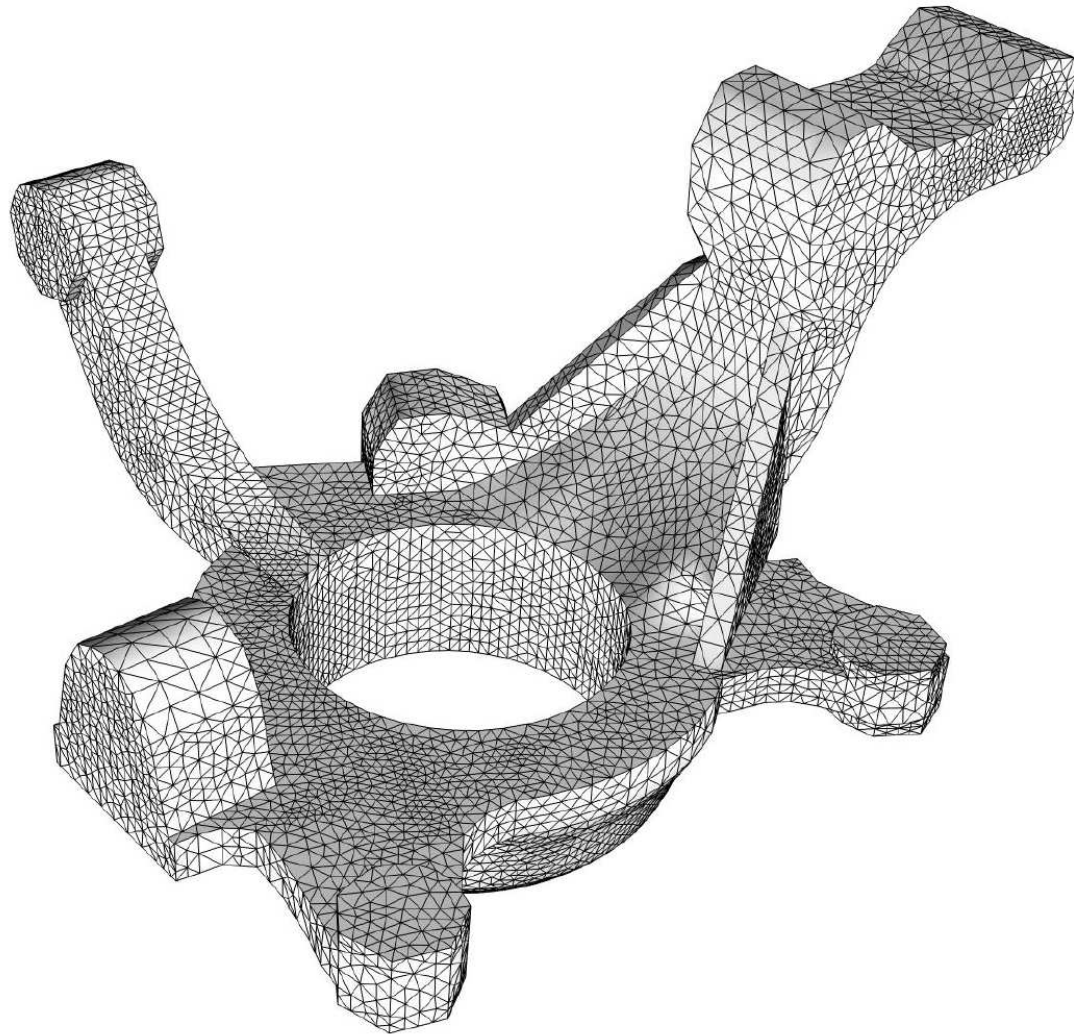Why are these assumptions unsatisfactory? Why bother?
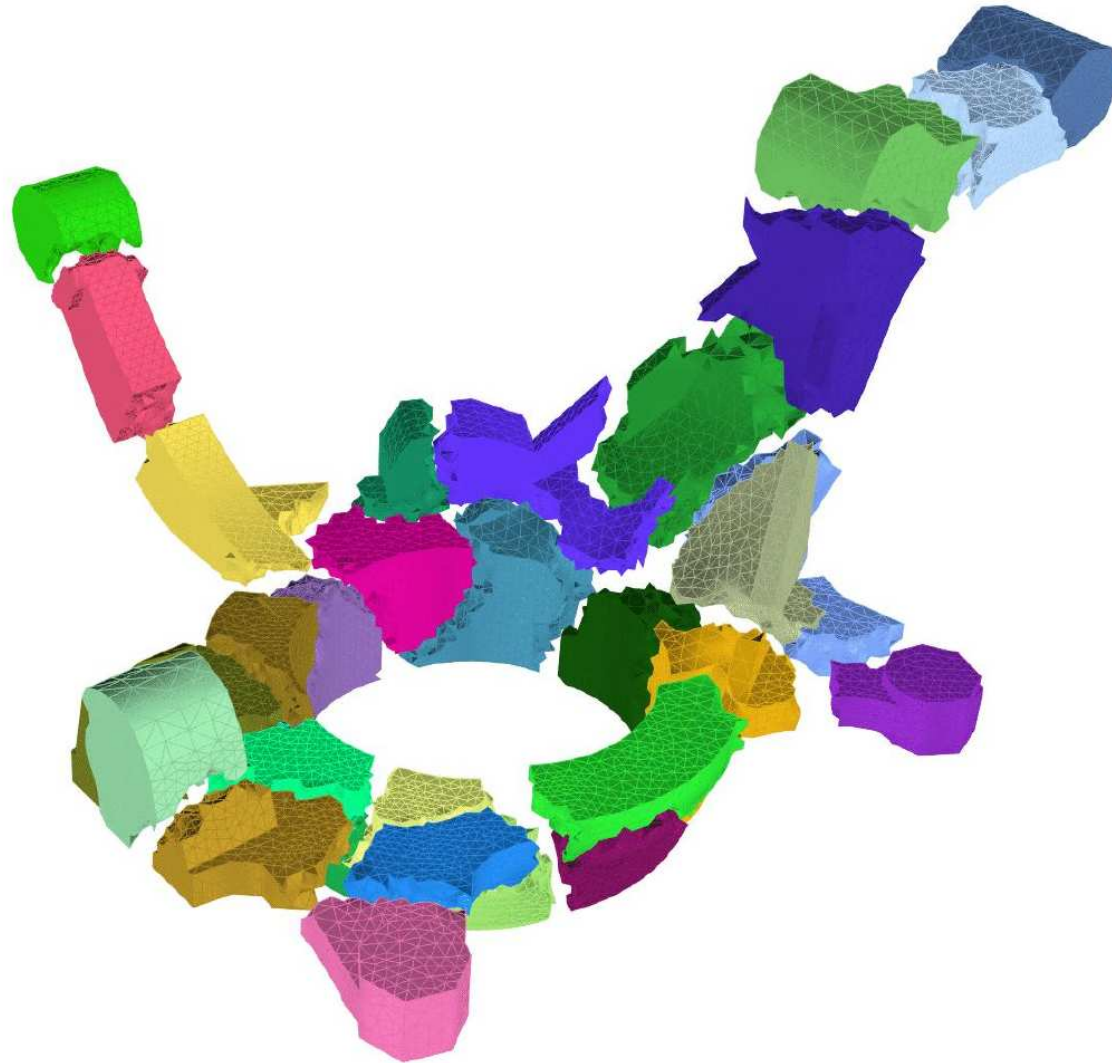
Figure 1: Finite element meshing of a mechanical object.

Figure 2: Partition into thirty subdomains. Courtesy Charbel Farhat.

Block Cholesky Elimination and Iterative Substructuring

Consider a block matrix, assumed positive definite, symmetric.

$$\left[\begin{array}{cc} A & B^T \\ B & C \end{array}\right].$$

It can be factored by block Cholesky:

$$\left[\begin{array}{cc} A & B^T \\ B & C \end{array}\right] = \left[\begin{array}{cc} I_A & \\ BA^{-1} & I_C \end{array}\right] \left[\begin{array}{cc} A & \\ & C - BA^{-1}B^T \end{array}\right] \left[\begin{array}{cc} I_A & A^{-1}B^T \\ & I_C \end{array}\right],$$

where I_A and I_C are appropriate identity matrices. $\mathcal{S} = C - BA^{-1}B^T$ is a Schur complement. Inverting, we find that

$$\left[\begin{array}{cc} A & B^T \\ B & C \end{array}\right]^{-1} = \left[\begin{array}{cc} I_A & -A^{-1}B^T \\ & I_C \end{array}\right] \left[\begin{array}{cc} A^{-1} & \\ & \mathcal{S}^{-1} \end{array}\right] \left[\begin{array}{cc} I_A & \\ -BA^{-1} & I_C \end{array}\right].$$

By changing basis, we can reduce matrix to a block diagonal form.

Cholesky's algorithm is used extensively in finite element practice. It is often helpful to apply the block ideas recursively by, e.g., writing the matrix A as a two-by-two block matrix; A is also often block diagonal and we can then use recursion to deal with each block separately.

Generally, the computation of and factoring of S can be very expensive and less amenable to parallelization than other parts. Explore the possibility of decreasing the size of the Schur complement. Then the solver will only provide an inexact inverse, a *preconditioner*. The preconditioner will be applied in each step of an iterative process of *conjugate gradient* type and it will be crucial that the preconditioned operator is well conditioned, which will translate into rapid convergence. (Research on domain decomposition theory is almost entirely focused on establishing bounds for the condition numbers.) Very desirable to work exclusively with positive definite matrices.

Two Subdomains

Consider a domain Ω subdivided into two nonoverlapping subdomains Ω_1 and Ω_2. In between the interface Γ.

Consider a finite element approximation of a selfadjoint elliptic problem on Ω (scalar elliptic, linear elasticity, or even an incompressible Stokes problem.)

Set up a load vector and a stiffness matrix for each subdomain

$$f^{(i)} = \begin{pmatrix} f_I^{(i)} \\ f_\Gamma^{(i)} \end{pmatrix}, \quad A^{(i)} = \begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix}, \quad i = 1, 2.$$

Homogeneous Dirichlet condition on $\partial\Omega_i \setminus \Gamma$, Neumann on Γ.

Subassemble:

$$A = \begin{pmatrix} A_{II}^{(1)} & 0 & A_{I\Gamma}^{(1)} \\ 0 & A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma} \end{pmatrix}, \quad u = \begin{pmatrix} u_I^{(1)} \\ u_I^{(2)} \\ u_\Gamma \end{pmatrix}, \quad f = \begin{pmatrix} f_I^{(1)} \\ f_I^{(2)} \\ f_\Gamma \end{pmatrix}.$$

$A_{\Gamma\Gamma} = A_{\Gamma\Gamma}^{(1)} + A_{\Gamma\Gamma}^{(2)}$. Degrees of freedom partioned into those internal to Ω_1, and Ω_2, and those on Γ.

Eliminate the interior unknowns. Gives two Schur complements:

$$S^{(i)} := A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)} A_{II}^{(i)^{-1}} A_{I\Gamma}^{(i)}, \quad i = 1, 2.$$

The given system is reduced to

$$S u_\Gamma = (S^{(1)} + S^{(2)}) u_\Gamma = g_\Gamma.$$

Working with One Constraint. Two Subdomains.

In 2D FETI–DP algorithms, we maintain continuity at subdomain vertices throughout the whole iteration. In 3D, we need to work with common averages over interface edges or faces. Here we consider a scalar elliptic problem, two subdomains and one average constraint. Same constraint can be used for BDDC.
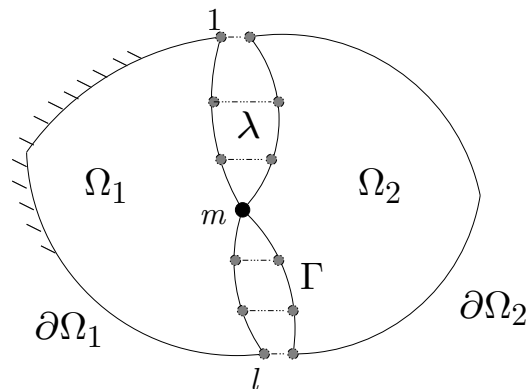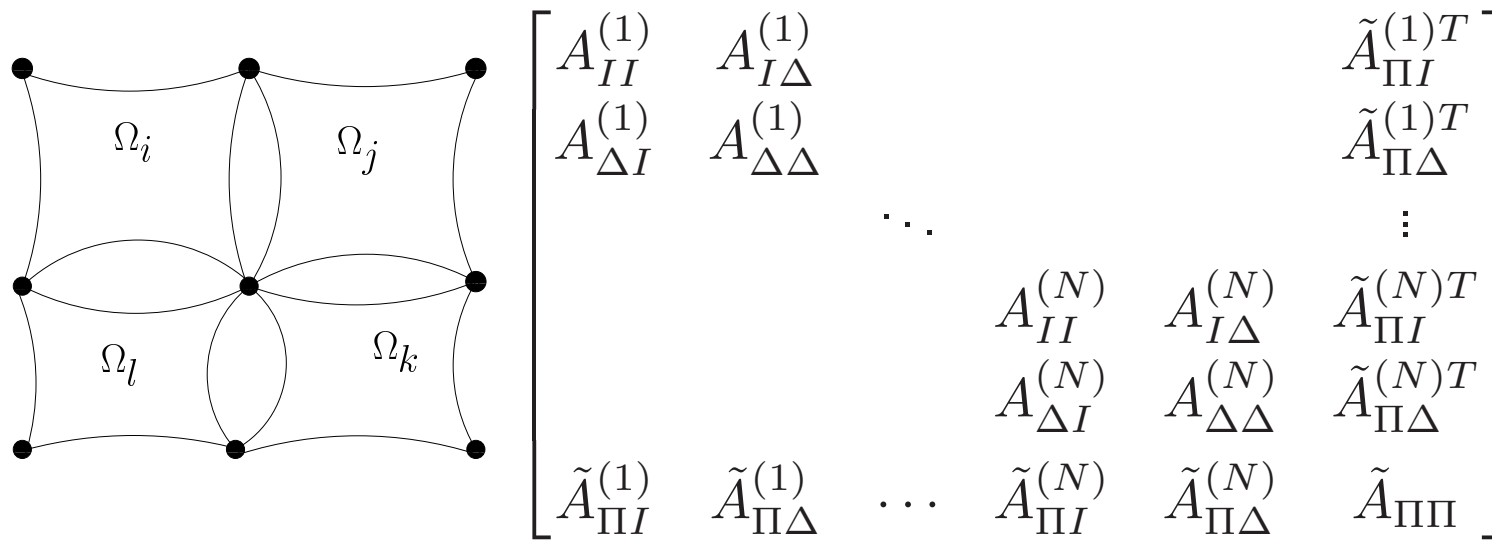

Figure 3: Partition into two subdomains, with Ω_2 floating, in the absence of a constraint.

Our lemon, which can be 2D or 3D, only has Dirichlet boundary conditions on part of one subdomain boundary. In 3D, we could work with a face average or an average over an edge of the face common to the two subdomains. We make a change of variables separating the edge average, the $primal$ displacement variable, from the $dual$ displacement variables. Any dual displacement variable has a zero edge average. (We can also think of each primal variable in terms of a constraint.) In the general case, the primal variables will provide a coarse component for our preconditioners and they live in the lower right corner of the block matrix.

In each BDDC iteration, solve local problems, then an equation with one variable, average the result across the interface, and finally minimize the energy in each subdomain by solving Dirichlet problems. The residual is then split to create a right hand side of the correct dimension and a symmetric preconditioner.

Two dimensions: Farhat, Lesoinne, LeTallec, Pierson, Rixen (2001): Keep continuity of primal variables at vertices (subassembly); other continuity constraints by Lagrange multipliers (the flux).

$$
\begin{bmatrix}
A_{II}^{(1)} & A_{I\Delta}^{(1)} & & & & & \tilde{A}_{\Pi I}^{(1)T} \\
A_{\Delta I}^{(1)} & A_{\Delta\Delta}^{(1)} & & & & & \tilde{A}_{\Pi\Delta}^{(1)T} \\
& & \ddots & & & & \vdots \\
& & & A_{II}^{(N)} & A_{I\Delta}^{(N)} & \tilde{A}_{\Pi I}^{(N)T} \\
& & & A_{\Delta I}^{(N)} & A_{\Delta\Delta}^{(N)} & \tilde{A}_{\Pi\Delta}^{(N)T} \\
\tilde{A}_{\Pi I}^{(1)} & \tilde{A}_{\Pi\Delta}^{(1)} & \cdots & \tilde{A}_{\Pi I}^{(N)} & \tilde{A}_{\Pi\Delta}^{(N)} & \tilde{A}_{\Pi\Pi}
\end{bmatrix}
$$

$$
\begin{bmatrix}
A_{BB} & \tilde{A}_{\Pi B}^{T} & B^{T} \\
\tilde{A}_{\Pi B} & \tilde{A}_{\Pi\Pi} & O \\
B & O & O
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_{B} \\
\tilde{\mathbf{u}}_{\Pi} \\
\boldsymbol{\lambda}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{f}_{B} \\
\tilde{\mathbf{f}}_{\Pi} \\
0
\end{bmatrix}
$$

History of FETI and FETI-DP

FETI introduced by Farhat/Roux (1991). Further algorithmic and theoretical work by Lesoinne, Pierson, Mandel, Tezaur, Rixen, Brenner, Langer, Steinbach, Klawonn, Widlund, Dryja, Toselli, Vasseur, LeTallec, Rheinbach, ... Earlier work on DDM with Lagrange multipliers by Dihn/Glowinski/Periaux (1984), Glowinski/Wheeler (1987).

FETI-DP introduced by Farhat/Lesoinne/LeTallec/Pierson/Rixen (2001). Further work by Farhat, Lesoinne, Pierson, Mandel, Tezaur, Brenner, Li, Toselli, Widlund, Dryja, Klawonn. h- and hp, BEM, mixed FEM, incompressible Stokes, mortar elements, Maxwell 3D, eigensolver ...

Structural mechanics 2nd and 4th order, acoustics, scalar diffusion problems, contact, Stokes, ... Tested for at least 100 million dof on thousands of processors (ASCI Option Red, Sandia National Laboratory, USA). Work by Farhat's group, also by Oliver Rheinbach, Essen in PETSC.

History of Neumann-Neumann and BDDC

Early work by Bourgat, Glowinski, De Roeck, LeTallec, and Vidrascu. Introduction of a second level by Mandel and Brezina and by Dryja and Widlund. Used extensively in many large scale applications, in particular, in France by LeTallec and Vidrascu et al. Extended, in various ways, to convection-diffusion equations by Achdou, LeTallec, Nataf, and Vidrascu, to incompressible Stokes by Pavarino and Widlund and in collaboration with Goldfeld, to almost incompressible elasticity and models of elastic bodies which in part are almost incompressible.

New coarse spaces by Dohrmann, Mandel, and Tezaur: BDDC. The spectra of the FETI-DP and BDDC operators are almost the same. First observed by Fragakis and Papadrakakis for older methods. Recent work on Stokes, flow in porous media, use of inexact subdomain solvers, spectral elements, and mortar finite elements.

Dual-Primal FETI in 3D

Good numerical results in 2D; not always very good in 3D. Therefore, in addition to (or instead of) continuity of primal variables at vertices, constrain certain average values (and moments) of primal variables over individual edges and faces to take common values across the interface; for 3D elasticity, minimally six constraints per subdomain. Change variables; gives thesame matrix formulation and robust performance.

Condition number estimate $C(1 + \log(H/h)^2)$ for some algorithms. Independent of jumps in coefficients, if scaling chosen carefully. Algorithms can have a small coarse problem; Klawonn, W., Dryja (2002). Numerical results for elasticity in 3D using vertex constraints and three face averages on each face (Farhat, Lesoinne, Pierson (2000)). Works numerically; might not be fully scalable. Scalability established for elasticity for somewhat richer primal space; Klawonn and W. (CPAM 2006.) PETSc codes by Rheinbach and several experimental papers.

N-N Methods of Same Flavor: BDDC

We can introduce a coarse basis function for each primal constraint; set one primal variable $= 1$ and all others $= 0$, one at a time. Extend with minimum energy for individual subdomains. Results in basis functions discontinuous across the interface Γ. Also one local subspace for each subdomain for which all relevant primal degrees vanish. Makes subdomain problems invertible.

Partially subassembled Schur complement of the system is block diagonal after this change of variables. Apply an operator E_D^T to residual. Solve linear systems corresponding to blocks exactly, and compute a weighted average, with operator E_D, of results, across the interface. Only one block, with a few variables for each subdomain assembled and factored. Compute residual, remove the interior residuals, and repeat coarse and local solves. Accelerate with conjugate gradient method. Theory focused on E_D.

Matrix Analysis of FETI-DP and BDDC

Consider three product spaces of finite element functions/vectors of nodal values.

$$\widehat{W}_\Gamma \subset \widetilde{W}_\Gamma \subset W.$$

W: no constraints; $\widehat{W}_\Gamma$: continuity at every point on Γ; $\widetilde{W}_\Gamma$: common values of primal variables and, effectively, a *nonconforming approximation*.

Change variables, explicitly introducing primal variables and complementary sets of dual displacement variables. Write Schur complements as

$$S^{(i)} = \begin{pmatrix} S^{(i)}_{\Delta\Delta} & S^{(i)}_{\Delta\Pi} \\ S^{(i)}_{\Pi\Delta} & S^{(i)}_{\Pi\Pi} \end{pmatrix}.$$

Let $\tilde{S}_\Gamma$ denote the partially assembled Schur complement. (In practice, work with interior variables as well when solving linear systems.)

BDDC matrices

For the BDDC method, we use the fully assembled Schur complement $\widehat{S} = \tilde{R}_\Gamma^T \tilde{S}_\Gamma \tilde{R}_\Gamma$; it is used to compute the residual. Using the preconditioner involves solving a system with the matrix $\tilde{S}_\Gamma$:

$$M_{BDDC}^{-1} = \tilde{R}_{D\Gamma}^T \tilde{S}_\Gamma^{-1} \tilde{R}_{D\Gamma},$$

where $\tilde{R}_{D\Gamma}$ is a scaled variant of $\tilde{R}_\Gamma$ with scale factors computed from the PDE coefficients.

Scaling chosen so that $E_D := \tilde{R}_\Gamma \tilde{R}_{D\Gamma}^T$ is a projection.

The matrix $\tilde{S}_\Gamma$ equally important for FETI-DP.

Condition number estimate

We wish to show for BDDC operator that

$$\langle u, u \rangle_{\widehat{S}_\Gamma} \leq \left\langle u, M^{-1}\widehat{S}_\Gamma u \right\rangle_{\widehat{S}_\Gamma} \leq \|E_D\|_{\widetilde{S}_\Gamma} \langle u, u \rangle_{\widehat{S}_\Gamma}.$$

Lower bound: Let

$$w = \widetilde{S}_\Gamma^{-1}\widetilde{R}_{D\Gamma}\widehat{S}_\Gamma u.$$

We have, since $\widetilde{R}_\Gamma^T \widetilde{R}_{D\Gamma} = \widetilde{R}_{D\Gamma}^T \widetilde{R}_\Gamma = I$,

$$\langle u, u \rangle_{\widehat{S}_\Gamma} = u^T \widehat{S}_\Gamma \widetilde{R}_{D\Gamma}^T \widetilde{R}_\Gamma u = u^T \widehat{S}_\Gamma \widetilde{R}_{D\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{S}_\Gamma \widetilde{R}_\Gamma u = \left\langle w, \widetilde{R}_\Gamma u \right\rangle_{\widetilde{S}_\Gamma}.$$

Therefore,

$$\langle u, u \rangle_{\widehat{S}_\Gamma} \leq \langle w, w \rangle_{\widetilde{S}_\Gamma}.$$

Since,

$$\langle w, w \rangle_{\widetilde{S}_\Gamma} = u^T \widehat{S}_\Gamma \widetilde{R}_{D\Gamma}^T \widetilde{S}_\Gamma^{-1} \widetilde{S}_\Gamma \widetilde{S}_\Gamma^{-1} \widetilde{R}_{D\Gamma} \widehat{S}_\Gamma u = \left\langle u, M^{-1} \widehat{S}_\Gamma u \right\rangle_{\widehat{S}_\Gamma},$$

we obtain $\langle u, u \rangle_{\widehat{S}_\Gamma} \leq \left\langle u, M^{-1} \widehat{S}_\Gamma u \right\rangle_{\widehat{S}_\Gamma}$. Smallest eigenvalue is in fact $= 1$.

Upper bound: Take w as before. We have, $\widetilde{R}_{D\Gamma}^T w = M^{-1} \widehat{S}_\Gamma u$. Since $\widehat{S}_\Gamma = \widetilde{R}_\Gamma^T \widetilde{S}_\Gamma \widetilde{R}_\Gamma$, we have

$$\left\langle M^{-1} \widehat{S}_\Gamma u, M^{-1} \widehat{S}_\Gamma u \right\rangle_{\widehat{S}_\Gamma} = \left\langle \widetilde{R}_{D\Gamma}^T w, \widetilde{R}_{D\Gamma}^T w \right\rangle_{\widehat{S}_\Gamma} = \left\langle \widetilde{R}_\Gamma \widetilde{R}_{D\Gamma}^T w, \widetilde{R}_\Gamma \widetilde{R}_{D\Gamma}^T w \right\rangle_{\widetilde{S}_\Gamma}$$

$$= \langle E_D w, E_D w \rangle_{\widetilde{S}_\Gamma}^2 \leq \|E_D\|_{\widetilde{S}_\Gamma}^2 \langle w, w \rangle_{\widetilde{S}_\Gamma}^2 .$$

The upper bound follows easily by using Cauchy-Schwarz.

Core estimate:

$$|E_D w|_{\widetilde{S}_\Gamma}^2 \le C(1 + \log(H/h))^2 |w|_{\widetilde{S}_\Gamma}^2 \quad \forall w \in \widetilde{W}.$$

Estimate a weighted norm of the discrete harmonic extension of half the jump in w across the interface. The estimate is local and involves individual subdomains, one at a time, and traces of the component of w of the subdomain and its neighbors across the subdomain boundary. The quality of bound depends crucially on choice of primal constraints. Work begun by Mandel and Sousedík to select primal constraints automatically.

This frame work has been ideal for developing inexact solvers and BDDC methods with three levels. Work by Jing Li and thesis by Xuemin Tu. All this theory assumes polyhedral subdomains.

Saddle Point Problems

Three studies have now been completed on saddle point problems. Two are by Xuemin Tu and concern flow in porous media solved as a saddle point problem and as a nonconforming, hybrid finite element problem, respectively. Work by Dohrmann as well.

Joint work with Jing Li has identified two core requirement for incompressible Stokes problems discretized by inf-sup stable pairs of finite element spaces with discontinuous pressure spaces.

1) The constraints should be chosen so that we have a strong bound on the norm of E_D; borrowed from recent joint work with Klawonn on linear elasticity.

2) All elements of the dual space, which is determined by the primal constraints, should have zero average normal velocities on each subdomain.

What is needed for more general subdomains?

In all theory for multi-level domain decomposition methods, we need a Poincaré inequality. Here is an interesting variant:

Theorem [Poincaré's Inequality and a Related Isoperimetric Inequality]
Let $\Omega \subset R^n$ be open, bounded and connected. Then,

$$\inf_{c \in R} \left(\int_\Omega |f - c|^{n/(n-1)} \, dx \right)^{(n-1)/n} \leq \gamma(\Omega, n) \int_\Omega |\nabla f| \, dx,$$

if and only if,

$$[\min(|A|, |B|)]^{1-1/n} \leq \gamma(\Omega, n)|\partial A \cap \partial B|. \tag{1}$$

Here $A \subset \Omega$, and arbitrary, and $B = \Omega \setminus A$.

This result can be found in a book by Lin and Yang, "Geometric Measure Theory – An Introduction".

Using Hölder's inequality several times, we find, for $n = 3$, that

$$\inf_{c \in R} \|u - c\|_{L^2(\Omega)} \leq \gamma(\Omega, n) \mathsf{Vol}(\Omega)^{1/3} \|\nabla u\|_{L^2(\Omega)}.$$

This is the conventional form of Poincaré's inequality. (Thanks to Fanghua Lin and Hyea Hyun Kim.)

The parameter in this inequality enters into all bounds of our result and it is closely related to the second eigenvalue of the Laplacian with Neumann boundary conditions.

We (re)learn from this result that we have to expect slow convergence if the subdomains are not shape regular. (Consider a slim bar.) We can also have problems if elements at the boundary are not shape regular.

But we also see that under reasonable assumptions on our subregions, we can expect a satisfactory parameter in the Poincaré inequality.

Another important tool is a simple trace theorem:

$$\beta \|u\|_{L^2(\partial\Omega)}^2 \le C(\beta^2 |u|_{H^1(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2).$$

The parameter β measures the thickness of Ω. This result is borrowed from Nečas' 1967 book and it is proven under the assumption that the region is Lipschitz; C is proportional to the Lipschitz constant.

One can easily construct subdomains which are not Lipschitz. What might be the right, more general family of domains?

In the literature, we find (Fritz) John domains, carrots, cigars, etc.

John Domains

Ω *is a John domain if there exists a constant $C_J \geq 1$ and a distinguished central point x_0 such that each $x \in \Omega$ can be joined by a curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and $dist(\gamma(t), \partial\Omega) \geq C_J^{-1}|x - \gamma(t)|$ for all $t \in [0,1]$.*

It is a twisted cone condition. Note that certain snow flake curves are John domains, i.e., the length of the boundary can be arbitrarily much larger than the diameter of Ω.

Extension of Tools for DD Theory

The technical tools necessary for the traditional analysis of the rate of convergence of iterative substructuring methods are collected in Section 4.6 of the T. & W. book. Among the tools necessary for the analysis of BDDC and FETI-DP in three dimensions is a bound on the energy of

$$I^h(\vartheta_{F^k} u)$$

and bounds on the corresponding edge functions. These bounds feature a second logarithmic factor. The old results on special subdomains can be extended to much more general subdomains; no new ideas are required.
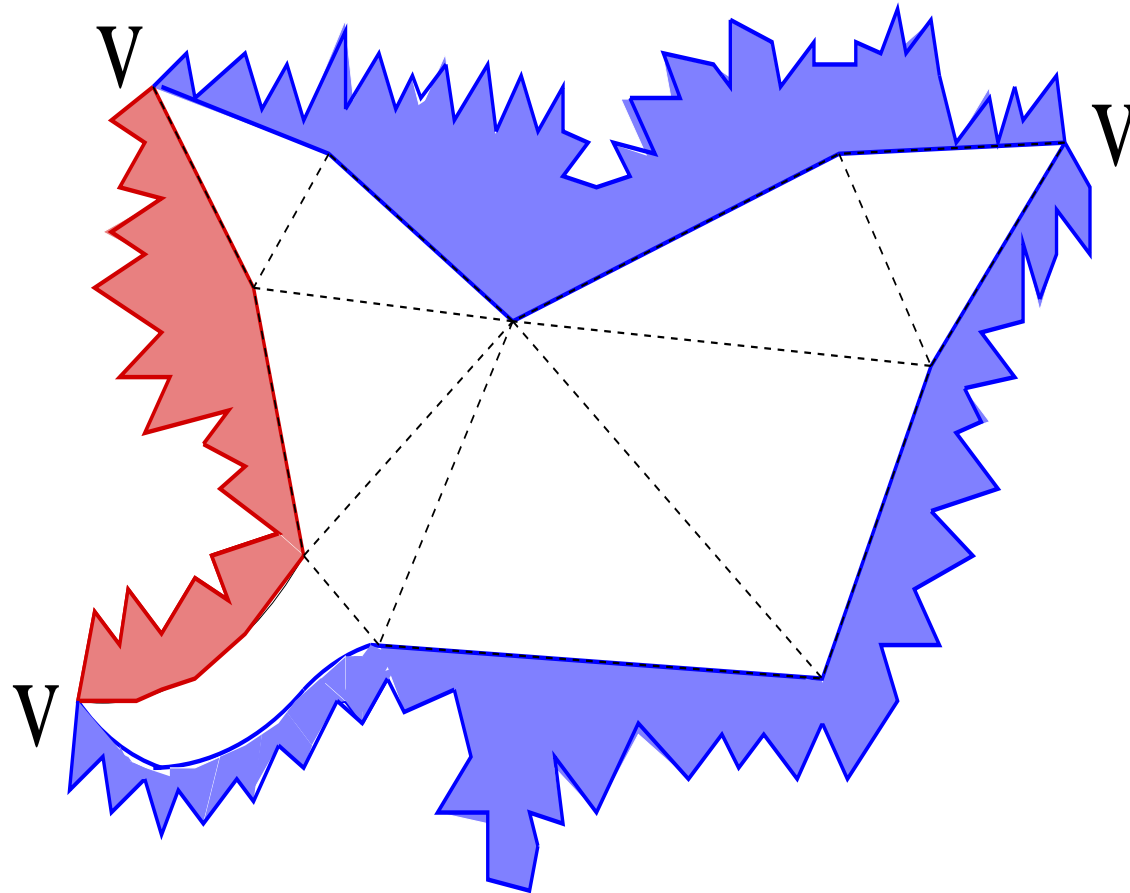
Figure 4: Construction of ϑ_E in 2D.

We know that the estimate of the condition numbers of BDDC (and FETI–DP) can be reduced to bound of an averaging operator E_D across the interface. On each subdomain face, e.g., we have a weighted average of the traces of functions defined in the relevant subset of subdomains. The weights depend on the coefficients of the elliptic problem. We have to cut the traces using ϑ_{F^k}, etc. We then estimate the energy of resulting components in terms of the energy of the functions, given on those set of the subdomains, from which the averages are computed. Two logarithmic factors result. These bounds have previously been developed, fully rigorously, for the case of simple polyhedral subdomains, for scalar elliptic problems, compressible elasticity, flow in porous media, Stokes and almost incompressible elasticity. For each of these cases, we have to select the coarse component and certain scale factors of the preconditioner quite carefully; that is not today's story. What is new is that we can obtain bounds, in many cases, which are of good quality for more general subdomains. Theory complete for two dimensions.

Overlapping Schwarz methods

Consider Poisson's equation on a bounded Lipschitz region Ω in two or three dimensions.

Two triangulations, a coarse and a fine, (which might be a refinement of the coarse.) The overall space is V^h, the space of continuous, piecewise linear finite element functions on the fine triangulation. There is also a covering of the region by overlapping subregions Ω_i'. Let δ/H measure the relative overlap between adjacent subregions, each of which is a union of elements. Assume shape regularity, but not necessarily quasi-uniformity of the elements. Spaces chosen for the Schwarz methods:

$$V_0 = V^H \quad \text{based on coarse triangulation,}$$

$$V_i = V^h \cap H_0^1(\Omega_i'), \quad i > 0.$$

Theorem. *The condition number of the additive Schwarz method satisfies*

$$\kappa(T_{as}) \leq C(1 + H/\delta).$$

The constant C is independent of the parameters H, h, and δ.

Result cannot be improved: Sue Brenner.

There are quite similar results for multiplicative algorithms as well.

Without a coarse space component, H/δ must be replaced by $1/(H\delta)$.

Several alternative coarse spaces have been considered; cf. Toselli and W. , Chapter 3.

To prove this bound, we must come up with a recipe of how to decompose any function in V.

We choose
$$u_0 = \tilde{I}^H u \in V_0, \tag{2}$$
where we use averages over neighborhoods of the nodes of coarse triangles and interpolation into the coarse space. Let

$$w = u - R_0^T u_0 = u - I^h u_0. \tag{3}$$

The local components are defined by

$$u_i = R_i(I^h(\theta_i w)) \in V_i, \quad 1 \le i \le N. \tag{4}$$

Here $\{\theta_i\}$ is a piecewise linear partition of unity associated with the overlapping partition.

The Case of Bad Coefficients

Now consider a scalar elliptic equation defined by a bilinear form

$$\sum \int_{\Omega_j} \rho_j \, \nabla u \cdot \nabla v \, dx.$$

The coefficients ρ_j are arbitrary positive constants and the Ω_j are quite general subdomains.

A natural coarse space is the range of the following interpolation operator

$$I_B^h u(x) = \sum_{V^k \in \Gamma} u(V^k) \theta_{V_k}(x) + \sum_{E^i \subset W} \bar{u}_{E^i} \theta_{E^i}(x) + \sum_{F^k \subset \Gamma} \bar{u}_{F^k} \theta_{F^k}(x).$$

Here $\bar{u}_{E^i}$ and $\bar{u}_{F^k}$ are averages over edges and faces of the subdomains.

$\theta_{V_k}(x)$ the standard nodal basis functions of the vertices of the subdomains, $\theta_{E^i}(x) = 1$ at the nodes of the edge E^i and vanishes at all other interface nodes, and $\theta_{F^k}(x)$ is a similar function defined for the face F^k. These functions are extended as discrete harmonic functions into the interior of the subdomains. Note that this interpolation operator, I_B, preserves constants. A slightly richer coarse space will preserve all linear functions; useful for elasticity.

Faces, edges, and vertices of quite general subdomains can be defined in terms of certain equivalence classes. We will now consider the energy of the face terms and estimate their energy in terms of the energy of the function interpolated. Can we estimate the averages $\bar{u}_{F^k}$ by Cauchy-Schwarz and the trace theorem?

Estimates of the energy of $\theta_{F^k}(x)$ well known for special regions, e.g., tetrahedra; bounds are $C(1 + \log(H/h)H$. We have learned, in detail so far only for two dimensions, how to construct functions ϑ_E forming a partition of unity with the same quality bound.

The bounds for the local components are done using a partition of unity related to the overlapping subregions and a Friedrichs inequality on patches of diameter δ. The patches are chosen so that the coefficient of the elliptic problem is constant in each of them; see also Chap. 3 of the T. & W. book. A second factor $(1 + H/\delta)$ comes from estimates of the local components; Brenner has shown that this factor cannot be improved.

Result on the two-level overlapping Schwarz method

Theorem. *The condition number κ of the preconditioned operator satisfies*

$$\kappa \leq C(1 + H/\delta)(1 + \log(H/h))^q.$$

Here C is independent of the mesh size, the number of subdomains, the coefficients ρ_i, etc. H/δ measures the relative overlap between neighboring overlapping subregions, but it depends on the Poincaré and John parameters. H/h measures the maximum number of elements across any subregion. If the coefficients are nice and the coarse space contains the linear functions, then $q = 1$. In two dimensions we, so far, have a result for any John domain with $q = 2$.