Hilbert Complexes and the Finite Element Exterior Calculus

D. N. Arnold, joint work with R. Falk and R. Winther

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Outline

1. Motivation
2. Hilbert complexes
3. Discretization
4. Finite element differential forms
5. Additional applications

reference:

Motivation

Vector Poisson equation

\[
\text{curl curl } u - \text{grad div } u = f \quad \text{in } \Omega \\
\quad u \cdot n = 0, \text{curl } u \times n = 0 \quad \text{on } \partial \Omega
\]

\( f \equiv 0 \) does not imply \( u \equiv 0 \):

\[
\dim \mathcal{H} = b_1
\]

harmonic forms (solutions for \( f = 0 \))

1st Betti number (number of holes)

curl curl \( u - \text{grad div } u = f \) (mod \( \mathcal{H} \)), \( u \nabla \mathcal{H} \), b.c.

\[
u = \arg \min_{H(\text{curl}) \cap H(\text{div}) \cap \mathcal{H}^1} \frac{1}{2} \int_{\Omega} |\text{curl } u|^2 + |\text{div } u|^2 \, dx - \int_{\Omega} f \cdot u \, dx
\]
Standard Galerkin does not work

\[ f = (0, x) \]

\[ P_1 \text{ elements} \]

\[ f = (-1, 0) \]

\[ P_1 \text{ elements} \]

A mixed formulation

\[
\sigma = -\text{div} u, \quad \text{grad} \sigma + \text{curl} \text{curl} u = f \pmod{\mathcal{N}}, \quad u \perp \mathcal{N}
\]

\[ \sigma \in H^1, \; u \in H(\text{curl}), \; p \in \mathcal{N}: \]

\[
\int_{\Omega} \sigma \tau \; dx - \int_{\Omega} u \cdot \text{grad} \tau = 0, \quad \tau \in H^1,
\]

\[
\int_{\Omega} \text{grad} \sigma \cdot v \; dx + \int_{\Omega} \text{curl} u \cdot \text{curl} v \; dx + \int_{\Omega} pv \; dx = \int_{\Omega} f \cdot v \; dx, \; v \in H(\text{curl})
\]

\[
\int_{\Omega} uq \; dx = 0, \quad q \in \mathcal{N}
\]

\[
\frac{1}{2} \int_{\Omega} |\sigma|^2 \; dx - \int_{\Omega} \text{grad} \sigma \cdot u \; dx - \frac{1}{2} \int_{\Omega} |\text{curl} u|^2 \; dx - \int_{\Omega} p \cdot u \; dx + \int_{\Omega} f \cdot u \; dx \rightarrow \text{saddle point}
\]

The Galerkin method works for this formulation, if we use stable els.

Hilbert complexes: basic definitions

- A cochain complex is a sequence of vector spaces and linear maps
  \[ \ldots \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \xrightarrow{d^{k+1}} \ldots \]
  with \( d^k \circ d^{k-1} = 0 \).

- Cycles, boundaries, cohomology:
  \[
  Z^k = \mathcal{N}(d^k), \quad B^k = \mathcal{R}(d^{k-1}), \quad H^k = Z^k / B^k
  \]

- A Hilbert complex is a sequence of Hilbert spaces and densely-defined closed operators
  \[ \ldots \xrightarrow{d^{k-1}} W^k \xrightarrow{d^k} W^{k+1} \xrightarrow{d^{k+1}} \ldots \]
  with \( d^k \circ d^{k-1} = 0 \).

- Closed Hilbert complex: \( \mathcal{B}^k \) is closed

  Bounded Hilbert complex: operators are bounded

- The domain complex:
  \[ V^k = \mathcal{D}(d^k) \]

  is a H-space with \( \| u \|_V^2 = \| u \|^2 + \| du \|^2 \):

  \[
  \ldots \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \xrightarrow{d^{k+1}} \ldots
  \]

  This is a bounded Hilbert complex (so a cochain complex).
Mixed formulation of the Hodge Laplacian

Given \( f \in W^k \), find \( \sigma \in V^{k-1}, u \in V^k, p \in \mathcal{S}^k \):

\[
\begin{align*}
\langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0 & \forall \tau \in V^{k-1} \\
\langle \sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle & \forall v \in V^k \\
\langle u, q \rangle &= 0 & \forall q \in \mathcal{S}^k
\end{align*}
\]

Equivalently \( \frac{1}{2} \langle \sigma, \sigma \rangle - \langle d\sigma, u \rangle - \frac{1}{2} \langle du, du \rangle - \langle p, u \rangle + \langle f, u \rangle \rightarrow \text{saddle point} \)

**Theorem**

\[ \forall f \exists! (\sigma, u, p) \text{ and } ||\sigma|| + ||u|| + ||v|| \leq c ||f|| \text{ with } c \text{ depending only on } c_P. \]

Need to control \( ||\sigma|| + ||u|| + ||v|| + ||p|| \) by a bounded choice of \( \tau, v, \) and \( q \). 
\( \tau = \sigma \) controls \( ||\sigma|| \), \( v = d\sigma \) controls \( ||d\sigma|| \), \( v = p \) controls \( ||p|| \).
\( v = u \) controls \( ||du|| \), \( u \rightarrow \text{controls} \) \( ||du|| \). \text{How to control} \( ||u|| ?? \)

Hodge decomposition:

\[ u = d\eta + s + z, \quad \eta \in V^{k-1}, s \in \mathcal{S}^k, z \in (3^k)^\perp \]

\( \tau = \eta \) controls \( ||d\eta|| \) and \( q = s \) controls \( ||s|| \). To bound \( ||z|| \) we use \text{Poincaré's inequality: } ||z|| \leq c_P ||dz|| = c_P ||du|| \text{ (which is under control)}

Hilbert complex:

\[ 0 \rightarrow L^2(\Omega) \xrightarrow{\text{grad}} L^2(\Omega; \mathbb{R}^3) \xrightarrow{\text{curl}} L^2(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} L^2(\Omega) \rightarrow 0 \]

Domain complex:

\[ 0 \rightarrow H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \rightarrow 0 \]

Dual domain complex:

\[ 0 \xleftarrow{\text{div}} \overset{\text{curl}}{L^2(\Omega)} \xleftarrow{\text{curl}} \overset{\text{grad}}{H(\text{curl}, \Omega)} \xleftarrow{\text{grad}} H^1(\Omega) \xleftarrow{0} \]

Mixed form of Hodge Laplacian:

0: Poisson equation, usual \( H^1 \) formulation \( (\sigma = 0) \)
1: Vector Poisson equation, scalar–vector mixed method \( (\sigma = - \text{div } u) \)
2: Vector Poisson equation, vector–vector mixed method \( (\sigma = \text{curl } u) \)
3: Poisson equation, mixed formulation \( (\sigma = - \text{grad } u) \)

Restricting \( f \) to \( \mathcal{B}^k \) or \( \mathcal{B}^k_* \) for \( k = 1, 2 \) we get additional problems

\[ \bullet \text{ curl curl } u = f, \text{ div } u = 0 \quad \bullet \text{ div } u = f, \text{ curl } u = 0 \]
Harmonic forms and Betti numbers

For the de Rham complex, \( \dim \delta^k = b_k \), the \( k \)th Betti number. These are the most basic topological invariants of the domain.

\[
b_k = \begin{cases} 
\# \text{ components of } \Omega, & k = 0 \\
\# \text{ tunnels thru } \Omega, & k = 1 \\
\# \text{ voids in } \Omega, & k = 2 \\
0, & k = 3 
\end{cases}
\]

The de Rham complex in \( n \)-dimensions

Hilbert complex:

\[
0 \to L^2 \Lambda^0(\Omega) \overset{d}{\to} L^2 \Lambda^1(\Omega) \overset{d}{\to} \cdots \overset{d}{\to} L^2 \Lambda^{n-1}(\Omega) \overset{d}{\to} L^2 \Lambda^n(\Omega) \to 0
\]

Domain complex:

\[
0 \to H \Lambda^0(\Omega) \overset{d}{\to} H \Lambda^1(\Omega) \overset{d}{\to} \cdots \overset{d}{\to} H \Lambda^{n-1}(\Omega) \overset{d}{\to} H \Lambda^n(\Omega) \to 0
\]

Dual domain complex:

\[
0 \leftarrow \check{H} \Lambda^0(\Omega) \overset{d^*}{\leftarrow} \check{H} \Lambda^1(\Omega) \overset{d^*}{\leftarrow} \cdots \overset{d^*}{\leftarrow} \check{H} \Lambda^{n-1}(\Omega) \overset{d^*}{\leftarrow} \check{H} \Lambda^n(\Omega) \leftarrow 0
\]

Mixed form of Hodge Laplacian:

0: Poisson equation, usual \( H^1 \) formulation (\( \sigma = 0 \))
1: Vector Poisson equation, scalar–vector mixed method (\( \sigma = - \text{div } u \))

\( n-1 \): Vector Poisson equation, skew-matrix–vector mixed method (\( \sigma = \text{curl } u \))
n: Poisson equation, mixed formulation (\( \sigma = - \text{grad } u \))

Discretization

We now want to discretize the mixed formulation with f.d. subspaces \( V_h^k \subset V^k \) indexed by \( h \) (Galerkin). Of course we assume

\[
\inf_{v_h \in V_h^k} \| v - v_h \|_V \to 0 \quad \text{as } h \to 0 \quad \forall v \in V \quad (A)
\]

It turns out that there are two more key assumptions.

Subcomplex assumption (SC):
\[
d(V_h^k) \subset V_{h+1}^k
\]

The subcomplex

\[
\cdots \overset{d^1}{\to} V_h^0 \overset{d}{\to} V_h^1 \overset{d}{\to} V_{h+1}^1 \overset{d^1}{\to} \cdots
\]

is itself an H-complex so we have (discrete) harmonic forms \( H_h^k \), Hodge decomposition, and Poincaré inequality with constant \( c_{P,h} \).

Bounded Cochain Projection assumption (BCP):
\[
\exists \pi_h^k : V^k \to V_h^k
\]

\[
\cdots \quad \overset{d}{\to} V^k \overset{d}{\to} V_{h+1}^k \quad \overset{d}{\to} \cdots
\]

\[
\pi_h^k : V^k \text{ bounded, uniform in } h
\]

\[
\pi_h^k : \text{ a projection}
\]

\[
\pi_{h+1}^k d^h = d^h \pi_h^k
\]
Stability theorem

**Theorem**

Let \((V^k, d^k)\) be a Hilbert complex and \(V^k_h\) finite dimensional subspaces satisfying A, SC, and BCP. Then

- \(\pi_h\) induces an isomorphism on cohomology for \(h\) small
- \(\text{gap}(\pi^k_h, \tilde{\pi}^k_h) \to 0\)
- The discrete Poincaré inequality \(\|\omega\| \leq c\|d\omega\|, \ \omega \in \mathcal{Z}^k_h\), holds with \(c\) independent of \(h\)

*Proof of discrete Poincaré inequality:* Given \(\omega \in \mathcal{Z}^k_h\), define \(\eta \in \mathcal{Z}^k \subset V_h\) by \(d\eta = d\omega\). By the Poincaré inequality, \(\|\eta\| \leq c_p\|d\omega\|\), whence \(\|\eta\|_V \leq c'\|d\omega\|\), so it is enough to show that \(\|\omega\| \leq c'\|\eta\|_V\). Now, \(\omega - \pi_h\eta \in V^k_h\) and, by SC and BCP, \(d(\omega - \pi_h\eta) = d\omega - \pi_h d\omega = 0\), so \(\omega - \pi_h\eta \in \mathcal{Z}^k_h\). Thus \(\omega = \pi_h\omega\), so \(\|\omega\| \leq \|\pi_h\eta\|\) by Pythagoras. Result follows since \(\pi_h\) is bounded. Note \(c_{P,h} \leq (c_p^2 + 1)^{1/2} \|\pi_h\|\).

Galerkin’s method

\[ \sigma \in V^{k-1}, \quad u \in V^k, \quad p \in \tilde{\pi}^k : \]
\[ \langle \sigma, \tau \rangle - \langle d\tau, u \rangle = 0 \quad \forall \tau \in V^{k-1} \]
\[ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle = \langle f, v \rangle \quad \forall v \in V^k \]
\[ \langle u, q \rangle = 0 \quad \forall q \in \tilde{\pi}^k \]

\[ \sigma_h \in V^{k-1}_h, \quad u_h \in V^k_h, \quad p_h \in \tilde{\pi}^k_h : \]
\[ \langle \sigma_h, \tau \rangle - \langle d\tau, u_h \rangle = 0 \quad \forall \tau \in V^{k-1}_h \]
\[ \langle d\sigma_h, v \rangle + \langle du_h, dv \rangle + \langle p_h, v \rangle = \langle f, v \rangle \quad \forall v \in V^k_h \]
\[ \langle u_h, q \rangle = 0 \quad \forall q \in \tilde{\pi}^k_h \]

If \(\tilde{\pi}^k \nsubseteq V^k\), then \(\tilde{\pi}^k \nsubseteq \tilde{\pi}\), so this is a nonconforming method. The consistency error is related to \(\text{gap}(\tilde{\pi}^k, \tilde{\pi}^k_h)\).

Convergence of Galerkin’s method

From stability we get an estimate which is quasi-optimal plus a small consistency error term.

**Notation:** for \(w \in V^k\) let \(E(w) := \inf_{w \in V^k_h} \|w - v\|_V\)

**Theorem**

Assume SC and BCP. Then

\[ \|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\|_V \leq c \{ E(\sigma) + E(u) + E(p) + \epsilon \} \]

where \(\epsilon \leq \inf_{v \in V^k_h} E(P_{\tilde{\pi}}u) \times \sup_{r \in \tilde{\pi}^k} E(r)\).

Improved error estimates

Duality gives improved estimates. Two additional assumptions:

- \(V^k \cap V^k_h\) is a dense subset of \(W^k\) with compact inclusion (CI).
- The projections \(\pi^k_h\) are \(W\)-bounded uniformly in \(h\).

Using compactness and best approximation we define two quantities, \(\eta\) and \(\mu\) which are \(o(1)\). In applications \(\eta = O(h), \mu = O(h^{k+1})\).

**Theorem**

\[ \|d\sigma - d\sigma_h\| \leq c E(d\sigma) \]
\[ \|\sigma - \sigma_h\| \leq c\{ E(\sigma) + \eta E(d\sigma) \} \]
\[ \|p - p_h\| \leq c\{ E(p) + \mu E(d\sigma) \} \]
\[ \|du - du_h\| \leq c\{ E(du) + \eta [E(d\sigma) + E(p)] \} \]
\[ \|u - u_h\| \leq c\{ E(u) + \eta [E(du) + E(\sigma)] + (\eta^2 + \delta)[E(d\sigma) + E(p)] + \mu E(P_{\tilde{\pi}}u) \} \]
Based on the abstract theory we want finite element subspaces of $H^k$ which satisfy SC and BCP. A key finding of FEEC is that for simplicial meshes there are precisely two natural families: there are precisely two natural families:

- $P_r^k(T)$
- $P_r^{k-1}(T)$

Shape functions, degrees of freedom, and unisolvence can be treated uniformly. Shape functions for $P_r^k$ are $k$-forms with $P_r$ coefficients.

Equality for $k = 0$

Equality for $k = n$

**FE subcomplexes of the de Rham complex**

SC and BCP are obtained if we choose:

\[
\begin{cases} 
P_r^{k-1}(T) \\
\text{or} \\
\end{cases} \xrightarrow{d} \begin{cases} 
P_r^k(T) \\
\text{or} \\
\end{cases} \]

This leads to $2^{n-1}$ subcomplexes for each $r$. Extreme cases:

\[
0 \rightarrow P^-_r \Lambda^0(T) \xrightarrow{d} P^-_r \Lambda^1(T) \xrightarrow{d} \cdots \xrightarrow{d} P^-_r \Lambda^n(T) \rightarrow 0
\]

\[
0 \rightarrow \Lambda^0(T) \xrightarrow{\text{div}} \Lambda^1(T) \xrightarrow{\text{div}} \cdots \xrightarrow{\text{div}} \Lambda^n(T) \rightarrow 0
\]

\[
0 \rightarrow P^-_r \Lambda^0(T) \xrightarrow{d} P^-_r \Lambda^1(T) \xrightarrow{d} \cdots \xrightarrow{d} P^-_r \Lambda^n(T) \rightarrow 0
\]

\[
0 \rightarrow \Lambda^0(T) \xrightarrow{\text{grad}} \Lambda^1(T) \xrightarrow{\text{grad}} \cdots \xrightarrow{\text{grad}} \Lambda^n(T) \rightarrow 0
\]

\[
0 \rightarrow \Lambda^0(T) \xrightarrow{\text{curl}} \Lambda^1(T) \xrightarrow{\text{curl}} \cdots \xrightarrow{\text{curl}} \Lambda^n(T) \rightarrow 0
\]
Convergence of mixed FE for the $k$-form Hodge Laplacian

Thus we have four stable families of mixed method for the $k$-form Laplacian:

\[
\begin{align*}
\mathcal{P}^r \Lambda^{k-1}(T) & \times \mathcal{P}^r \Lambda^k(T) \\
\mathcal{P}^r \Lambda^{k-1}(T) & \times \mathcal{P}^{r-1} \Lambda^k(T) \\
\mathcal{P}^{r-1} \Lambda^{k-1}(T) & \times \mathcal{P}^r \Lambda^k(T) \\
\mathcal{P}^{r-1} \Lambda^{k-1}(T) & \times \mathcal{P}^{r-1} \Lambda^k(T)
\end{align*}
\]

For each one, the improved estimates are optimal order in each individual quantity (assuming sufficient smoothness).

More on $\mathcal{P}^r \Lambda^k$ and $\mathcal{P}^{r-1} \Lambda^k$: Rick Falk’s talk

More on BCP: Ragnar Winther’s talk

Additional applications

Eigenvalue problems

Returning to the abstract setting we can consider the eigenvalue problem: Find $\lambda \in \mathbb{R}$, $0 \neq (\sigma, u, p) \in V^{k-1} \times V^k \times S^k$:

\[
\begin{align*}
\langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0 & \forall \tau & \in V^{k-1} \\
\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \lambda \langle u, v \rangle & \forall v & \in V^k \\
\langle u, q \rangle &= 0 & \forall q & \in S^k
\end{align*}
\]

This includes, e.g., the Maxwell eigenvalue problems

\[
\text{curl curl } u = \lambda u, \quad \text{div } u = 0.
\]

Under the same assumptions as for the source problem (SC, BCP, CI), we obtain a complete convergence theory. (We do not explicitly use the Discrete Compactness Property or Fortin property.)

Other Hilbert complexes

- Variable coefficients. Let $W^k = L^2 \Lambda^k$ but with a weighted $L^2$ inner product. Take $V^k = H\Lambda^k$ and $d^*$ as before. The Hodge Laplacian is now a differential operator with variable coefficients. This allows us to treat, e.g., problems like the Maxwell eigenvalue problem for a general dielectric medium

\[
\text{curl } \mu^{-1} \text{curl } \mu = \lambda \epsilon u, \quad \text{div } \epsilon u = 0
\]

(with tensor coefficients).

- Boundary conditions. Again $W^k = L^2 \Lambda^k$, but now take $V^k = \tilde{H}\Lambda^k$:

\[
0 \rightarrow \tilde{H}\Lambda^0(\Omega) \xrightarrow{d} \tilde{H}\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \tilde{H}\Lambda^{n-1}(\Omega) \xrightarrow{d} \tilde{H}\Lambda^n(\Omega) \rightarrow 0
\]

In this way we can treat essential boundary conditions.
For $\Omega \subset \mathbb{R}^2$, consider the Hilbert complex

$$0 \rightarrow L^2(\Omega) \overset{\text{airy}}{\longrightarrow} L^2(\Omega; \mathbb{S}) \overset{\text{div}}{\longrightarrow} L^2(\Omega; \mathbb{R}^2) \rightarrow 0,$$

where \( \text{airy} = \left( \begin{array}{cc} \partial_y^2 & -\partial_x \partial_y \\ -\partial_x \partial_y & \partial_x^2 \end{array} \right) \).

The domain complex is

$$0 \rightarrow H^2(\Omega) \overset{\text{airy}}{\longrightarrow} H(\text{div}, \Omega; \mathbb{S}) \overset{\text{div}}{\longrightarrow} L^2(\Omega; \mathbb{R}^2) \rightarrow 0$$

This is a closed Hilbert complex. If we weight the $L^2(\Omega; \mathbb{S})$ inner product by the compliance tensor, the rightmost Hodge Laplacian is the mixed elasticity system.

### Conclusions

- Exterior calculus, de Rham cohomology, and Hodge theory capture the structure behind the well-posedness of various elliptic PDE.
- Hilbert complexes isolate the most relevant features, and provide a framework for studying Galerkin discretizations.
- Two basic properties, SC & BCP, are the key to capturing cohomology and to obtaining stability of discretizations.
- The finite element spaces $P_r \Lambda^k$ and $P_r \Lambda^k$ spaces are the natural finite element discretizations of $H\Lambda^k$ and can be combined into finite element de Rham subcomplexes satisfying SC & BCP.
- These spaces can be used to solve the Hodge Laplacian and many related problems, including Maxwell’s equations. They unify, clarify, and refine many known finite element methods, and allow for a clean, uniform, rigorous analysis.
- Applied to another complex, the FEEC approach has lead to the solution of the long-standing search for mixed finite elements for elasticity.