

Scalar products and reconstructions in MFD

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Non-Standard Numerical Methods for PDE's - INdAM - 3

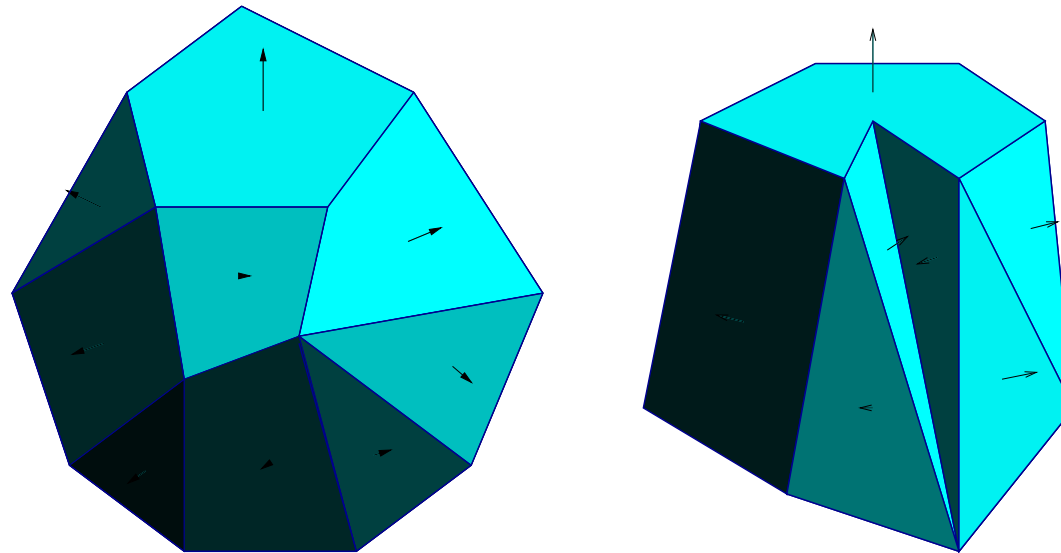
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PLAN OF THE TALK

- General shapes
- The unknowns
- The local stiffness matrix
- Conforming and Nonconforming Reconstructions
- What has been done - What has to be done
- Conclusions

PATAHEDRA

In all this presentation we shall deal with **elements** of a **very general shape** (as the **pata-hedra** here below):



Two examples of pata-hedral elements

REGULARITY OF THE PATAHEDRA IN \mathcal{T}_h

We assume that **there exists** a *compatible* sub-decomposition \mathcal{S}_h into *shape-regular* tetrahedra, and two positive numbers N and ρ , such that

- every patahedron $P \in \mathcal{T}_h$ admits a decomposition $\mathcal{S}_h|_P$ made of less than N tetrahedra.
- The shape-regularity of the tetrahedra $K \in \mathcal{S}_h$ is defined as follows: the ratio between the radius r_K of the inscribed sphere and the diameter h_K of the element K is bounded below by ρ :

$$\frac{r_K}{h_K} \geq \rho > 0.$$

TOY PROBLEMS

$$\int_{\Omega} \mu \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{v} \, d\Omega + \int_{\Omega} \omega^2 \mathbf{u} \cdot \mathbf{v} \, d\Omega = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\Omega \quad \forall \mathbf{v} \in H(\operatorname{curl}; \Omega).$$

$$\int_{\Omega} \mathbb{K} \nabla p \cdot \nabla q \, dx = \int_{\Omega} b q \, dx \quad \forall q \in H_0^1(\Omega).$$

$$\int_{\Omega} \mathbb{K}^{-1} \mathbf{u} \cdot \mathbf{v} \, dx = \int_{\Omega} p \operatorname{div} \mathbf{v} \, dx \quad \forall \mathbf{v} \in H(\operatorname{div}; \Omega).$$

$$\int_{\Omega} \operatorname{div} \mathbf{u} q \, dx = \int_{\Omega} b q \, dx \quad \forall q \in L^2(\Omega).$$

CHOICE OF Finite Element SPACES

If you had a decomposition into *triangles* or *tetrahedra* you would choose

- Piecewise linear functions for approximating $H_0^1(\Omega)$
- Edge (e.g. Nédélec) elements for approximating $H(\text{curl}; \Omega)$
- Face (e.g. RT or BDM) elements for approximating $H(\text{div}; \Omega)$
- Piecewise constant functions for approximating $L^2\Omega$

On *hexahedra* you will survive will similar choices.

What to do on *patahedra*?

0-COCHAINS

Let us start from the simplest case. The old good variational formulation for second order linear elliptic problems.

Setting $\mathcal{V} := H_0^1(\Omega)$ define as usual

$$a(p, q) := \int_{\Omega} \mathbb{K} \nabla p \cdot \nabla q \, dx.$$

We then formulate the problem as *find* $p \in \mathcal{V}$ *such that*:

$$a(p, q) = \int_{\Omega} b q \, dx \quad \forall q \in \mathcal{V}$$

and proceed with the discretization.

DISCRETIZATION OF THE PRIMAL FORMULATION

Assume now that we are given a decomposition of Ω into *patahedra*. Let \mathcal{N} be the set of **vertices** of the decomposition, and let N be its cardinality. Let finally \mathcal{V}_h be *a finite dimensional subspace of \mathcal{V}* such that **the mapping**

$$q \rightarrow \text{value of } q \text{ at the vertices}$$

is an isomorphism from \mathcal{V}_h to \mathbb{R}^N (these are the *shape functions*).

Then the discrete problem **would be find $p_h \in \mathcal{V}_h$ such that:**

$$a(p_h, q_h) = \int_{\Omega} b q_h dx \quad \forall q_h \in \mathcal{V}_h.$$

The only *little trouble* is that *we don't have \mathcal{V}_h ...*

THE COCHAIN POINT OF VIEW

An alternative point of view would be to consider that the space \mathcal{V}_h *does not exist*, and the unknown p_h is in fact a **nodal function** as is done in the **finite difference** framework.

The differences with a purely finite difference approach are that

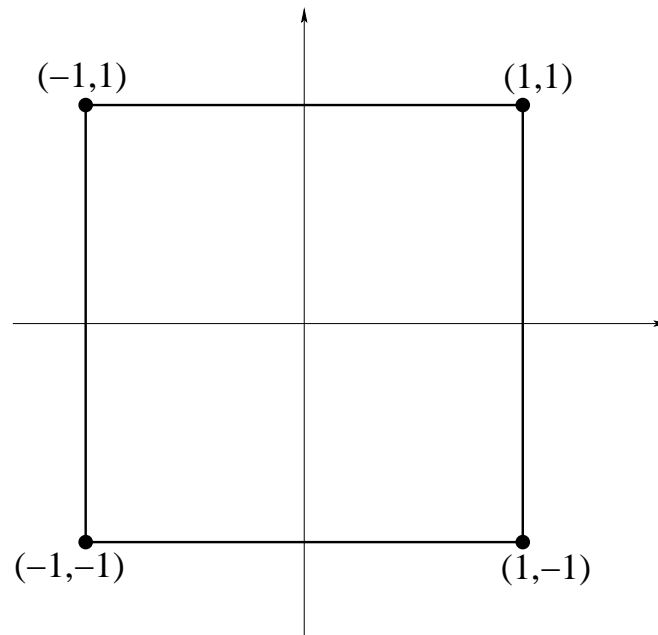
- we have much more freedom in the choice of the nodes within the domain Ω (no *regular grid*)
- we shall pay the above generality with the necessity of *cooking up* a sort of H_0^1 -*scalar product* among nodal functions.

We shall see how to cook up such a scalar problem on a very simple example.

SPECIAL CASE

For the sake of simplicity (and, I hope, clarity) in the next slides we shall consider a particularly simple case.

We assume that $\mathbb{K} = \text{identity}$, and we shall discuss the local stiffness matrix just on the **element** $[-1, 1]^2$.



THE PATCH-TEST

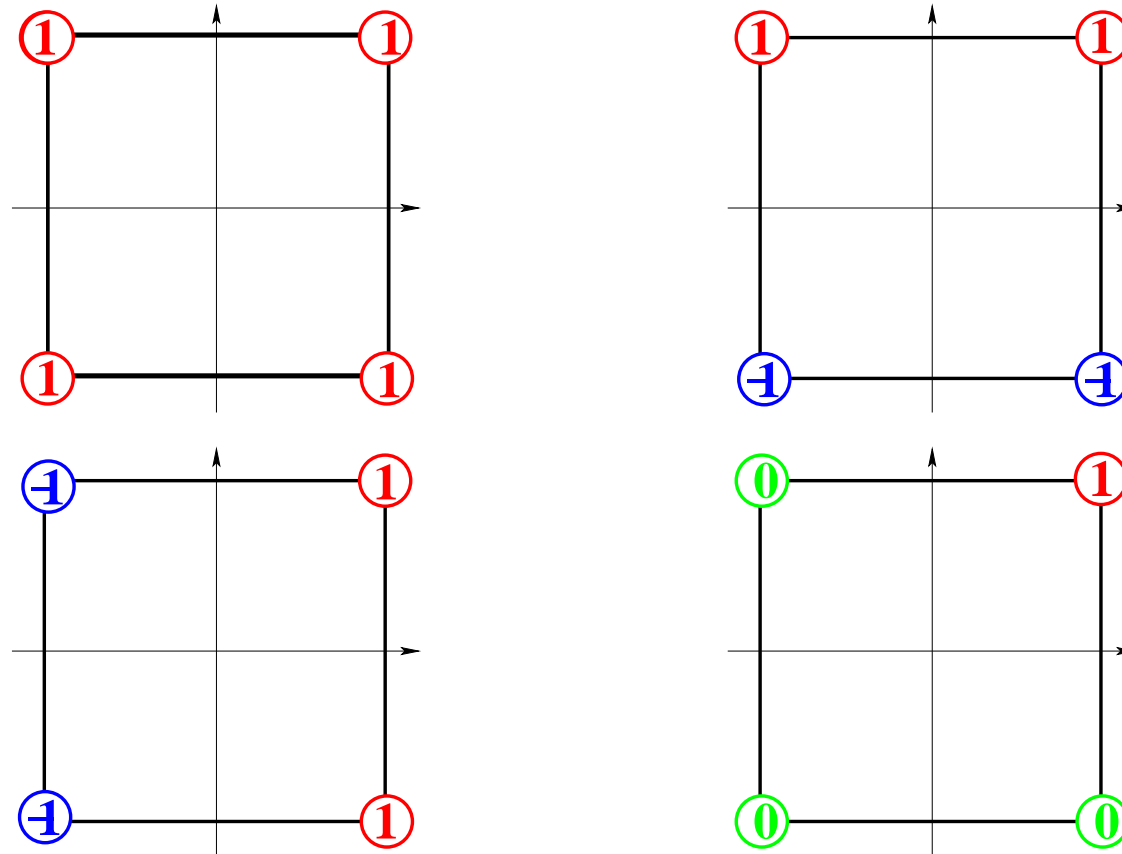
The patch test (introduced by B. Irons) states (vaguely) that in order to be good a Finite Element Method **must** reproduce the **exact solution** on a general patch of elements whenever the exact solution corresponds to a state of **constant stresses**

Its interpretation is reasonably clear in elasticity problems, with some doubts here and there, but much less clear in more general cases.

Here however it seems clear that we must reproduce correctly the exact solution whenever the exact solution is **linear**.

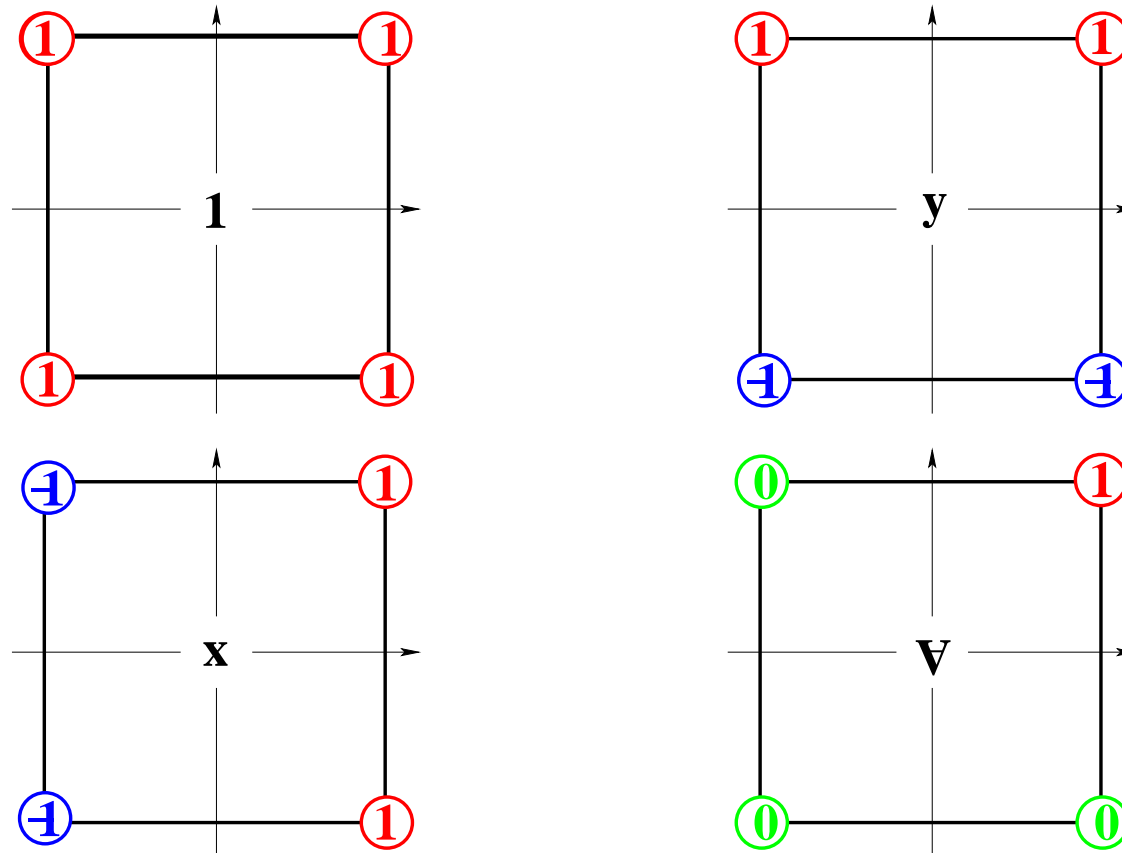
CONSTRUCTION OF THE LOCAL STIFFNESS MATRIX

The **nodal values of linear functions** will play a special role



CONSTRUCTION OF THE LOCAL STIFFNESS MATRIX

We might sometimes consider that we have a function inside, but the basic idea is that our unknowns are the NODAL values



CONSTRUCTION OF THE LOCAL STIFFNESS MATRIX

We want to preserve the *good* scalar product

$$\int_Q \nabla u \cdot \nabla v \, dQ$$

for the (nodal values of the) linear functions.

$$\mathbf{v}^{(1)}=(1,1,1,1) \quad \mathbf{v}^{(2)}=(-1,-1,1,1) \quad \mathbf{v}^{(3)}=(1,-1,-1,1) \quad \mathbf{v}^{(4)}=(0,0,0,1)$$

0	0	0	0
0	 Q 	0	?
0	0	 Q 	?
0	?	?	?

We also note that the result of

$$\int_Q \nabla p_1 \cdot \nabla \varphi \, dQ$$

for p_1 polynomial of degree ≤ 1 and φ generic in $H^1(Q)$ depends only on the boundary values of φ , or, rather, on the **mean value** of φ on each edge. More precisely, if

$$\int_{\ell_i} \varphi \, dl = 0 \quad \text{for every edge } \ell_i$$

then

$$\int_Q \nabla p_1 \cdot \nabla \varphi \, dQ = \sum_i \int_{\ell_i} \frac{\partial p_1}{\partial \mathbf{n}_i} \varphi \, dl = 0.$$

CONSTRUCTION OF THE LOCAL STIFFNESS MATRIX

We assume (for the moment) that our functions are **linear on each edge**. Hence their mean value on each edge depends solely on the nodal values and the scalar product between a linear function and any other nodal function can be computed

$$v^{(1)}=(1,1,1,1) \quad v^{(2)}=(-1,-1,1,1) \quad v^{(3)}=(1,-1,-1,1) \quad v^{(4)}=(0,0,0,1)$$

0	0	0	0
0	 Q 	0	?
0	0	 Q 	?
0	?	?	?

0	0	0	0
0	 Q 	0	e/2
0	0	 Q 	e/2
0	e/2	e/2	?

CONSTRUCTION OF THE LOCAL STIFFNESS MATRIX

At this point, for a polygon with k nodes, it is not difficult to take a basis in \mathbb{R}^k that includes the (nodal values of the) linear functions plus $k - 3$ vectors that are "orthogonal" to the previous three.

$$\mathbf{v}^{(1)} = (1, 1, 1, 1)$$

$$\mathbf{v}^{(2)} = (-1, -1, 1, 1)$$

0	0	0	0
0	 Q 	0	?
0	0	 Q 	?
0	?	?	?

$$\mathbf{v}^{(3)} = (1, -1, -1, 1)$$

$$\mathbf{v}^{(4)} = (-1, 1, -1, 1)$$

0	0	0	0
0	 Q 	0	0
0	0	 Q 	0
0	0	0	?

CONSTRUCTION OF THE LOCAL STIFFNESS MATRIX

The scalar product among the $k - 3$ "additional" vectors remains uncertain. If we think to "nodal values of given functions", it will obviously depend on the choice of such functions.

$$\mathbf{v}^{(1)} = (1, 1, 1, 1) \text{ function } 1$$

$$\mathbf{v}^{(3)} = (1, -1, -1, 1) \text{ function } y$$

$$\mathbf{v}^{(2)} = (-1, -1, 1, 1) \text{ function } x$$

$$\mathbf{v}^{(4)} = (-1, 1, -1, 1) \text{ function } xy + \varphi$$

0	0	0	0
0	 Q 	0	0
0	0	 Q 	0
0	0	0	?

CONSTRUCTION OF THE LOCAL STIFFNESS MATRIX

On our super-special case (of the square $[-1, 1]^2$) we can take the fourth function to be xy

$$\mathbf{v}^{(1)} = (1, 1, 1, 1) \text{ function } 1$$

$$\mathbf{v}^{(3)} = (1, -1, -1, 1) \text{ function } y$$

$$\mathbf{v}^{(2)} = (-1, -1, 1, 1) \text{ function } x$$

$$\mathbf{v}^{(4)} = (-1, 1, -1, 1) \text{ function } xy$$

0	0	0	0
0	 Q 	0	0
0	0	 Q 	0
0	0	0	8/3

CONSTRUCTION OF THE LOCAL STIFFNESS MATRIX

However, if we want to think that $(-1, 1, -1, 1)$ are the nodal values of a function that is linear on each boundary edge, then $8/3$ is the *minimum* value that we can take. Why?

$$v^{(1)} = (1, 1, 1, 1) \text{ function } 1$$

$$v^{(3)} = (1, -1, -1, 1) \text{ function } y$$

$$v^{(2)} = (-1, -1, 1, 1) \text{ function } x$$

$$v^{(4)} = (-1, 1, -1, 1) \text{ function } xy + \varphi$$

0	0	0	0
0	 Q 	0	0
0	0	 Q 	0
0	0	0	$\frac{8}{3} + \ \nabla\varphi\ _0^2$

VIRTUAL BASIS FUNCTIONS

Assume that we decide to take the inner product of $(-1, 1, -1, 1)$ with itself **equal to 8**. It is immediate to show that there exists a function ψ (of the form $\psi = xy + \varphi$ for $\varphi \in H_0^1(Q)$) such that

$$\int_Q |\nabla \psi|^2 dQ = 8.$$

$v^{(1)}=(1,1,1,1)$ **function 1**

$v^{(3)}=(1,-1,-1,1)$ **function y**

$v^{(2)}=(-1,-1,1,1)$ **function x**

$v^{(4)}=(-1,1,-1,1)$ **function xy+φ**

0	0	0	0
0	 Q 	0	0
0	0	 Q 	0
0	0	0	$\frac{8}{3} + \nabla \varphi _0^2$

VIRTUAL BASIS FUNCTIONS?

Assume now that we decide to take the inner product of $(-1, 1, -1, 1)$ with itself equal to 2. As $2 < 8/3$ there are **no** functions ψ (of the form $\psi = xy + \varphi$ for $\varphi \in H_0^1(Q)$) such that

$$\int_Q |\nabla \psi|^2 dQ = 2.$$

$v^{(1)}=(1,1,1,1)$ **function 1**

$v^{(3)}=(1,-1,-1,1)$ **function y**

$v^{(2)}=(-1,-1,1,1)$ **function x**

$v^{(4)}=(-1,1,-1,1)$ **function $xy+\varphi$**

0	0	0	0
0	 Q 	0	0
0	0	 Q 	0
0	0	0	$\frac{8}{3} + \nabla \varphi _0^2$

NONCONFORMING BASIS FUNCTIONS

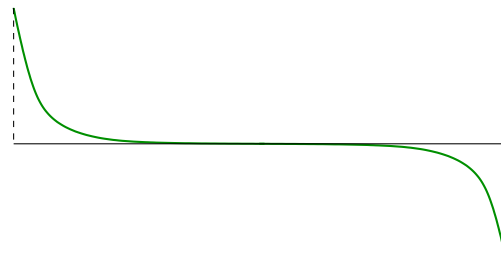
However one can prove that: For every $\alpha > 0$ there exists a function $\psi = xy + \varphi$ with

$$\int_{l_i} \varphi \, dl = 0 \quad \text{for all edges } l_i \quad (i = 1, \dots, 4).$$

such that

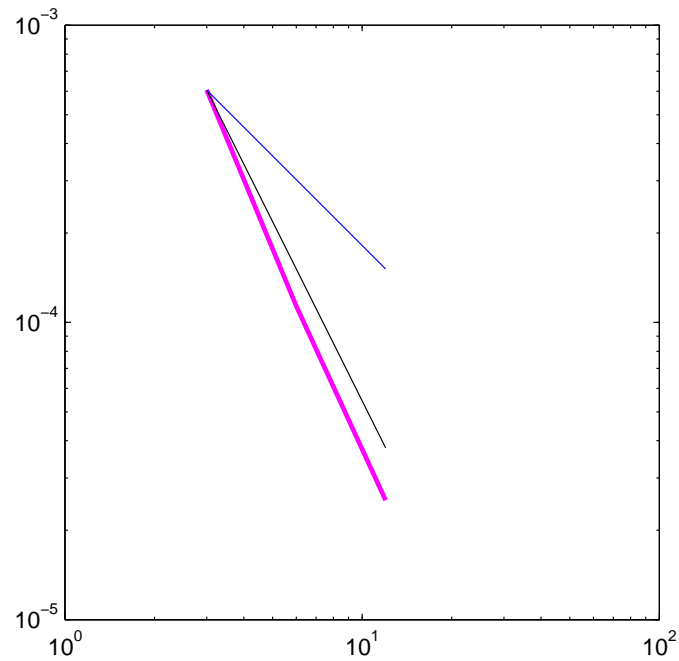
$$\int_Q |\nabla \psi|^2 = \alpha.$$

Hint: use the density of $\mathcal{D}(l_i)$ into $H^{1/2}(l_i)$...



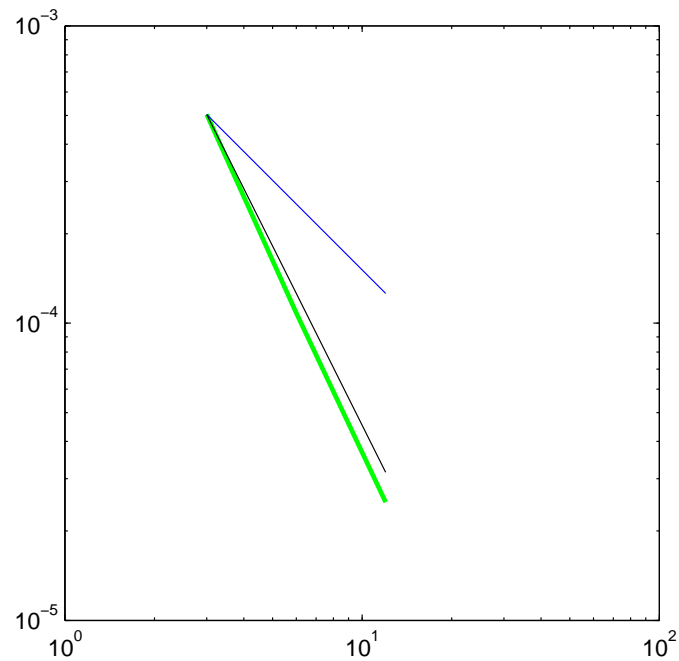
DIFFERENT CHOICES OF THE COEFFICIENT

The coefficient $8/3$ corresponds to the function xy (and to the use of Q_1 shape functions).



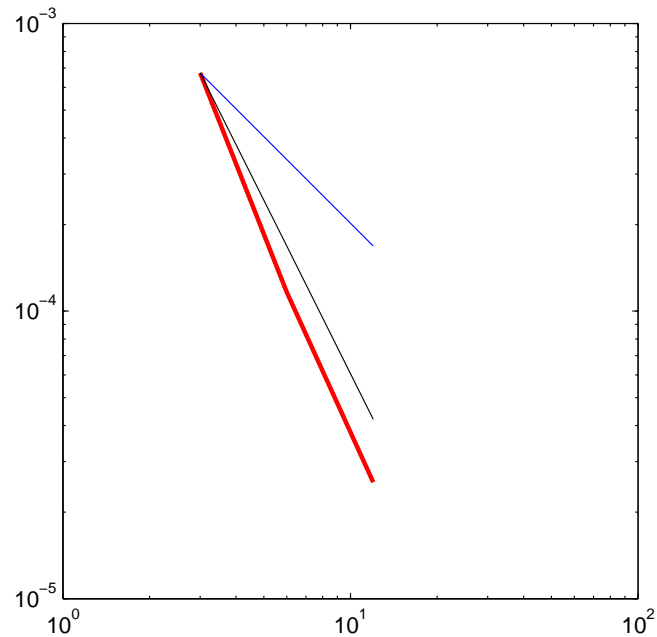
DIFFERENT CHOICES OF THE COEFFICIENT

The coefficient δ (giving the five-points scheme) corresponds to a suitable function φ in $H_0^1(Q)$ (and to *some* conforming shape functions).

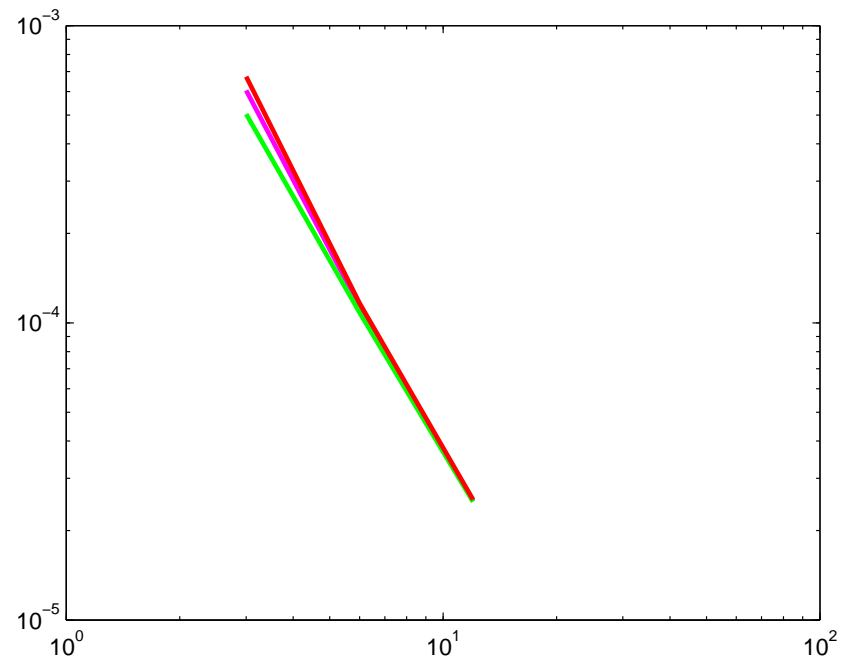


DIFFERENT CHOICES OF THE COEFFICIENT

The coefficient 0 is the **limit** of functions φ not in $H_0^1(Q)$ (but having zero mean value on each edge). In general a coefficient $0 < \alpha < 8/3$ corresponds to the use of *some* nonconforming shape functions.

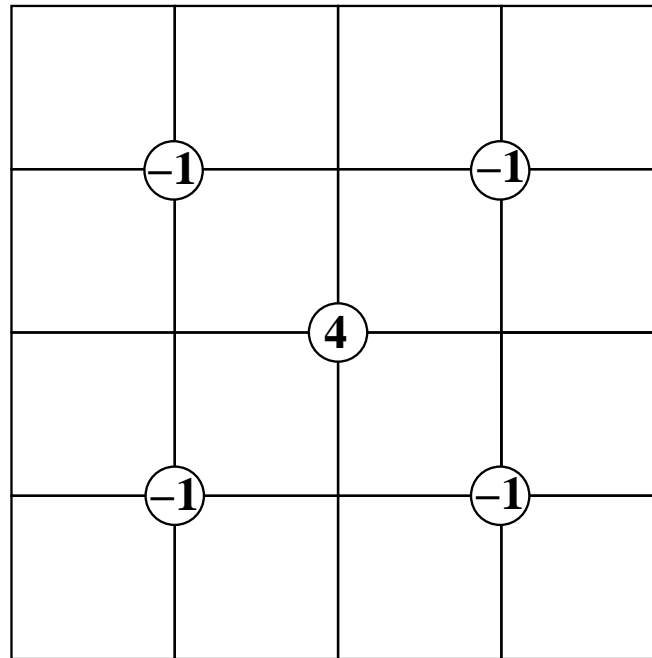


COMPARISON



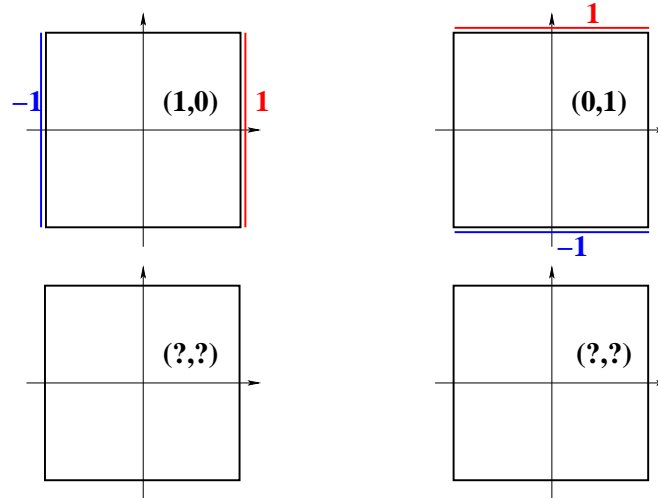
DIFFERENT CHOICES OF THE COEFFICIENT

Actually, the coefficient 0 gives rise to the (unusual, but not totally crazy) diagonal stencil



What to do for **face** or **edge** cochains?

We start with the face elements. Here the unknowns are the fluxes (= integral of the normal component) of vectors on the different faces. Still considering the square $] - 1, 1[^2$ we start from **the fluxes of constant vectors**. In 2 dimensions the constant vectors are $(1, 0)$ and $(0, 1)$:



LOCAL STIFFNESS MATRIX

The local stiffness matrix now looks like

$$\mathbf{v}^{(1)} = (0, -1, 0, 1) \quad \mathbf{v}^{(2)} = (1, 0, -1, 0) \quad \mathbf{v}^{(3)} = (?, ?, ?, ?) \quad \mathbf{v}^{(4)} = (?, ?, ?, ?)$$

 Q 	0	?	?
0	 Q 	?	?
?	?	?	?
?	?	?	?

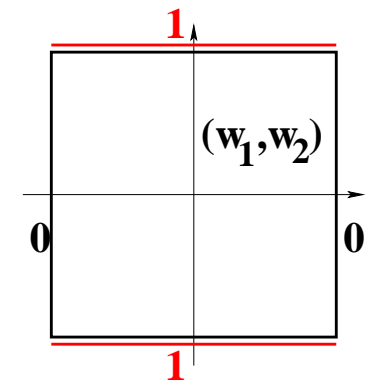
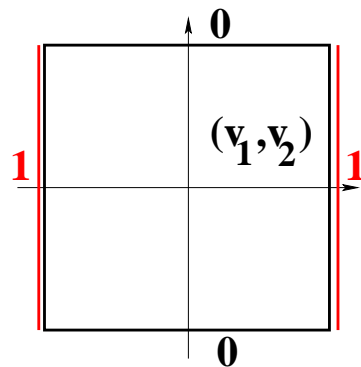
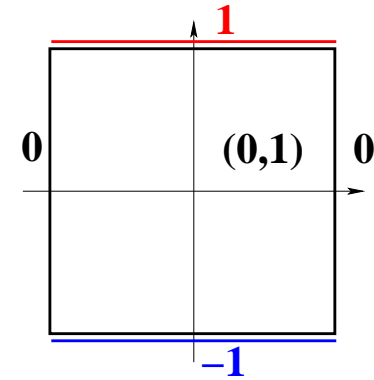
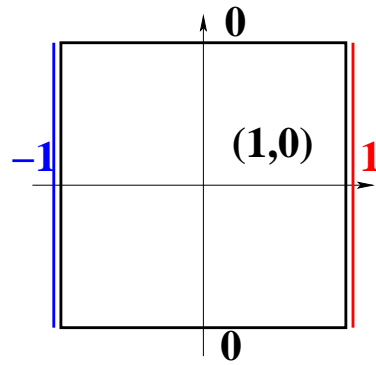
Again, we observe that the scalar product of a constant vector (say, $(1, 0)$) with any other vector $\boldsymbol{\tau}$ *having constant divergence and constant normal component at the boundary edges* depends solely **on the fluxes of $\boldsymbol{\tau}$** . Indeed, noting first that $(1, 0) = \text{grad}x$ we have

$$\begin{aligned} \int_Q \text{grad}x \cdot \boldsymbol{\tau} dx dy &= - \int_Q x \text{div} \boldsymbol{\tau} dx dy + \int_{\partial Q} x \boldsymbol{\tau} \cdot \mathbf{n} dl \\ &= \int_{\partial Q} x \boldsymbol{\tau} \cdot \mathbf{n} dl = \int_{x=1} \tau_1 dl + \int_{x=-1} \tau_1 dl \end{aligned}$$

We observe that we just need $\text{div} \boldsymbol{\tau}$ even in x and in y .

FACE COCHAINS

We can easily choose the fluxes of two additional vectors that are "orthogonal" to the constant vectors



LOCAL STIFFNESS MATRIX

The local stiffness matrix will now look like

$$\mathbf{v}^{(1)} = (0, -1, 0, 1) \quad \mathbf{v}^{(2)} = (1, 0, -1, 0) \quad \mathbf{v}^{(3)} = (0, 1, 0, 1) \quad \mathbf{v}^{(4)} = (1, 0, 1, 0)$$

 Q 	0	0	0
0	 Q 	0	0
0	0	?	?
0	0	?	?

LOCAL STIFFNESS MATRIX

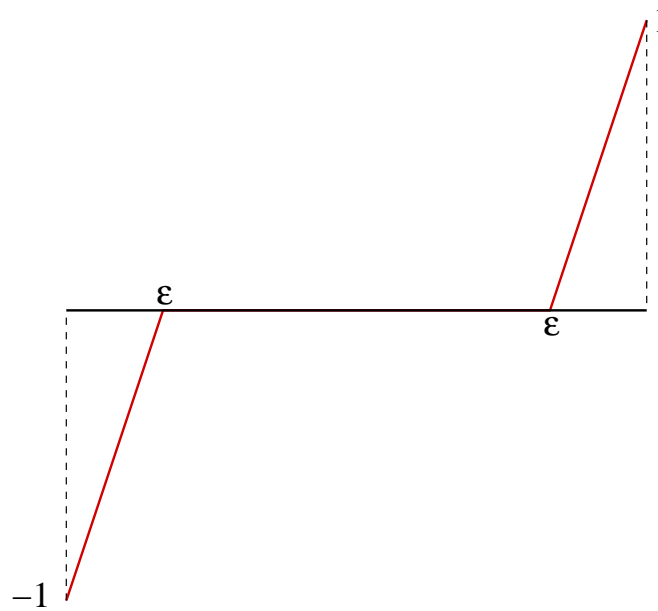
We may ask if there are minimum values for ε that ensure *conforming reconstructions*

$$\mathbf{v}^{(1)} = (0, -1, 0, 1) \quad \mathbf{v}^{(2)} = (1, 0, -1, 0) \quad \mathbf{v}^{(3)} = (0, 1, 0, 1) \quad \mathbf{v}^{(4)} = (1, 0, 1, 0)$$

$ \mathbf{Q} $	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{0}$	$ \mathbf{Q} $	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$	ε	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	ε

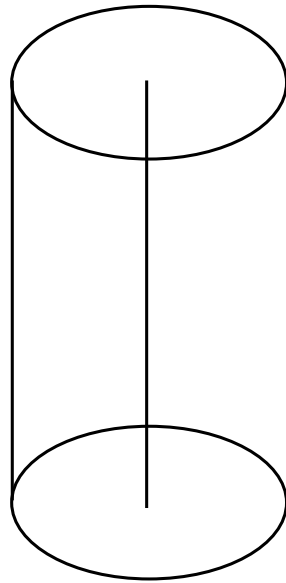
LOCAL STIFFNESS MATRIX

Can we have functions with small L^2 norm and boundary values ± 1 ?
At best, the derivative can be *bounded in L^1*

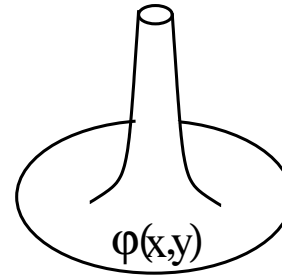


SMALL EDGE ELEMENTS

Similarly, one can have small edge vectors, with curl bounded in L^2



$$\mathbf{V} = (0, 0, \varphi(x, y)\chi(z))$$



MOSTLY DONE

- Basic scalar products for Cochains in 2 and 3 dimensions.
- Laplacian: Primal formulation, mixed formulation, eigenvalues.
- Laplacian: Basic *a posteriori* estimates.
- Stokes: (u, p) formulation
- Linear elasticity: Mixed formulation with reduced symmetry.
- Curved Faces: only for Laplacian in mixed formulation (very good results!)

MOSTLY TO BE DONE

- Curved Faces in general situations.
- Laplacian: advanced a posteriori estimates and adaptive strategies.
- Nonlinear elasticity, Elastoplasticity.
- Advection dominated passive transport.
- Oseen.
- Magnetostatic problems: direct and eigenvalues.
- **Best use of the freedom** in the choice of the coefficients.

CONCLUSIONS

- The Mimetic approach allows very general geometries. The treatment of **non-matching grids** is also allowed, and the **refining** and **coarsening** procedures are facilitated.
- The cases when conforming (or nonconforming) reconstructions exists are now better understood, but not fully solved.
- We still have to understand how to make the **best use** of the freedom allowed by it. This includes the **choice of the geometry**, as well as the **choice of the *free part* of the local stiffness matrix**.
- The treatment of **curved faces** is the possibly the biggest success of the approach so far, but it implies a number of open questions.