Mimetic Finite Difference methods for convection-diffusion problems

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with

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Outline

1. Mixed low-order MFD
    - MFD discretization
    - Error estimates
2. Nodal first-order MFD method
    - MFD discretization
    - SD-MFD
    - Numerical tests
3. A FE interpretation of nodal MFDs

Convection-diffusion in mixed form

[C. & Manzini & Russo, SINUM, 2009]

Model probelm: $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ Lipschitz bounded domain

$$\begin{aligned} \vec{F} &= -(\mathbf{K}\,\nabla p - \beta p) && \text{in} \quad \Omega \\ \mathrm{div}\,\vec{F} + \mathrm{c}p &= \mathrm{b} && \text{in} \quad \Omega \\ p &= 0 && \text{on} \quad \partial\Omega \end{aligned}$$

Variational form $W = L^2(\Omega),\ V = H(\mathrm{div};\Omega)$

Find $(p, \vec{F}) \in W \times V$ s.t.

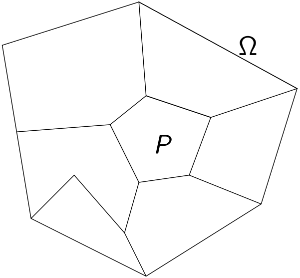
$$\begin{aligned} (\alpha\vec{F}, \vec{G}) - (p, \mathrm{div}\vec{G}) - (\alpha\beta p, \vec{G}) &= \quad 0 && \forall \vec{G} \in V \\ (\mathrm{div}\vec{F}, q) \qquad\qquad + (\mathrm{c}p, q) &= (\mathrm{b}, q) && \forall q \in W \end{aligned}$$

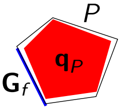
where $\alpha := \mathbf{K}^{-1}$.

Mixed MFD formulation

[Brezzi, Lipnikov, Shashkov, SINUM, 2005]

- $\mathcal{T}_h$ MFD-type partitions of Ω into polygonal (polyhedral) elements.





- Discrete pressure space Q_h

$$Q_h = \{\mathbf{q} = \{q_P \in \mathbb{R}\}_{P\in\mathcal{T}_h}\}. \qquad \text{For } q \in W, q^{\mathrm{I}} \in Q_h : \; (q^{\mathrm{I}})_P = \frac{1}{|P|}\int_P q$$

- Discrete flux space X_h

$$X_h = \{\mathbf{G} = \{G_f \in \mathbb{R}\}_{f\in\partial\mathcal{T}_h}\}. \qquad \text{For } \vec{G} \in V, \vec{G}^{\mathrm{I}} \in X_h : \; \left(\vec{G}^{\mathrm{I}}\right)_f = \frac{1}{|f|}\int_f \vec{G}\cdot\vec{n}_f$$

$\vec{n}_f$ unit normal vector assigned to f

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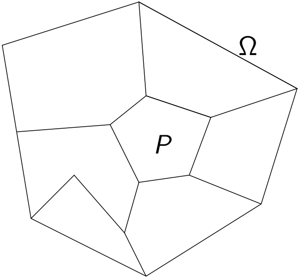

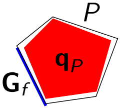

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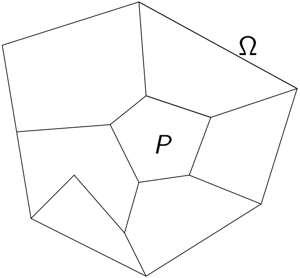

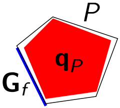

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MFD discretization

Mimetic Finite Difference (MFD): find $(\mathbf{p}, \mathbf{F}) \in Q_h \times X_h$ such that

$$\begin{aligned} &[\mathbf{F}, \mathbf{G}]_{X_h} - [\mathbf{p}, \mathrm{div}_h \mathbf{G}]_{Q_h} + \sum_{P \in \mathcal{T}_h} p_P \left[\beta^{\mathrm{I}}, \mathbf{G}\right]_P = 0 && \forall \mathbf{G} \in X_h \\ &[\mathrm{div}_h \mathbf{F}, \mathbf{q}]_{Q_h} + [\mathrm{c}^{\mathrm{I}} \mathbf{p}, \mathbf{q}]_{Q_h} = [\mathrm{b}^{\mathrm{I}}, \mathbf{q}]_{Q_h} && \forall \mathbf{q} \in Q_h \end{aligned}$$

- Discrete divergence $\mathrm{div}_h : X_h \to Q_h$ given on each $P \in \mathcal{T}_h$ by

$$\mathrm{div}_h \mathbf{G}|_P = \frac{1}{|P|} \sum_{f \in \partial P} \sigma_f^P |f| G_f$$

- Pressure space Q_h scalar product

$$[\mathbf{p}, \mathbf{q}]_{Q_h} = \sum_{E \in \Omega_h} |P| p_P q_P$$

- Flux space X_h scalar product

$$[\mathbf{F}, \mathbf{G}]_{X_h} = \sum_{E \in \Omega_h} [\mathbf{F}, \mathbf{G}]_P$$

in terms of any stable and consistent local scalar product $[\cdot, \cdot]_P$.

$\mathcal{P}_0$-compatible reconstructions

Set $X_P = X_h|_P$. We name $\mathcal{P}_0$-compatible reconstruction a linear map $\mathcal{R}^P : X_P \to L^2(P)$ such that for all $\mathbf{G} \in X_E$,

1. $\mathcal{R}^P \mathbf{G}|_f \cdot \mathbf{n}_f = G_f, \qquad \forall f \in \partial P,$
2. $\mathrm{div} \mathcal{R}^P \mathbf{G} = \mathrm{div}_h \mathbf{G}|_P,$

and, for all constant vector $\vec{C}$,

3. $\mathcal{R}_P \vec{C}^{\mathrm{I}} = \vec{C}.$

Let M_P be the (symmetric and positive definite) matrix defining the local flux scalar product, and let $\lambda_{\min}$ be its smallest eigenvalue.

Theorem [Brezzi, Lipnikov, Shashkov, Simoncini, CMAME, 2007]. There exists $\alpha_0 > 0$ such that if $\lambda_{\min} \geq \alpha$ then there exists a $\mathcal{P}_0$-compatible reconstruction $\mathcal{R}^P$ such that

$$[\mathbf{F}, \mathbf{G}]_P = \int_P (\overline{\mathbf{K}})^{-1} \mathcal{R}^P \mathbf{F} \cdot \mathcal{R}^P \mathbf{G} \, dP.$$

Error estimates

Assume that the scalar product for X_h is given by a $\mathcal{P}_0$-compatible reconstruction. Then, we can re-write the MFD method in terms of L^2:

$$\begin{aligned}
&(\bar\alpha\mathcal{R}\mathbf{F},\mathcal{R}\mathbf{G}) - (\bar{\mathbf{p}},\mathrm{div}\ \mathcal{R}\mathbf{G}) - (\bar\alpha\mathcal{R}\beta^{\mathrm{I}}\bar{\mathbf{p}},\mathcal{R}\mathbf{G}) &&= 0 &&\forall \mathbf{G}\in X_h\\
&(\mathrm{div}\ \mathcal{R}\mathbf{F},\bar{\mathbf{q}}) \qquad\qquad\qquad\qquad + ((\overline{\mathrm{c}^{\mathrm{I}}\mathbf{p}}),\bar{\mathbf{q}}) &&= (\overline{\mathrm{b}^{\mathrm{I}}},\bar{\mathbf{q}}) &&\forall \mathbf{q}\in Q_h
\end{aligned}$$

and analyze it *as* a mixed finite-element method (a la Douglas & Roberts '85).

Theorem. Let Ω convex Lipschitz continuous domain, $\mathcal{T}_h$ MFD-type partitions of Ω, $\mathbf{K}$ strongly elliptic, $(1/2)\mathrm{div}\beta + c \geq 0$.
If the scalar product of X_h is related to a stable $\mathcal{P}_0$-compatible reconstruction then, for h small enough,

$$||p-\bar{\mathbf{p}}||_0 + ||\vec{F}-\mathcal{R}\mathbf{F}||_{div} \leq C\,h\left(||p||_1 + h||p||_2 + ||\mathrm{b}-\mathrm{b}^{\mathrm{I}}||_0\right)$$

If, moreover, $\mathbf{K}$ is piecewise constant and $\beta = \mathcal{R}\beta^{\mathrm{I}}$, then

$$||p^{\mathrm{I}}-\bar{\mathbf{p}}||_0 \leq C\,h^2\left(h||p||_2 + ||\mathrm{b}||_1\right) \qquad \text{(pressure super-convergence)}$$

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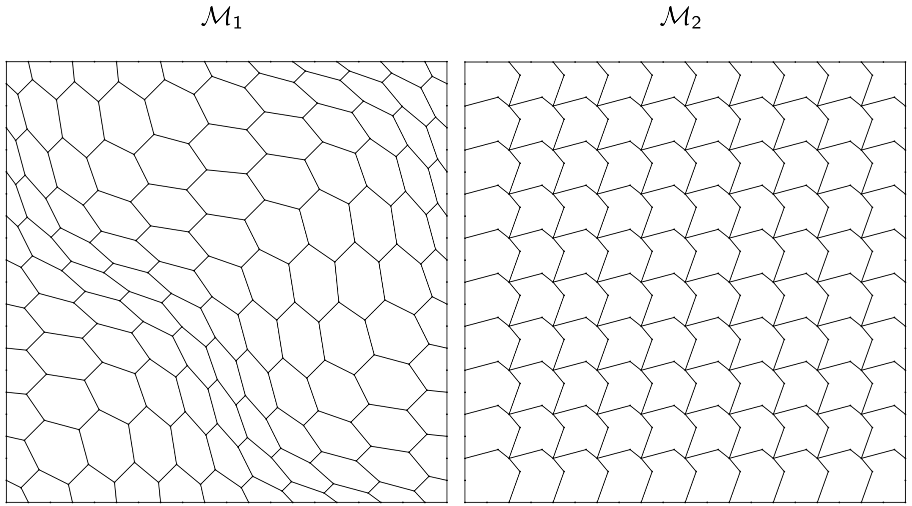
Meshes used for numerical tests



Pressure error

$\Omega = (0,1)^2$, $\mathbf{K} = \mathbf{I}$, $\beta = (1,3)^T$, $c = xy^2$, and $p = \sin(2\pi x)\sin(2\pi y) + x^2 + y^2 + 1$

	Mesh $\mathcal{M}_1$		Mesh $\mathcal{M}_2$	
n	$\|\|p - \bar{\mathbf{p}}\|\|_0^{rel}$	rate	$\|\|p - \bar{\mathbf{p}}\|\|_0^{rel}$	rate
10	$9.134\,10^{-2}$	$--$	$8.095\,10^{-2}$	$--$
20	$4.630\,10^{-2}$	1.007	$3.964\,10^{-2}$	1.029
40	$2.315\,10^{-2}$	1.012	$1.966\,10^{-2}$	1.011
80	$1.164\,10^{-2}$	0.995	$9.810\,10^{-3}$	1.003
160	$5.841\,10^{-3}$	0.995	$4.902\,10^{-3}$	1.000
320	$2.927\,10^{-3}$	0.996	$2.451\,10^{-3}$	1.000
	$\|\|p^1 - \bar{\mathbf{p}}\|\|_0^{rel}$	rate	$\|\|p^1 - \mathbf{p}\|\|_0^{rel}$	rate
10	$3.069\,10^{-2}$	$--$	$1.813\,10^{-2}$	$--$
20	$1.078\,10^{-2}$	1.551	$5.250\,10^{-3}$	1.787
40	$2.807\,10^{-3}$	1.964	$1.362\,10^{-3}$	1.946
80	$7.483\,10^{-4}$	1.913	$3.464\,10^{-4}$	1.975
160	$1.904\,10^{-4}$	1.975	$8.720\,10^{-5}$	1.990
320	$4.796\,10^{-5}$	1.989	$2.182\,10^{-5}$	1.998

- Solution post-processing [C. & Manzini, CMAME, 2008]
- A posteriori error estimators [da Veiga, NM, 2008, da Veiga, Manzini, Int. J. Numer. Meth. Eng, 2008]
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Model problem and nodal MFD spaces

Model problem $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ Lipschitz bounded domain

$$\begin{aligned} -\nabla \cdot (\mathbf{K}\, \nabla u) + \beta \cdot \nabla u = \mathrm{b} \quad & \text{in} \quad \Omega \\ u = 0 \quad & \text{on} \quad \partial\Omega \end{aligned}$$

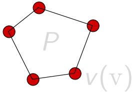
Variational form find $u \in H_0^1(\Omega)$ such that

$$(\mathbf{K}\, \nabla u, \nabla v) + (\beta \cdot \nabla u, v) = (\mathrm{b}, v) \quad \forall v \in H_0^1(\Omega)$$

Assumptions $\mathbf{K}$ strongly elliptic, $-\frac{1}{2}\nabla \cdot \beta \geq 0$, solution in $\mathcal{H}^1(\Omega) = H_0^1(\Omega) \cap C^0(\overline{\Omega})$

Nodal MFD space $\mathcal{T}_h$ MFD-type partitions of Ω, $\mathcal{V}(\mathcal{T}_h)$ set of vertices of $\mathcal{T}_h$ [Brezzi,Buffa,Lipnikov, M2AN, 2009]

$$\mathcal{N}_0 = \{\mathbf{v} = \{v(\mathrm{v})\}_{\mathrm{v} \in \mathcal{V}(\mathcal{T}_h)} : v(\mathrm{v}) = 0 \quad \forall \mathrm{v} \in \partial\Omega\}$$



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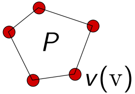

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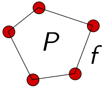

Quadrature formulas Let $P \in \mathcal{T}_h$ and $f \subset \partial P$.

- ω_P weights of a vertex-based 0-th order quadrature formula
- ω_f weights of a vertex-based 1st order quadrature formula

$$(\bar{\mathrm{b}}, \mathbf{v})_{\mathcal{N}} := \sum_{P\in\mathcal{T}_h} \bar{\mathrm{b}}|_P(\omega_P \cdot \mathbf{v}_P) \qquad \text{with } \bar{\mathrm{b}}|_P \text{ average of b over } P$$

Mimicking the formula $\int_P \nabla v \, dP = \sum_{f\in\partial P} \vec{n}_f^P \int_f v \, dS$, define the

Averaged gradient

$$\nabla^* \mathbf{v}|_P := \frac{1}{|P|} \sum_{f\in\partial P} \vec{n}_f^P \, \omega_f \cdot \mathbf{v}_f \qquad (\nabla^* p^{\mathrm{I}}|_P = \nabla p|_P \quad \forall p \in \mathcal{P}_1(P))$$

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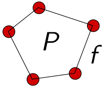

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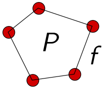

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Nodal MFD formulation

$$[\![\mathbf{v}, p^{\mathrm{I}}]\!]_P := |P|\,(\overline{\mathbf{K}}\nabla^* \mathbf{v}) \cdot (\nabla^* p^{\mathrm{I}}) \qquad \forall p \in \mathcal{P}_1(P) \quad \text{and} \quad \mathbf{v} \in \mathcal{N}|_P$$

yields consistency and can be extended to all of $\mathcal{T}|_P$ to yield a symmetric bilinear form verifying an appropriate stability condition. Then,

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Nodal MFD   Find $\mathbf{u}\in\mathcal{N}_0$:

$$[\![\mathbf{u}, \mathbf{v}]\!] + (\bar{\beta}\cdot\nabla^*\mathbf{u}, \mathbf{v})_{\mathcal{N}} = (\bar{\mathrm{b}}, \mathbf{v})_{\mathcal{N}} \quad \forall\, \mathbf{v}\in\mathcal{N}_0.$$

Nodal MFD formulation

$$[\![\mathbf{v}, p^{\mathrm{I}}]\!]_P := |P|\,(\overline{\mathbf{K}}\nabla^*\mathbf{v})\cdot(\nabla^* p^{\mathrm{I}}) \qquad \forall p \in \mathcal{P}_1(P) \quad \text{and} \quad \mathbf{v}\in\mathcal{N}|_P$$

yields consistency and can be extended to all of $\mathcal{T}|_P$ to yield a symmetric bilinear form verifying an appropriate stability condition. Then,

$$[\![\mathbf{u},\mathbf{v}]\!] := \sum_{P\in\mathcal{T}_h} [\![\mathbf{u},\mathbf{v}]\!]_P$$

Nodal SD-MFD Find $\mathbf{u}\in\mathcal{N}_0$:

$$[\![\mathbf{u},\mathbf{v}]\!] + (\bar\beta\cdot\nabla^*\mathbf{u},\mathbf{v})_{\mathcal{N}} + \sum_{P\in\mathcal{T}_h}\tau_P\,|P|(\bar{\mathrm{b}} - \bar\beta\cdot\nabla^*\mathbf{u})\,(\bar\beta\cdot\nabla^*\mathbf{v}) = (\bar{\mathrm{b}},\mathbf{v})_{\mathcal{N}} \quad \forall\,\mathbf{v}\in\mathcal{N}_0.$$

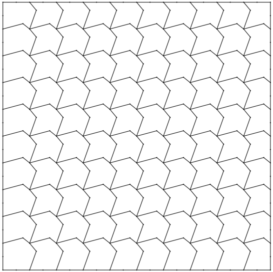
$$\tau_P = c_{\mathrm{sd}}\begin{cases} h_P & \text{if } Pe_P > 1\\ h_P^2/\varepsilon & \text{otherwise}\end{cases} \qquad \text{with} \quad Pe_P := \frac{|\bar\beta|\, h_P}{2\varepsilon}$$

Note. SD-MFD $\equiv \mathcal{P}_1$-FEM on triangular (tetrahedral) partitions

Convergence test: MFD method

$$\mathbf{K} = \varepsilon\, \mathbb{I} \qquad \beta = (2,3)$$

$$u = \left(x - e^{\frac{2(x-1)}{\varepsilon}}\right)\left(y^2 - e^{\frac{3(y-1)}{\varepsilon}}\right)$$

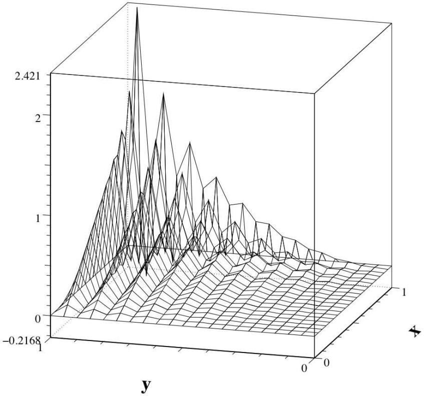


$\varepsilon = 1$ (diffusion dominated regime)

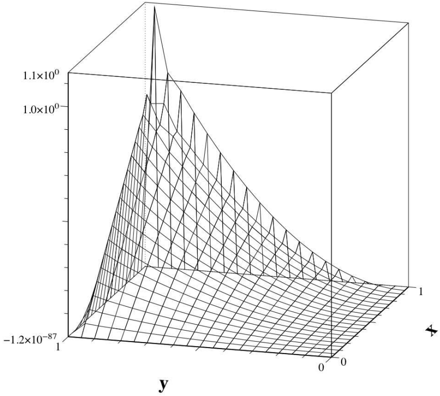
h	$\|\|u^I - \mathbf{u}\|\|^{rel}_{1,h}$	rate	$\|\|u^I - \mathbf{u}\|\|^{rel}_{0,h}$	rate
$1.458\,10^{-1}$	$2.014\,10^{-1}$	$--$	$1.327\,10^{-2}$	$--$
$7.289\,10^{-2}$	$1.043\,10^{-1}$	0.948	$3.468\,10^{-3}$	1.936
$3.644\,10^{-2}$	$5.298\,10^{-2}$	0.977	$8.810\,10^{-4}$	1.976
$1.822\,10^{-2}$	$2.667\,10^{-2}$	0.990	$2.218\,10^{-4}$	1.989

Convection-dominated test ($\varepsilon = 10^{-2}$)

MFD

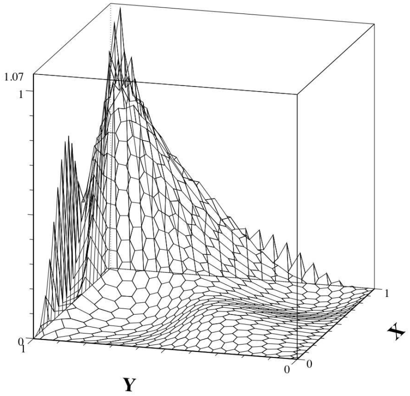


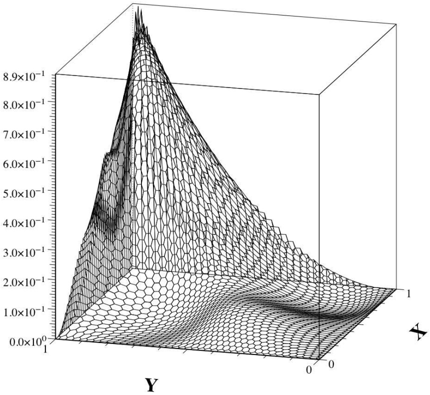
SD-MFD
$c_{\mathrm{sd}} = 0.1$



Convection-dominated test ($\varepsilon = 10^{-2}$)

Mainly-exagonal meshes $\mathcal{M}_1$



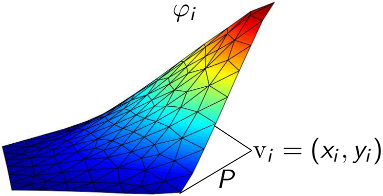


Harmonic basis functions

For $P \in \mathcal{T}_h$, let

$$H(P) = \left\{ v : \begin{array}{ll} \Delta v = 0 & \text{in } P \\ v|_f \in \mathcal{P}_1(f) & \forall f \in \partial P \end{array} \right\}$$

$H(P) = \langle \{\varphi_i\}_{i=1}^{\mathrm{v}_P} \rangle$, with $\varphi_i(\mathrm{v}_j) = \delta_{ij}$



Note that $\mathcal{P}_1(P) \subset H(P)$ and

- $1 \equiv \sum_{i=1}^{\mathrm{v}_P} \varphi_i$
- $x = \sum_{i=1}^{\mathrm{v}_P} x_i \varphi_i, \quad y = \sum_{i=1}^{\mathrm{v}_P} y_i \varphi_i$

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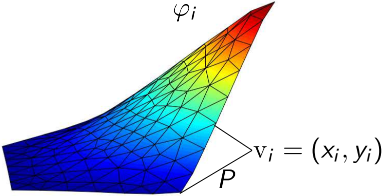

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MFD scalar product ($\mathbf{K} \equiv \mathbb{I}$)

For $u_H, v_H \in H(P)$

$$\begin{aligned} [u_H, v_H]_P &:= (\nabla u_H, \nabla v_H)_P = \sum_{i,j=1}^{\mathrm{v}_P} u_H(\mathrm{v}_i) \left(\int_P \nabla \varphi_i \cdot \nabla \varphi_j dP \right) v_H(\mathrm{v}_j) \\ &= (u_H^{\mathrm{I}})^T M_{\mathrm{ex}} v_H^{\mathrm{I}} \end{aligned}$$

Note that $m_{ij} := \displaystyle\int_P \nabla \varphi_i \cdot \nabla \varphi_j dP = \int_{\partial P} \frac{\partial \varphi_i}{\partial \vec{n}^P} \varphi_j$

Consistency: approximate m_{ij} so that the resulting

approximate bilinear form $[\![u_H, v_H]\!]_P = (u_H^{\mathrm{I}})^T M v_H^{\mathrm{I}}$ is exact when $\begin{cases} u_H \in \mathcal{P}_1(P) \\ v_H \in H(P) \end{cases}$

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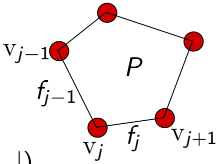
MFD consistency condition

Let, for instance, $u_H = x$. Testing with φ_j, $j = 1, \ldots, \mathrm{v}_P$

$$\int_P \nabla x \cdot \nabla \varphi_j = \int_P \nabla\Big(\sum_{i=1}^{\mathrm{v}_P} x_i \varphi_i\Big) \cdot \nabla \varphi_j = \sum_{i=1}^{\mathrm{v}_P} x_i m_{ij}$$

On the other hand,



$$\int_P \nabla x \cdot \nabla \varphi_j = \int_{\partial P} n_x^P \varphi_j = \sum_{f \in \partial P} n_x^f \int_f \varphi_j$$

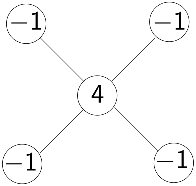
$$= n_x^{f_{j-1}} \int_{f_{j-1}} \varphi_{j-1} + n_x^{f_j} \int_{f_j} \varphi_j = \frac{1}{2}\big(n_x^{f_{j-1}} |f_{j-1}| + n_x^{f_j} |f_{j-}|\big)$$

Defining the $\mathrm{v}_P \times 2$ matrices R and N by row

$$R^j = [\, x_j \quad y_j \,] \qquad N^j = \frac{1}{2}\big[|f_{j-1}| \vec{n}^{f_{j-1}} + \vec{n}^{f_j} |f_j|\big] \qquad \text{yields} \quad {\color{blue} R^T M = N^T}$$

Comments

- MFD as harmonic FEM plus quadrature
    - Triangular partitions base is $\{1, x, y\}$
    - Quadrilateral partitions base is $\{1, x, y, xy\}$ [Hansbo, A new approach to quadrature for finite elements incorporating hourglass control as a special case CMAME, 1998]



- Extends to higher-order nodal MFD
- Polyhedral interpolation and FE literature
  "Finite element simulations in computer graphics are typically based on tetrahedral or hexahedral elements, which enables simple and efficient implementations, but in turn requires complicated remeshing in case of topological changes or adaptive refinement. We propose a flexible finite element method for arbitrary polyhedral elements, thereby effectively avoiding the need for remeshing"
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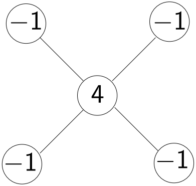

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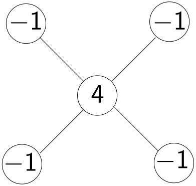

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