Regularized Poincaré operator and *p* version approximation of the Maxwell eigenvalue problem

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Main references

M. COSTABEL, A. MCINTOSH
On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains
Math. Z. 265, No 2 (June 2010) 297–320.

D. BOFFI, M. COSTABEL, M. DAUGE, L. DEMKOWICZ, R. HIPTMAIR
Discrete compactness for the *p*-version of discrete differential forms
IRMAR-Preprint 09-39, Universite Rennes 1, September 2009.
arXiv : 0909.5079.

Approximation of the Maxwell eigenvalue problem

Maxwell eigenvalue problem (PEC boundary, homogeneous isotropic material)

Find $\omega \neq 0$, $(\boldsymbol{E}, \boldsymbol{H}) \neq 0$ such that

(Maxwell EVP) $$\begin{cases} \mathbf{curl}\, \boldsymbol{E} - i\omega \boldsymbol{H} = 0 \quad \& \quad \mathbf{curl}\, \boldsymbol{H} + i\omega \boldsymbol{E} = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{E} \times \boldsymbol{n} = 0 \quad \& \quad \boldsymbol{H} \cdot \boldsymbol{n} = 0 \qquad \text{on} \quad \partial\Omega \end{cases}$$

Simplest variational formulation

Find $\omega \neq 0$, $\boldsymbol{E} \in H_0(\mathbf{curl}, \Omega) \setminus \{0\}$ such that

$$\forall \widetilde{\boldsymbol{E}} \in H_0(\mathbf{curl}, \Omega) \ : \quad \int_\Omega \mathbf{curl}\, \boldsymbol{E} \cdot \mathbf{curl}\, \widetilde{\boldsymbol{E}} = \omega^2 \int_\Omega \boldsymbol{E} \cdot \widetilde{\boldsymbol{E}}$$

Energy space: $H_0(\mathbf{curl}, \Omega) = \{\boldsymbol{u} \in L^2(\Omega)^3 \mid \mathbf{curl}\, \boldsymbol{u} \in L^2(\Omega)^3; \boldsymbol{u} \times \boldsymbol{n} = 0\}$

Galerkin discretization:

Restriction to finite-dimensional subspace V_N, $N \to \infty$.

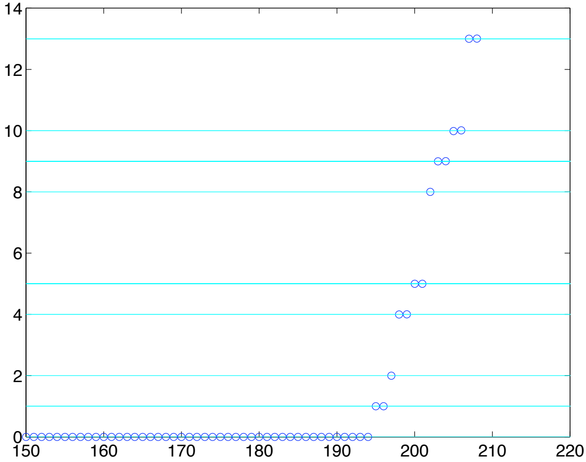
Eigenfrequencies are non-negative, discrete.

Problem: $\omega = 0$ has infinite multiplicity

Kernel: Electrostatic fields: **gradients** of all $\phi \in H_0^1(\Omega)$ (+ harmonic forms).

Approximation on the square $\Omega = (0, \pi) \times (0, \pi)$

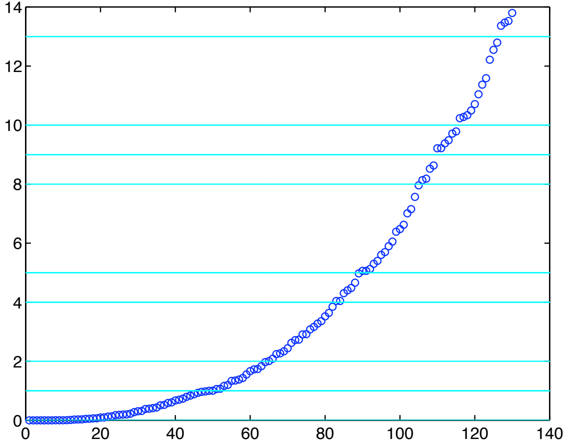
Good approximation: Triangular edge elements (15 nodes per side, $\mathbb{P}_1$)



Eigenvalue ω_k^2 vs. rank k

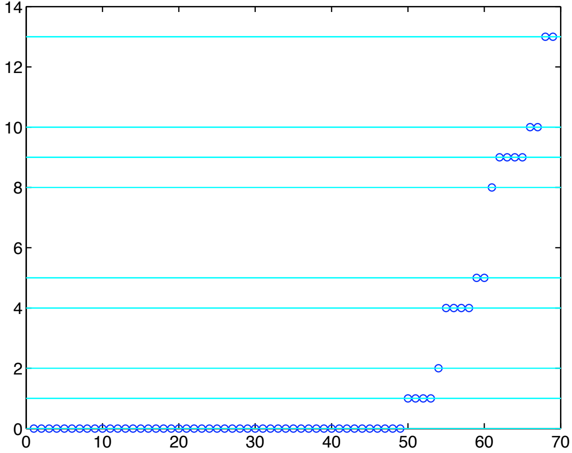
Approximation on the square $\Omega = (0,\pi) \times (0,\pi)$

Bad approximation: Nodal triangular elements (15 nodes per side, $\mathbb{P}_1$)



Not enough gradients

Another bad approximation: One square element $((\mathbb{Q}_8)^2 \cap H_0(\mathbf{curl}, \Omega))$



Wrong multiplicities!

Too many discrete divergence free functions

Theorem [BCDDH]

Using the *p*-version of Nédélec edge finite elements on

- ∗ triangles or tetrahedra (first or second family) or on
- ∗ quadrilaterals or affine hexahedra (first family)

we obtain a spurious free spectrally correct approximation of Maxwell eigenvalues and eigenfunctions.

This means: Enumerate the positive exact eigenvalues (with multiplicity)

$$0 < \omega^{(1)} \le \omega^{(2)} \le \ldots \le \omega^{(j)} \le \ldots$$

and the positive approximate eigenvalues

$$0 < \omega_N^{(1)} \le \omega_N^{(2)} \le \ldots \le \omega_N^{(j)} \le \ldots$$

Then $\omega_N^{(j)}$ converges to $\omega^{(j)}$, together with the eigenspaces.

The convergence result

Theorem [BCDDH]

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$d = 2$: The De Rham complex

$$H_0(d,\Omega,\Lambda^0) \xrightarrow{\mathrm{d}_0} H_0(d,\Omega,\Lambda^1) \xrightarrow{\mathrm{d}_1} H_0(d,\Omega,\Lambda^2) \xrightarrow{\mathrm{d}_2} 0$$

coincides with

$$H_0^1(\Omega) \xrightarrow{\textbf{grad}} H_0(\mathrm{curl},\Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega) \xrightarrow{0} 0$$

$d = 3$: The De Rham complex

$$H_0(d,\Omega,\Lambda^0) \xrightarrow{\mathrm{d}_0} H_0(d,\Omega,\Lambda^1) \xrightarrow{\mathrm{d}_1} H_0(d,\Omega,\Lambda^2) \xrightarrow{\mathrm{d}_2} H_0(d,\Omega,\Lambda^3) \xrightarrow{\mathrm{d}_3} 0$$

coincides with

$$H_0^1(\Omega) \xrightarrow{\textbf{grad}} H_0(\textbf{curl},\Omega) \xrightarrow{\textbf{curl}} H_0(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} 0$$

Some vocabulary about the de Rham complex

cochain projections $\equiv$ pr. satisfying the commutative diagram property

$$W^k = L^2(\Omega, \Lambda^k)$$

$$V^k = H_0(d, \Omega, \Lambda^k) = H_0^1(\Omega), H_0(\mathbf{curl}, \Omega), H_0(\mathrm{div}, \Omega), L^2(\Omega)$$

$$V_k^* = H(\delta, \Omega, \Lambda^k) = \{0\}, H(\mathrm{div}, \Omega), H(\mathbf{curl}, \Omega), H^1(\Omega)$$

$$\mathfrak{B}^k = \mathrm{im}\, d = dV^{k-1} \subset \mathfrak{Z}^k = \ker d = H(d0, \Omega, \Lambda^k)$$

$$\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{B}^{k\perp} : \text{ harmonic forms}$$

V_h^k, V_p^k, V_N : finite element subspaces, h, p version, generic

for example:

$\mathfrak{B}_N^{1\perp}$: discrete divergence free fields, here $= \mathfrak{Z}_N^{1\perp}$

$\mathfrak{B}^{1\perp}$: divergence free fields, here $= \mathfrak{Z}^{1\perp}$

in general: $\mathfrak{B}_N^{1\perp} \not\subset \mathfrak{B}^{1\perp}$

Necessary and sufficient conditions for convergence [Caorsi et al 2000]

(CAS)

Completeness of the Approximating Subspaces

$$\forall \mathbf{u} \in H_0(\mathbf{curl}, \Omega) : \qquad \lim_{N\to\infty} \inf_{\mathbf{u}_N \in V_N^1} \|\mathbf{u} - \mathbf{u}_N\|_{H(\mathbf{curl},\Omega)} = 0$$

(CAS) $\Longrightarrow$ any eigenvector can be approximated by V_N^1 as $N \to \infty$.

But $\omega = 0$ has infinite multiplicity

$\Longrightarrow$ All discrete eigenvalues will converge to 0 !

We need to handle the kernel. Two different possible directions:

1. Blow up of the kernel
   Regularization [old], weighted regularization [Co.-Dauge 2002]

$$(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) \longrightarrow (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + s(\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v})_{L^2_w(\Omega)}$$

2. Separation of the kernel
   Commuting diagrams (“cochain projections”) + some conditions...

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(CDK)

Completeness of the Discrete Kernels

$$\forall \mathbf{k} \in \mathfrak{Z}^1 : \qquad \lim_{N\to\infty} \inf_{\mathbf{k}_N \in \mathfrak{Z}_N^1} \|\mathbf{k} - \mathbf{k}_N\|_{L^2(\Omega)} = 0.$$

(DCP) [KIKUCHI 1989]

Discrete Compactness Property

Any sequence $\{\mathbf{u}_N\}_{N\to\infty}$ with

$$\mathbf{u}_N \in V_N^1 \cap (\mathfrak{Z}_N^1)^\perp \quad \text{and} \quad \|\mathbf{u}_N\|_{H(\mathbf{curl},\Omega)} \le 1$$

contains a subsequence that *converges in* $L^2(\Omega)$

Spurious-free spectrally correct approximation

The ideal situation of eigenvalue approximation is to have (SFA) + (SCA):
With the exact eigenvalues

$$0 < \omega^{(1)} \leq \omega^{(2)} \leq \ldots \leq \omega^{(i)} \leq \ldots$$

and the discrete eigenvalues

$$0 < \omega_N^{(1)} \leq \omega_N^{(2)} \leq \ldots \leq \omega_N^{(i)} \leq \ldots$$

$\omega_N^{(i)}$ converges to $\omega^{(i)}$, together with the eigenspaces.

Theorem [Caorsi - Fernandes - Raffetto 2000]

(CAS) + (CDK) + (DCP) $\Longrightarrow$ (SFA) + (SCA)

It remains to show (DCP)...

Projection-based interpolation [Demkowicz et al]

From [Demkowicz08; Theorem 5.3]

There exist densely defined projection operators in the commuting diagram

$$
\begin{array}{ccccccc}
H_0^1(\Omega) & \xrightarrow{\textbf{grad}} & H_0(\textbf{curl},\Omega) & \xrightarrow{\textbf{curl}} & H_0(\text{div},\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \\
\big\downarrow \pi_p^0 & & \big\downarrow \pi_p^1 & & \big\downarrow \pi_p^2 & & \big\downarrow \pi_p^3 \\
V_p^0 = W_p & \xrightarrow{\textbf{grad}} & V_p^1 = \mathbf{Q}_p & \xrightarrow{\textbf{curl}} & V_p^2 = \mathbf{V}_p & \xrightarrow{\text{div}} & V_p^3 = Y_p \\
\text{Lagrange} & & \text{Nedelec} & & \text{Raviart-Thomas} & & \text{discont.}
\end{array}
$$

Approximation properties [Demkowicz-Buffa]:

$$
\begin{aligned}
\left\| u - \pi_p^0 u \right\|_{H^1} &\leq C \log^2 p \, p^{1-r} \, \|u\|_{H^r} \,, \; r > \tfrac{3}{2} \\
\left\| u - \pi_p^1 u \right\|_{H(\mathbf{curl})} &\leq C \log p \, p^{-r} \, \|u\|_{H^r(\mathbf{curl})} \,, \; r > \tfrac{1}{2} \\
\left\| u - \pi_p^2 u \right\|_{H(\mathrm{div})} &\leq C \log p \, p^{-r} \, \|u\|_{H^r(\mathrm{div})} \,, \; r > 0
\end{aligned}
$$

The Integral Operators

- M. E. BOGOVSKIĬ (1979)
- G. P. GALDI (1994)
- R. G. DURAN, M. A. MUSCHIETTI (2001–2004)
- M. CROUZEIX (Sém IRMAR 2004)
- M. MITREA, D. MITREA, S. MONNIAUX (2005–2009)
- M. COSTABEL, A. MCINTOSH Math. Z. 265, No 2 (2010) 297–320.
- M. COSTABEL, A. MCINTOSH, R. TAGGART (2010)

The Poincaré operators: Definition

Let $D \subset \mathbb{R}^3$ be star-shaped with respect to $a \in D$

$$\mathfrak{R}_a^{\mathbf{grad}} \boldsymbol{u}(x) \quad = \quad (x-a)\cdot \int_0^1 \boldsymbol{u}\big(a + t(x-a)\big)\, dt = \int_a^x \boldsymbol{u}\cdot \mathbf{ds}$$

Known:

The Poincaré operators: Definition

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$$\begin{aligned}
\mathfrak{R}_a^{\mathbf{grad}} \boldsymbol{u}(x) &= (x-a)\cdot \int_0^1 \boldsymbol{u}\big(a+t(x-a)\big)\,dt = \int_a^x \boldsymbol{u}\cdot \mathbf{ds} \\
\mathfrak{R}_a^{\mathbf{curl}} \boldsymbol{u}(x) &= -(x-a)\times \int_0^1 t\,\boldsymbol{u}\big(a+t(x-a)\big)\,dt \\
\mathfrak{R}_a^{\mathrm{div}} u(x) &= (x-a) \int_0^1 t^2\,u\big(a+t(x-a)\big)\,dt
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\end{aligned}$$

Known: **1.** Polynomials are mapped to polynomials:

$$\mathbb{P}^p \xrightarrow{\mathfrak{R}_a^{\mathrm{div}}} \mathbf{RT}^p \xrightarrow{\mathfrak{R}_a^{\mathbf{curl}}} \boldsymbol{W}^{p+1} \xrightarrow{\mathfrak{R}_a^{\mathbf{grad}}} \mathbb{P}^{p+1}$$

Raviart-Thomas Nedelec

Let $D \subset \mathbb{R}^3$ be star-shaped with respect to $a \in D$

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Known: 2. Homotopy relations:

$$\mathfrak{R}_a^{\mathbf{grad}}\,\mathbf{grad}\,u = u - u(a)$$

$$\mathfrak{R}_a^{\mathbf{curl}}\,\mathbf{curl}\,\boldsymbol{u} + \mathbf{grad}\,\mathfrak{R}_a^{\mathbf{grad}}\boldsymbol{u} = \boldsymbol{u}$$

$$\mathfrak{R}_a^{\mathrm{div}}\,\mathrm{div}\,\boldsymbol{u} + \mathbf{curl}\,\mathfrak{R}_a^{\mathbf{curl}}\boldsymbol{u} = \boldsymbol{u}$$

$$\mathrm{div}\,\mathfrak{R}_a^{\mathrm{div}} u = u$$

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$$\mathfrak{R}_a^{\mathbf{curl}} \boldsymbol{u}(x) = -(x-a) \times \int_0^1 t\, \boldsymbol{u}\big(a + t(x-a)\big)\, dt$$

$$\mathfrak{R}_a^{\mathrm{div}} u(x) = (x-a) \int_0^1 t^2\, u\big(a + t(x-a)\big)\, dt$$

Known: **3.** Continuity [Gopalakrishnan, Demkowicz 2004]:

$$\mathfrak{R}_a^{\mathbf{curl}}, \mathfrak{R}_a^{\mathrm{div}} : L^2(D) \to L^2(D)$$

We want more: $L^2(D) \;\to\; H^1(D)$!

The Poincaré operators: Definition

Let $D \subset \mathbb{R}^3$ be star-shaped with respect to $a \in D$

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$$\mathfrak{R}_a^{\mathrm{div}} u(x) = (x-a) \int_0^1 t^2\, u\big(a+t(x-a)\big)\, dt$$

Known: **4.** Application: Discrete Friedrichs Inequality [GoDe04]

$$(DFI) \qquad \forall \mathbf{u} \in (\mathfrak{Z}_p^1)^\perp : \qquad \|\mathbf{u}\|_{L^2(\Omega)} \leq C_F \, \|\mathbf{curl}\, \mathbf{u}\|_{L^2(\Omega)}$$

$$u \in \boldsymbol{W}^p \cap (\mathbf{grad}\, \mathbb{P}^p)^\perp \quad \Longrightarrow \quad u - \mathfrak{R}_a^{\mathbf{curl}}\, \mathbf{curl}\, u = \mathbf{grad}\, \mathfrak{R}_a^{\mathbf{grad}} u \quad \perp \quad u$$

$\Longrightarrow$

$$\|u\| \leq \|u - (u - \mathfrak{R}_a^{\mathbf{curl}}\, \mathbf{curl}\, u)\| = \|\mathfrak{R}_a^{\mathbf{curl}}\, \mathbf{curl}\, u\| \leq \|\mathfrak{R}_a^{\mathbf{curl}}\|_{\mathscr{L}(L^2)} \, \|\mathbf{curl}\, u\|$$

Add boundary conditions with the help of a trace lifting operator.

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\end{aligned}
$$

Known: **5.** Application: Trace lifting in $H(\mathbf{curl})$ [C-Dauge-Demkowicz08]

Construction of a p-uniformly bounded polynomial trace lifting in $H(\mathbf{curl})$ from trace liftings in H^1 and $H(div)$.

$$
\begin{array}{ccccccc}
W_p(\Omega) & \xrightarrow{\mathbf{grad}} & \mathbf{Q}_p(\Omega) & \underset{\mathfrak{R}_a^{\mathbf{curl}}}{\overset{\mathbf{curl}}{\rightleftarrows}} & \mathbf{V}_p & \xrightarrow{\mathrm{div}} & Y_p(\Omega) \\
\gamma_0 \downarrow\uparrow \mathscr{L}_0^{(p)} & & \gamma_t \downarrow\uparrow \mathscr{L}_t^{(p)} & & \gamma_n \downarrow\uparrow \mathscr{L}_n^{(p)} & & \gamma_{avg} \downarrow\uparrow \mathscr{L}_{avg} \\
W_p(\partial\Omega) & \xrightarrow{\mathbf{grad}_t} & \mathbf{Q}_p(\partial\Omega) & \xrightarrow{\mathrm{curl}_t} & V_p(\partial\Omega) & \xrightarrow{\gamma_{avg}} & \mathbb{R}
\end{array}
$$

The Poincaré operators: Definition

Regularized Poincaré operator:

$\theta \in C_0^\infty(B)$, D star-shaped with respect to B, $\int \theta(a)da = 1$

$$\mathfrak{R}^{\mathbf{curl}} u(x) = -\int_B \theta(a)(x-a) \times \int_0^1 t\, u\big(a + t(x-a)\big)\, dt\, da$$

$$\mathfrak{R}^{\mathrm{div}} u(x) = \int_B \theta(a)(x-a) \cdot \int_0^1 t^2\, u\big(a + t(x-a)\big)\, dt\, da$$

and for differential ℓ-forms u

$$\mathfrak{R}_\ell u(x) = \int_B \theta(a)(x-a) \lrcorner \int_0^1 t^{\ell-1}\, u\big(a + t(x-a)\big)\, dt\, da$$

Change of variables... Weakly singular kernel

$$\mathfrak{R}^{\mathbf{curl}} u(x) = \int \int_0^\infty \Big(r^2 \frac{x-y}{|x-y|^3} + r \frac{x-y}{|x-y|^2} \Big) \theta\Big(y - r\frac{x-y}{|x-y|}\Big)\, dr \times u(y)\, dy$$

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Regularized Poincaré operator:

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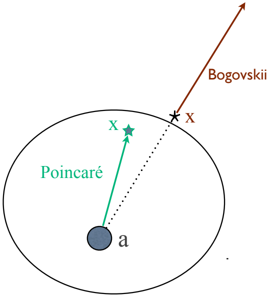
Bogovskiĭ integral operator:

$$\mathfrak{T}_\ell u(x) = -\int_B \theta(a)(x-a) \lrcorner \int_1^\infty t^{\ell-1} u\big(a+t(x-a)\big)\, dt\, da$$

Duality: $\mathfrak{T}_\ell = \star\,(\mathfrak{R}_{n-\ell+1})'\star$



Support properties:

- For $x \in D$, $\mathfrak{R}_\ell u(x)$ depends only on $u\big|_D$
- If $u = 0$ on $\mathbb{R}^d \setminus D$, then $\mathfrak{T}_\ell u = 0$ on $\mathbb{R}^d \setminus D$.

Theorem [Co.-McIntosh]

* $\mathfrak{R}_\ell, \mathfrak{T}_\ell$ are pseudodifferential operators of order -1 on $\mathbb{R}^d$
* $\mathfrak{R}_\ell$ maps polynomials to polynomials
* $d_{\ell-1}\mathfrak{R}_\ell u + \mathfrak{R}_{\ell+1} d_\ell u = u$
* $d_{\ell-1}\mathfrak{T}_\ell u + \mathfrak{T}_{\ell+1} d_\ell u = u$
* $\mathfrak{R}_\ell : H^s(D, \Lambda^\ell) \to H^{s+1}(D, \Lambda^{\ell-1}) \quad \forall s \in \mathbb{R}$
* $\mathfrak{T}_\ell : \widetilde{H}^s(D, \Lambda^\ell) \to \widetilde{H}^{s+1}(D, \Lambda^{\ell-1}) \quad \forall s \in \mathbb{R}$

$$\widetilde{H}^s(D) = H^s_{\overline{D}}(\mathbb{R}^d)$$

Many other spaces possible: $W^{s,p}$, $B^s_{p,q}$ (Besov), $F^s_{p,q}$ (Triebel-Lizorkin)

Proof: $\mathfrak{T}_\ell \in \mathrm{OpS}^{-1}_{1,0}(\mathbb{R}^d)$

$(x-y)G_\ell(x,y) = k(x,x-y); \quad k(x,z) = z\int_0^\infty s^{n-\ell}(s+1)^{\ell-1}\theta(x+sz)\,ds$

$k = k_0 + k_1; \quad k_0 = \int_0^1 \dots ds \in C^\infty(\mathbb{R}^{2d})$

$k_1 = \int_1^\infty \dots ds \Longrightarrow k_1(x,z) = 0$ for $|z| \ge |x| + \mathrm{diam}(\mathrm{supp}\,\theta)$

Symbol:

$$
\begin{aligned}
\hat{k}_1(x,\xi) &= \int_{\mathbb{R}^d} e^{-i\langle\xi,z\rangle} k_1(x,z)\,dz \\
&= \int_1^\infty s^{n-\ell}(s+1)^{\ell-1} \int e^{-i\langle\xi,z\rangle} z\theta(x+sz)\,dz\,ds\,, \\
&= \int_0^1 (t+1)^{\ell-1} e^{it\langle\xi,x\rangle} \int e^{-it\langle\xi,y\rangle}(y-x)\theta(y)\,dy\,dt \\
&= \int_0^1 (t+1)^{\ell-1} e^{it\langle\xi,x\rangle} \Big(i(\nabla\hat{\theta})(t\xi) - x\hat{\theta}(t\xi)\Big)\,dt \\
&= |\xi|^{-1} \int_0^{|\xi|} \Big(1+\frac{\tau}{|\xi|}\Big)^{\ell-1} e^{i\tau\langle\omega,x\rangle} \Big(i(\nabla\hat{\theta})(\tau\omega) - x\hat{\theta}(\tau\omega)\Big)\,d\tau \\
&\le C_\theta \min\{1, |\xi|^{-1}\}(1+|x|)
\end{aligned}
$$

Similarly for $\partial_x^\alpha \partial_\xi^\beta k_1(x,\xi)$: $\quad\Longrightarrow\quad \hat{k}_1 \in \mathrm{S}^{-1}_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$

On a star-shaped domain D:

$$d_{\ell-1}\mathfrak{R}_\ell u + \mathfrak{R}_{\ell+1} d_\ell u = u$$
$$d_{\ell-1}\mathfrak{T}_\ell u + \mathfrak{T}_{\ell+1} d_\ell u = u$$

and $\mathfrak{R}_\ell$, $\mathfrak{T}_\ell$ have support properties with respect to D.

Consequence 1

$$d_\ell u = 0 \quad \Longrightarrow \quad u = d_{\ell-1}\mathfrak{R}_\ell u = d_{\ell-1}\mathfrak{T}_\ell u$$

For any $s \in \mathbb{R}$:

$$u \in H^s(D,\Lambda^\ell) \text{ and } d_\ell u = 0 \quad \Longrightarrow \quad \exists v \in H^{s+1}(D,\Lambda^{\ell-1}): \quad u = d_{\ell-1} v$$
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Star-shaped domain D vs general bounded Lipschitz domain Ω

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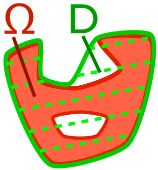

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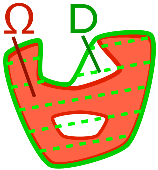

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Consequence 2. ???

New result: On a bounded Lipschitz domain Ω:
There exist pseudodifferential operators $\mathfrak{R}_\ell$, $\mathfrak{T}_\ell$ of order -1 and infinitely smoothing integral operators $\mathfrak{K}_\ell$, $\mathfrak{L}_\ell$ such that

$$d_{\ell-1}\mathfrak{R}_\ell u + \mathfrak{R}_{\ell+1} d_\ell u = u + \mathfrak{K}_\ell u$$
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$$d_\ell u = 0 \quad \Longrightarrow \quad (1 + \mathfrak{K}_\ell) u = d_{\ell-1}\mathfrak{R}_\ell u$$
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Construction of $\mathfrak{R}_\ell$ & $\mathfrak{K}_\ell$ on a Lipschitz domain Ω

Star-shaped partition of unity:

$\overline{\Omega} \subset \bigcup_{i=1}^m U_i, \chi_i \in C_0^\infty(U_i), \sum \chi_i = 1$ on $\overline{\Omega}$, U_i star-shaped $/B_i, \theta_i \in C_0^\infty(B_i), \int \theta_i = 1$

$\leadsto \mathfrak{R}_{\ell,i}, \mathfrak{T}_{\ell,i}$ constructed from θ_i

Definition

$$\mathfrak{R}_\ell = \sum \chi_i \mathfrak{R}_{\ell,i}\ ; \quad \mathfrak{K}_\ell = \sum [d_{\ell-1}, \chi_i] \mathfrak{R}_{\ell,i}$$

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But: $\mathfrak{K}_\ell$ is of order -2 only ! We want $-\infty$!

Therefore: k star-shaped partitions of unity

$$(U_i^{(j)})_{i=1,\dots,m^{(j)}} \quad \leadsto \quad \mathfrak{R}_\ell^{(j)}, \mathfrak{K}_\ell^{(j)} \quad (j = 1,\dots,k)$$

Definition

$$\mathfrak{R}_\ell = \mathfrak{R}_\ell^{(1)} + \mathfrak{K}_{\ell-1}^{(1)}\mathfrak{R}_\ell^{(2)} + \mathfrak{K}_{\ell-1}^{(1)}\mathfrak{K}_{\ell-1}^{(2)}\mathfrak{R}_\ell^{(3)} + \dots + \mathfrak{K}_{\ell-1}^{(1)} \cdots \mathfrak{K}_{\ell-1}^{(k-1)}\mathfrak{R}_\ell^{(k)}$$

$$\mathfrak{K}_\ell = \mathfrak{K}_\ell^{(1)} \cdots \mathfrak{K}_\ell^{(k)}.$$

Observation: If x_0 is such that there exists a j for which all of the functions $\chi_i^{(j)}$, $i = 1,\dots,m^{(j)}$, are constant in a neighborhood of x_0, then $\mathfrak{K}_\ell$ is of order $-\infty$ near x_0.

Lemma: There exists a finite number k of star-shaped partitions of unity of Ω such that for every $x_0 \in \mathbb{R}^d$ the above property holds.

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Corollary 1

For any $s \in \mathbb{R}$ we have:

(a) $u \in H^s(\Omega, \Lambda^\ell)$, $u = d_{\ell-1} v$, any $v \in H^t(\Omega, \Lambda^{\ell-1})$, $t \in \mathbb{R}$

$$\Longrightarrow \quad \exists\, w \in H^{s+1}(\Omega, \Lambda^{\ell-1}) : \; u = d_{\ell-1} w$$

$$\|w\|_{H^{s+1}(\Omega)} \leq C \left(\|u\|_{H^s(\Omega)} + \|v\|_{H^t(\Omega)} \right) .$$

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Proof:

$$v = d_{\ell-2} \mathfrak{R}_{\ell-1} v + \mathfrak{R}_\ell d_{\ell-1} v - \mathfrak{K}_{\ell-1} v$$

$$\Longrightarrow$$

$$u = d_{\ell-1} v = d_{\ell-1} \left(\mathfrak{R}_\ell u - \mathfrak{K}_{\ell-1} v \right) =: d_{\ell-1} w$$

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(b) $u \in \widetilde{H}^s(\Omega, \Lambda^\ell)$, $u = d_{\ell-1} v$, any $v \in \widetilde{H}^t(\Omega, \Lambda^{\ell-1})$, $t \in \mathbb{R}$

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Regularity of the kernel

Corollary 2

For any $s \in \mathbb{R}$ we have:

(a) $u \in H^s(\Omega, \Lambda^\ell)$, $d_\ell u = 0$ in $\Omega \implies u = d_{\ell-1}\mathfrak{R}_\ell u + \mathfrak{K}_\ell u$ in Ω

$$\mathfrak{R}_\ell u \in H^{s+1}(\Omega, \Lambda^{\ell-1}), \quad \mathfrak{K}_\ell u \in C^\infty(\overline{\Omega}, \Lambda^\ell)$$

(b) $u \in \widetilde{H}^s(\Omega, \Lambda^\ell)$, $d_\ell u = 0$ in $\mathbb{R}^d \implies u = d_{\ell-1}\mathfrak{T}_\ell u + \mathfrak{L}_\ell u$ in $\mathbb{R}^d$

$$\mathfrak{T}_\ell u \in \widetilde{H}^{s+1}(\Omega, \Lambda^{\ell-1}), \quad \mathfrak{L}_\ell u \in \widetilde{C}^\infty(\Omega, \Lambda^\ell)$$

Corollary 3, Regularity of cohomology spaces

$$\ker\left(d_\ell \Big|_{H^s(\Omega,\Lambda^\ell)}\right) / \operatorname{im}\left(d_{\ell-1} \Big|_{H^{s+1}(\Omega,\Lambda^{\ell-1})}\right)$$

$$\ker\left(d_\ell \Big|_{\widetilde{H}^s(\Omega,\Lambda^\ell)}\right) / \operatorname{im}\left(d_{\ell-1} \Big|_{\widetilde{H}^{s+1}(\Omega,\Lambda^{\ell-1})}\right)$$

are of finite dimension independent of s and can be represented by C^∞ functions.

Some applications in vector analysis

Question: $\phi \in H^{-1}(\Omega)$, $\mathbf{grad}\,\phi \in H^{-1}(\Omega) \Longrightarrow \phi \in L^2(\Omega)$

$$\|\phi\|_0 \leq C\left(\|\mathbf{grad}\,\phi\|_{-1} + \|\phi\|_{-1}\right)$$

Application: Korn's inequality

Proof: Lions ca. 1958 (Ω smooth), Nečas 1967 (Ω Lipschitz), Nitsche 1980

$$\phi \in H^t(\Omega),\ \mathbf{grad}\,\phi \in H^s(\Omega) \Longrightarrow \phi \in H^{s+1}(\Omega)\ \forall s,t \in \mathbb{R}$$

Gradient in negative Sobolev spaces

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Curl with Dirichlet conditions

Question: Ω simply connected Lipschitz,

$$\boldsymbol{u} \in H_0^1(\Omega),\ \operatorname{div} \boldsymbol{u} = 0 \Longrightarrow \exists \boldsymbol{v} \in H_0^2(\Omega) : \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{v}$$

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Application: Proof of Korn's inequality, elasticity

Proof: Ciarlet jr. & Ciarlet 2005

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Regularity of the cohomology forms h_j? $h_j \in C^\infty(\overline{\Omega})$
Application: As before, for general bounded Lipschitz domains
Proof: New
There exists finite-dimensional $\mathscr{H}_{\Omega,2} \subset \widetilde{C}^\infty(\Omega, \Lambda^\ell)$ such that

$$H_0^s(\operatorname{div} 0, \Omega) = \operatorname{\mathbf{curl}} H_0^{s+1}(\Omega) \oplus \mathscr{H}_{\Omega,2} \quad \forall s \in \mathbb{R}$$

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Question: $u \in L^2(\Omega)$, $\int_\Omega u = 0$, $\Longrightarrow \exists \boldsymbol{v} \in H_0^1(\Omega) : u = \operatorname{div} \boldsymbol{v}$

$$\|\boldsymbol{v}\|_1 \leq C\, \|u\|_0$$

Application: Inf-sup condition, Stokes, Maxwell, ...
Proof: Old

Question: $m \geq 0$, $1 < p < \infty$,
$u \in W_0^{m,p}(\Omega)$, $\int_\Omega u = 0$, $\Longrightarrow \exists \boldsymbol{v} \in W_0^{m+1,p}(\Omega) : u = \operatorname{div} \boldsymbol{v}$

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Application: Stokes
Proof: Bogovskii 1979, book by G.P. Galdi 1994,
but still conjectured in 2002...

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Question: K reference element, $p \in \mathbb{N}$,

$$u \in \mathbb{P}^p(K), \Longrightarrow \exists \boldsymbol{v} \in \mathbf{RT}^p(K) : u = \operatorname{div} \boldsymbol{v}$$

$$\|\boldsymbol{v}\|_0 \leq C \, \|u\|_{-1} \,, \qquad C \text{ independent of } p$$

Application: Uniform *hp*-efficiency of residual-based error estimator

Proof: Braess, Pillwein, Schöberl 2009 for rectangles K

For simplex K, general polyhedral K: New

$$\boldsymbol{v} = \mathfrak{R}^{\operatorname{div}} u \quad \Longrightarrow \quad \|\boldsymbol{v}\|_{s+1} \leq C(s) \, \|u\|_s \quad \forall s \in \mathbb{R}$$

Divergence in polynomial spaces, L^2 - H^{-1} estimate

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Curl in polynomial spaces

Question: K simplex, $p \in \mathbb{N}$, $W^p(K)$ edge elements of degree p, $0 < r < 1$

$$\boldsymbol{z} \in H^r(K),\ \textbf{curl}\,\boldsymbol{z} \in \textbf{curl}\, W^p(K)$$

$$\Longrightarrow \quad \exists\, \boldsymbol{v} \in W^p(K): \quad \textbf{curl}\,\boldsymbol{z} = \textbf{curl}\,\boldsymbol{v}, \quad \boldsymbol{z} = \boldsymbol{v} + \textbf{grad}\,\phi$$

$$\|\phi\|_{1+r} \leq C\, \|\boldsymbol{z}\|_r$$

C independent of p

Application: Discrete compactness

Proof: New [BCDDH] :

$$\boldsymbol{z} = \mathfrak{R}_a^{\mathrm{curl}}\,\textbf{curl}\,\boldsymbol{z} + \textbf{grad}\,\mathfrak{R}_a^{\mathrm{grad}}\,\boldsymbol{z} = \boldsymbol{v} + \textbf{grad}\,\phi$$

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$$\begin{aligned} \boldsymbol{z} &= \mathfrak{R}_a^{\mathbf{curl}}\,\mathbf{curl}\,\boldsymbol{z} &+& \quad \mathbf{grad}\,\mathfrak{R}_a^{\mathbf{grad}}\,\boldsymbol{z} \\ &= \boldsymbol{v} &+& \quad \mathbf{grad}\,\phi \end{aligned}$$

From [Demkowicz08; Theorem 5.3]
There exist densely defined projection operators in the commuting diagram

$$\begin{array}{ccccccc}
H_0^1(\Omega) & \xrightarrow{\mathbf{grad}} & H_0(\mathbf{curl},\Omega) & \xrightarrow{\mathbf{curl}} & H_0(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\downarrow \pi_p^0 & & \downarrow \pi_p^1 & & \downarrow \pi_p^2 & & \downarrow \pi_p^3 \\
V_p^0 = W_p & \xrightarrow{\mathbf{grad}} & V_p^1 = \mathbf{Q}_p & \xrightarrow{\mathbf{curl}} & V_p^2 = \mathbf{V}_p & \xrightarrow{\mathrm{div}} & V_p^3 = Y_p \\
\text{Lagrange} & & \text{Nedelec} & & \text{Raviart-Thomas} & & \text{discont.}
\end{array}$$

Approximation properties [Demkowicz-Buffa]:

$$\left\|u-\pi_p^0 u\right\|_{H^1} \leq C \log^2 p \, p^{1-r} \, \|u\|_{H^r}, \; r > \tfrac{3}{2}$$

$$\left\|u-\pi_p^1 u\right\|_{H(\mathbf{curl})} \leq C \log p \, p^{-r} \, \|u\|_{H^r(\mathbf{curl})}, \; r > \tfrac{1}{2}$$

$$\left\|u-\pi_p^2 u\right\|_{H(\mathrm{div})} \leq C \log p \, p^{-r} \, \|u\|_{H^r(\mathrm{div})}, \; r > 0$$

Proof of discrete compactness

Nedelec's trick + regularized Poincaré operator give (DCP):

Let $u \in (\mathfrak{Z}_p^k)^\perp \subset V_p^k$ and let $z \in V^k$ be defined by

$$d\,z = d\,u \quad \text{and} \quad z \in (\mathfrak{Z}^k)^\perp$$

Then

$$d(u - \pi_p^k z) = d\pi_p^k(u - z) = \pi_p^{k+1} d(u - z) = 0$$

$$\Longrightarrow \qquad u - \pi_p^k z \in \mathfrak{Z}_p^k \quad \perp_{W,V} \begin{cases} u \\ z \end{cases}$$

$\Longrightarrow$

$$\|u - z\| \le \|u - z - (u - \pi_p^k z)\| = \|z - \pi_p^k z\|$$

On each element: $\quad z = d\mathfrak{R}z + \mathfrak{R}dz = d\phi + \mathfrak{R}du \Longrightarrow$

$$\|z - \pi_p^k z\| = \|d(\phi - \pi_p^{k-1}\phi)\| \le C p^{-r} \|\phi\|_{1+r} \le C p^{-r} \|z\|_r$$

for some $r > 0$. Also holds globally.

If u varies in a bounded set of V, z varies in a bounded set of $V \cap (\mathfrak{Z}^k)^\perp \subset H^r$, which is compactly embedded in L^2.
Since $\|u - z\| = O(p^{-r})$, u also remains in a compact subset of L^2.

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An open problem

Proposition (puL2p)

For the p version edge elements, there exist cochain projections

$$\pi_p^k : W^k \longrightarrow V_p^k$$

whose L^2 operator norm is uniformly bounded in p

The corresponding result (huL2p) for the h version is true: Construction by SCHÖBERL-CHRISTIANSEN-ARNOLD-FALK-WINTHER [AFW10]

Conjecture (M. C., 2010)

Proposition (puL2p) is not true

One can get away with less:
In [BCDD06], discrete compactness (for 2D rectangles) was proved based on a $O(\sqrt{p})$ estimate for the L^2 norm of a certain (uniformly $H(\mathbf{curl})$-bounded) projector.

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Thank you for your attention!