Skin effect in electromagnetism

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Outline

1. Framework

2. Equations

3. 3D Multiscale Asymptotic Expansion

4. Axisymmetric Problems

5. Numerical simulations of skin effect

6. Exponential rates
The Skin Effect: A 3-D Problem

\[ \sigma + \Omega - \sigma \gg 1 \Omega - \Omega + \Sigma \]

- \( \Omega - \) Highly Conducting body \( \subset \subset \Omega \): Conductivity \( \sigma - \equiv \sigma \gg 1 \)
- \( \Sigma = \partial \Omega - \): Interface
- \( \Omega + \) Insulating or Dielectric body: Conductivity \( \sigma + = 0 \)

The Skin Effect: rapid decay of electromagnetic fields inside the conductor.

The classical Skin Depth: \( \ell(\sigma) = \sqrt{2/\omega \mu_0 \sigma} \)
Our references

V. Péron (PhD thesis, Université Rennes 1, 2009)  
*Modélisation mathématique de phénomènes électromagnétiques dans des matériaux à fort contraste.*

M. Dauge, E. Faou, V. Péron (Note CRAS, 2010)  
*Comportement asymptotique à haute conductivité de l’épaisseur de peau en électromagnétisme*

G. Caloz, M. Dauge, V. Péron (Article JMAA, 2010)  
*Uniform estimates for transmission problems with high contrast in heat conduction and electromagnetism*

*On the influence of the geometry on skin effect in electromagnetism*

M. Dauge, V. Péron, C. Poignard (In preparation, 2010)  
*Asymptotic expansion for the solution of a stiff transmission problem in electromagnetism with a singular interface*

Aim: Understanding the influence of the geometry on the skin effect.
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Maxwell Problem

Maxwell equations with perfectly insulating exterior b.c.

\[
\begin{align*}
(P_{\sigma}) \quad \begin{cases}
\text{curl } E - i\omega \mu_0 H = 0 \quad \text{and} \quad \text{curl } H + (i\omega \varepsilon_0 - \sigma)E = J \\
E \cdot n = 0 \quad \text{and} \quad H \times n = 0 \quad \text{on} \quad \partial \Omega
\end{cases}
\end{align*}
\]

with the piecewise constant conductivity

\[
\sigma = (\sigma_+, \sigma_-) = (0, \sigma \gg 1)
\]

and the rhs

\[
J \in H_0(\text{div}, \Omega) = \{ u \in L^2(\Omega)^3 \mid \text{div } u \in L^2(\Omega), \ u \cdot n = 0 \text{ on } \partial \Omega \}
\]
Existence of solutions

Hypothesis (SH)

The angular frequency $\omega$ is not an eigenfrequency of the problem

\[
\begin{align*}
\text{curl } E - i\omega \mu_0 H &= 0 \quad \text{and} \quad \text{curl } H + i\omega \varepsilon_0 E &= 0 \quad \text{in} \quad \Omega_+ \\
E \times n &= 0 \quad \text{and} \quad H \cdot n &= 0 \quad \text{on} \quad \Sigma \\
E \cdot n &= 0 \quad \text{and} \quad H \times n &= 0 \quad \text{on} \quad \partial \Omega
\end{align*}
\]

Theorem (CALOZ, DAUGE, PÉRON, 2009)

If the surface $\Sigma$ is Lipschitz, under Hypothesis (SH), there exist $\sigma_0$ and $C > 0$, such that for all $\sigma \geq \sigma_0$, $(P_{\sigma})$ with B.C. and $J \in H_0(\text{div}, \Omega)$ has a unique solution $(E, H)$ in $L^2(\Omega)^6$, and

\[
\|E\|_{0,\Omega} + \|H\|_{0,\Omega} + \sqrt{\sigma} \|E\|_{0,\Omega} \leq C \|J\|_{H(\text{div}, \Omega)}
\]

Application: Convergence of asymptotic expansion for large conductivity.
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Earlier related references for asymptotics when $\sigma \to \infty$

- **S. M. Rytov.**
  
  Calcul du skin effect par la méthodes des perturbations.

  *Journal of Physics* (1940)

- **E. Stephan.**
  
  Solution procedures for interface problems in [...] electromagnetics.


- **R. C. MacCamy, E. Stephan.**
  
  Solution procedures for three-dimensional eddy current problems.


- **R. C. MacCamy, E. Stephan.**
  
  A skin effect approximation for eddy current problems.


- **H. Haddar, P. Joly, H.-M. Nguyen.**
  
  Generalized impedance [...] for strongly absorbing obstacles [...] 

Asymptotic Expansion

Hypothesis

1. $\Sigma$ is a smooth surface, with $(y_\beta, y_3)$ “normal coordinates” to $\Sigma$
2. $\omega$ satisfies the Spectral Hypothesis (SH)
3. $J$ is smooth and $J = 0$ in $\Omega_-$

Small parameter

$$\delta := \sqrt{\omega \varepsilon_0 / \sigma} \to 0 \quad \text{as} \quad \sigma \to \infty$$

$\text{Pb} \left( P_{\sigma} \right)$ has a unique sol. $H_{(\delta)}$ for $\delta$ small enough. Expansion:

$$H_{(\delta)}^+(x) = H_0^+(x) + \delta H_1^+(x) + \delta^2 H_2^+(x) + \cdots + O(\delta^N) \quad \text{in} \quad \Omega_+$$

$$H_{(\delta)}^-(x) = \mathcal{H}_0(y_\beta, \frac{y_3}{\delta}) + \delta \mathcal{H}_1(y_\beta, \frac{y_3}{\delta}) + \delta^2 \mathcal{H}_2(y_\beta, \frac{y_3}{\delta}) + \cdots + O(\delta^N) \quad \text{in} \quad \Omega_-$$

The fields $\mathcal{H}_j \in H(\text{curl}, \Sigma \times \mathbb{R}_+)$ are exponentially decreasing profiles

\[ \cdots / \cdots \]
Profiles of the Magnetic Field

Exponential decrease rate $\lambda$ in coordinate $Y_3$ with $Y_3 = \frac{V_3}{\delta}$

$$\lambda = \omega \sqrt{\varepsilon_0 \mu_0} e^{-i\pi/4}$$

1. Denote $h_0(y_\beta) := (n \times H_0^+) \times n(y_\beta, 0)$. Profile $\tilde{H}_0$ is tangential:

$$\tilde{H}_0(y_\beta, Y_3) = h_0(y_\beta) e^{-\lambda Y_3}$$

2. Denote by $\tilde{H}_1^\alpha$ and $\tilde{H}_1^3$ the tangential and normal components of $\tilde{H}_1$.

$$\tilde{H}_1^\alpha(y_\beta, Y_3) = \left[ h_1^\alpha + Y_3 \left( \mathcal{H} h_0^\alpha + b_\sigma^\alpha h_0^\sigma \right) \right](y_\beta) e^{-\lambda Y_3}$$

$$\tilde{H}_1^3(y_\beta, Y_3) = \lambda^{-1} D_\alpha h_0^\alpha(y_\beta) e^{-\lambda Y_3}$$

Here, $b_\sigma^\alpha$ is the symmetric curvature tensor of $\Sigma$, and $\mathcal{H} = \frac{1}{2} b_\alpha^\alpha$ its mean curvature, and $D_\alpha$ is the covariant derivative. Finally,

$$h_j^\alpha(y_\beta) := (H_j^+)^\alpha(y_\beta, 0) \quad \text{(tangential traces)}.$$
A new definition of the skin depth (smooth interface $\Sigma$)

Denote $\mathcal{H}(\delta)(y_\alpha, y_3) := \mathbf{H}(\delta)(x)$, for $y_\alpha \in \Sigma$ and $0 \leq y_3$ small enough.

Recall the relation $\delta = \sqrt{\omega \varepsilon_0 / \sigma}$.

**Definition**

Let $y_\alpha \in \Sigma$ and $\sigma \geq \sigma_0$. Assume $\mathcal{H}(\delta)(y_\alpha, 0) \neq 0$.

The **skin depth** $\mathcal{L}(\sigma, y_\alpha)$ is the smallest length s.t.

$$\| \mathcal{H}(\delta)(y_\alpha, \mathcal{L}(\sigma, y_\alpha)) \| = \| \mathcal{H}(\delta)(y_\alpha, 0) \| e^{-1}$$

**Theorem (Dauge, Faou, Péron, 2010)**

*Recall: $\mathcal{H}$ mean curvature and $\ell(\sigma) = \sqrt{2 / \omega \mu_0 \sigma}$ the classical skin depth.*

Assume $h_0(y_\alpha) \neq 0$.

$$\mathcal{L}(\sigma, y_\alpha) = \ell(\sigma) \left( 1 + \mathcal{H}(y_\alpha) \ell(\sigma) + O(\sigma^{-1}) \right), \quad \sigma \to \infty$$
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Axisymmetric domains
The meridian domain

Figure: The meridian domain $\Omega^m = \Omega^- \cup \Omega^+ \cup \Sigma^m$
Case of orthoradial data: a scalar problem

The curl in cylindrical coordinates:

\[
\begin{align*}
(curl \mathbf{H})_r &= \frac{1}{r} \partial_\theta H_z - \partial_z H_\theta, \\
(curl \mathbf{H})_\theta &= \partial_z H_r - \partial_r H_z, \\
(curl \mathbf{H})_z &= \frac{1}{r} \left( \partial_r (rH_\theta) - \partial_\theta H_r \right).
\end{align*}
\]

The Maxwell problem is axisymmetric.

\( \mathbf{H} \) is axisymmetric iff \( \vec{\mathbf{H}} := (H_r, H_\theta, H_z) \) does not depend on \( \theta \).

\( \mathbf{H} \) is orthoradial iff \( \vec{\mathbf{H}} = (0, H_\theta, 0) \).

Assume that the right-hand side is axisymmetric and orthoradial.

Then, \( \mathbf{H}(\delta) \) is axisymmetric and orthoradial

\[
\vec{\mathbf{H}}(\delta)(r, \theta, z) = (0, h_\theta(\delta)(r, z), 0).
\]
Configurations chosen for computations
Configuration A (Cylinder)

Figure: The meridian domain $\Omega^m$ in configuration A
**Meshes**

**Configuration A**

**Figure:** Meshes $M_2$, and $M_3$ in configuration A
Configurations chosen for computations

Configuration B (Spheroid)

Figure: The meridian domain $\Omega^m$ in configuration B
Figure: The meshes $M_3$ and $M_6$
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Finite Element Method

In FEM computations, we use

1. the angular frequency $\omega = 3 \cdot 10^7$.
2. the rhs $g = r$ (trace on $\Gamma^m$). It is real.
3. the high order quadrangular elements available in the finite element library MÉLINA

We compute $h_\theta(\delta)$. Denote the discrete solution by

$$\tilde{h}_\theta(\delta) =: \tilde{h}_{\theta,\sigma}$$

with

$$\delta = \sqrt{\omega \varepsilon_0 / \sigma}.$$  

We note that

1. The first term $h_{\theta,0}^+$ of the asymptotics of $h_\theta(\delta)$ is real.
2. Hence, the imaginary part $\text{Im} h_\theta(\delta)$ is $O(\delta)$ in the dielectric $\Omega_+^m$.
3. Therefore the imaginary part of the computed field is expected to be larger in the conductor and to show the skin effect.
Skin effect in configuration B

Figure: Configuration B. $| \text{Im } \tilde{h}_{\theta,\sigma} |$ when $\sigma = 5$ and $\sigma = 80$
Skin effect in configuration A

Figure: Configuration A. $|\text{Im} \tilde{h}_{\theta,\sigma}|$ when $\sigma = 5$ and $\sigma = 80$
Influence of the geometry on the skin effect
Configuration B and swapped configuration B

$\mathcal{H} > 0$ on the left, and $\mathcal{H} < 0$ on the right

Figure: $|\text{Im} \tilde{h}_{\theta, \sigma}|, \sigma = 5$
Influence of the geometry on the skin effect
Configuration B2 and swapped configuration B2

\( \mathcal{H} > 0 \) on the left, and \( \mathcal{H} < 0 \) on the right, and more prolate ellipsoids

Figure: \( \left| \text{Im} \tilde{h}_{\theta,\sigma} \right|, \sigma = 5 \)
Influence of the geometry on the skin effect
Configuration B2 and swapped configuration B2

Zoom of the previous figures.

Figure: $|\text{Im } \tilde{h}_{\theta, \sigma}|$, $\sigma = 5$
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Configuration B

We extract values of $\log_{10} |\tilde{h}_{\theta,\sigma}|$ in $\Omega^m$ along the axis $z = 0 : y_3 = 2 - r$.

**Figure:** On the left $\sigma = 20$. On the right, $\sigma = 80$.

The curves exactly behave like lines: the exponential decay shows up.
Configuration A

We extract values of \( \log_{10} |\hat{h}_{\theta,\sigma}| \) in \( \Omega_m \) along the diagonal axis \( r = z \)

**Figure:** On the left \( \sigma = 20 \). On the right, \( \sigma = 80 \).

The exponential decay is less obvious.
Rates of exponential decay

We plot the *slopes* in the 4 previous figures.
Conclusion

- In config. B, slopes tend to positive limits as $y_3 \to 0$ (exponential decay).
- The values of the slopes are very close to theoretical ones.
- In config. A, slopes tend to 0 as $\rho \to 0$ (no exponential decay at corner $a$).
- But exponential decay is restored further away from $a$.
- The principal asymptotic contribution inside the conductor is a profile $v_0$ globally defined on a sector $S$ solving the model Dirichlet pb:

$$
\begin{cases}
(\partial_X^2 + \partial_Y^2)v_0 - \lambda^2 v_0 &= 0 \quad \text{in} \quad S, \\
v_0 &= h_0^+(a) \quad \text{on} \quad \partial S,
\end{cases}
$$

instead the 1D problem in configuration B

$$
\begin{cases}
\partial_Y^2 v_0 - \lambda^2 v_0 &= 0 \quad \text{for} \quad 0 < Y < +\infty, \\
v_0 &= h_0^+ \quad \text{for} \quad Y = 0.
\end{cases}
$$