

Skin effect in electromagnetism

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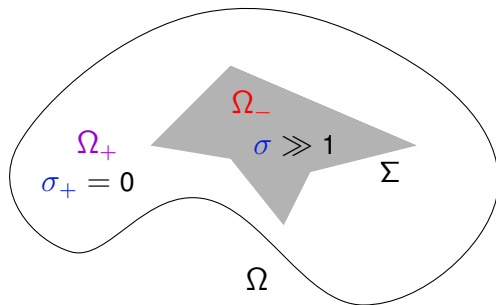
² Projet MC2, INRIA Bordeaux Sud-Ouest

Workshop “Non-Standard Numerical Methods for PDE’s”
(Pavia, June 29-July 2, 2010)

Outline

- 1 Framework**
- 2 Equations
- 3 3D Multiscale Asymptotic Expansion
- 4 Axisymmetric Problems
- 5 Numerical simulations of skin effect
- 6 Exponential rates

The Skin Effect : A 3-D Problem



- Ω_- Highly Conducting body $\subset\subset \Omega$: Conductivity $\sigma_- \equiv \sigma \gg 1$
- $\Sigma = \partial\Omega_-$: Interface
- Ω_+ Insulating or Dielectric body: Conductivity $\sigma_+ = 0$

The **Skin Effect** : rapid decay of electromagnetic fields inside the conductor.

The classical **Skin Depth** : $\ell(\sigma) = \sqrt{2/\omega\mu_0\sigma}$



Our references



V. PÉRON (PhD thesis, Université Rennes 1, 2009) [Link](#)

Modélisation mathématique de phénomènes électromagnétiques dans des matériaux à fort contraste.



M. DAUGE, E. FAOU, V. PÉRON (Note CRAS, 2010) [Link](#)

Comportement asymptotique à haute conductivité de l'épaisseur de peau en électromagnétisme



G. CALOZ, M. DAUGE, V. PÉRON (Article JMAA, 2010) [Link](#)

Uniform estimates for transmission problems with high contrast in heat conduction and electromagnetism



G. CALOZ, M. DAUGE, E. FAOU, V. PÉRON (Preprint, 2010) [Link](#)

On the influence of the geometry on skin effect in electromagnetism



M. DAUGE, V. PÉRON, C. POIGNARD (In preparation, 2010)

Asymptotic expansion for the solution of a stiff transmission problem in electromagnetism with a singular interface

- **Aim : Understanding the influence of the geometry on the skin effect.**

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Maxwell Problem

Maxwell equations with perfectly insulating exterior b.c.

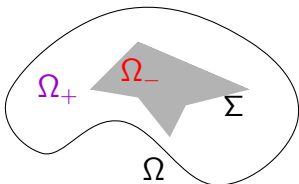
$$(\mathbf{P}_{\underline{\sigma}}) \quad \begin{cases} \operatorname{curl} \mathbf{E} - i\omega\mu_0 \mathbf{H} = 0 & \text{and} & \operatorname{curl} \mathbf{H} + (i\omega\varepsilon_0 - \underline{\sigma}) \mathbf{E} = \mathbf{J} \\ \mathbf{E} \cdot \mathbf{n} = 0 & \text{and} & \mathbf{H} \times \mathbf{n} = 0 & \text{on} & \partial\Omega \end{cases}$$

with the piecewise constant conductivity

$$\underline{\sigma} = (\sigma_+, \sigma_-) = (0, \sigma \gg 1)$$

and the rhs

$$\mathbf{J} \in H_0(\operatorname{div}, \Omega) = \{\mathbf{u} \in L^2(\Omega)^3 \mid \operatorname{div} \mathbf{u} \in L^2(\Omega), \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$



Existence of solutions

Hypothesis (SH)

The angular frequency ω is not an eigenfrequency of the problem

$$\begin{cases} \operatorname{curl} \mathbf{E} - i\omega\mu_0\mathbf{H} = 0 & \text{and} & \operatorname{curl} \mathbf{H} + i\omega\varepsilon_0\mathbf{E} = 0 & \text{in } \Omega_+ \\ \mathbf{E} \times \mathbf{n} = 0 & \text{and} & \mathbf{H} \cdot \mathbf{n} = 0 & \text{on } \Sigma \\ \mathbf{E} \cdot \mathbf{n} = 0 & \text{and} & \mathbf{H} \times \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

Theorem (CALOZ, DAUGE, PÉRON, 2009)

If the surface Σ is Lipschitz, under Hypothesis (SH), there exist σ_0 and $C > 0$, such that for all $\sigma \geq \sigma_0$, (\mathbf{P}_σ) with B.C. and $\mathbf{J} \in H_0(\operatorname{div}, \Omega)$ has a unique solution (\mathbf{E}, \mathbf{H}) in $L^2(\Omega)^6$, and

$$\|\mathbf{E}\|_{0,\Omega} + \|\mathbf{H}\|_{0,\Omega} + \sqrt{\sigma} \|\mathbf{E}\|_{0,\Omega_-} \leq C \|\mathbf{J}\|_{H(\operatorname{div},\Omega)}$$

Application: Convergence of asymptotic expansion for large conductivity.

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Earlier related references for asymptotics when $\sigma \rightarrow \infty$



S. M. RYTOV.

Calcul du skin effect par la méthodes des perturbations.

Journal of Physics (1940)



E. STEPHAN.

Solution procedures for interface problems in [...] electromagnetics.

CISM Courses and Lectures, **277**, 291–348 (1983).



R. C. MACCAMY, E. STEPHAN.

Solution procedures for three-dimensional eddy current problems.

J. Math. Anal. Appl. **101**(2) (1984) 348–379.



R. C. MACCAMY, E. STEPHAN.

A skin effect approximation for eddy current problems.

Arch. Rational Mech. Anal. **90**(1) (1985) 87–98.



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Generalized impedance [...] for strongly absorbing obstacles [...]

Math. Models Methods Appl. Sci. **18**(10) (2008) 1787–1827.

Asymptotic Expansion

Hypothesis

- 1 Σ is a smooth surface, with (y_β, y_3) “normal coordinates” to Σ
- 2 ω satisfies the Spectral Hypothesis (SH)
- 3 \mathbf{J} is smooth and $\mathbf{J} = 0$ in Ω_-

Small parameter

$$\delta := \sqrt{\omega \varepsilon_0 / \sigma} \longrightarrow 0 \quad \text{as} \quad \sigma \rightarrow \infty$$

Pb (\mathbf{P}_σ) has a unique sol. \mathbf{H}_δ for δ small enough. Expansion:

$$\mathbf{H}_\delta^+(\mathbf{x}) = \mathbf{H}_0^+(\mathbf{x}) + \delta \mathbf{H}_1^+(\mathbf{x}) + \delta^2 \mathbf{H}_2^+(\mathbf{x}) + \dots + \mathcal{O}(\delta^N) \quad \text{in } \Omega_+$$

$$\mathbf{H}_\delta^-(\mathbf{x}) = \mathfrak{H}_0(y_\beta, \frac{y_3}{\delta}) + \delta \mathfrak{H}_1(y_\beta, \frac{y_3}{\delta}) + \delta^2 \mathfrak{H}_2(y_\beta, \frac{y_3}{\delta}) + \dots + \mathcal{O}(\delta^N) \quad \text{in } \Omega_-$$

The fields $\mathfrak{H}_j \in H(\text{curl}, \Sigma \times \mathbb{R}_+)$ are exponentially decreasing profiles

... / ...

Profiles of the Magnetic Field

Exponential decrease rate λ in coordinate Y_3 with $Y_3 = \frac{y_3}{\delta}$

$$\lambda = \omega \sqrt{\varepsilon_0 \mu_0} e^{-i\pi/4}$$

- ① Denote $\mathbf{h}_0(y_\beta) := (\mathbf{n} \times \mathbf{H}_0^+) \times \mathbf{n}(y_\beta, 0)$. Profile \mathfrak{H}_0 is *tangential*:

$$\mathfrak{H}_0(y_\beta, Y_3) = \mathbf{h}_0(y_\beta) e^{-\lambda Y_3}$$

- ② Denote by \mathfrak{H}_1^α and \mathfrak{H}_1^3 the *tangential and normal components* of \mathfrak{H}_1 .

$$\begin{aligned} \mathfrak{H}_1^\alpha(y_\beta, Y_3) &= \left[h_1^\alpha + Y_3 \left(\mathcal{H} h_0^\alpha + b_\sigma^\alpha h_0^\sigma \right) \right] (y_\beta) e^{-\lambda Y_3} \\ \mathfrak{H}_1^3(y_\beta, Y_3) &= \lambda^{-1} D_\alpha h_0^\alpha (y_\beta) e^{-\lambda Y_3} \end{aligned}$$

Here, b_σ^α is the symmetric curvature tensor of Σ , and $\mathcal{H} = \frac{1}{2} b_\alpha^\alpha$ its mean curvature, and D_α is the covariant derivative. Finally,

$$h_j^\alpha(y_\beta) := (\mathbf{H}_j^+)^\alpha(y_\beta, 0) \quad (\text{tangential traces}).$$

A new definition of the skin depth (smooth interface Σ)

Denote $\mathfrak{H}_{(\delta)}(y_\alpha, y_3) := \mathbf{H}_{(\delta)}^-(\mathbf{x})$, for $y_\alpha \in \Sigma$ and $0 \leq y_3$ small enough.

Recall the relation $\delta = \sqrt{\omega \varepsilon_0 / \sigma}$.

Definition

Let $y_\alpha \in \Sigma$ and $\sigma \geq \sigma_0$. Assume $\mathfrak{H}_{(\delta)}(y_\alpha, 0) \neq 0$.

The *skin depth* $\mathcal{L}(\sigma, y_\alpha)$ is the smallest length s.t.

$$\|\mathfrak{H}_{(\delta)}(y_\alpha, \mathcal{L}(\sigma, y_\alpha))\| = \|\mathfrak{H}_{(\delta)}(y_\alpha, 0)\| e^{-1}$$

Theorem (DAUGE, FAOU, PÉRON, 2010)

Recall: \mathcal{H} mean curvature and $l(\sigma) = \sqrt{2/\omega\mu_0\sigma}$ the classical skin depth.
Assume $\mathbf{h}_0(y_\alpha) \neq 0$.

$$\mathcal{L}(\sigma, y_\alpha) = l(\sigma) \left(1 + \mathcal{H}(y_\alpha) l(\sigma) + \mathcal{O}(\sigma^{-1}) \right), \quad \sigma \rightarrow \infty$$

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Axisymmetric domains

The meridian domain

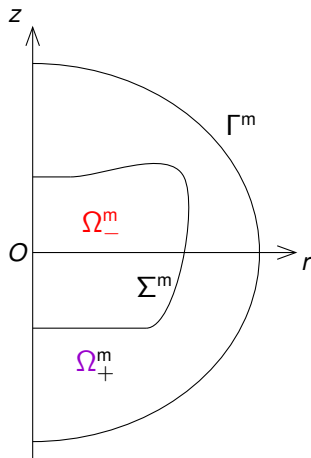


Figure: The meridian domain $\Omega^m = \Omega_-^m \cup \Omega_+^m \cup \Sigma^m$

Case of orthoradial data: a scalar problem

The curl in cylindrical coordinates:

$$\begin{cases} (\operatorname{curl} \mathbf{H})_r = \frac{1}{r} \partial_\theta H_z - \partial_z H_\theta, \\ (\operatorname{curl} \mathbf{H})_\theta = \partial_z H_r - \partial_r H_z, \\ (\operatorname{curl} \mathbf{H})_z = \frac{1}{r} (\partial_r (r H_\theta) - \partial_\theta H_r). \end{cases}$$

The Maxwell problem is axisymmetric.

\mathbf{H} is axisymmetric iff $\check{\mathbf{H}} := (H_r, H_\theta, H_z)$ does not depend on θ .

\mathbf{H} is orthoradial iff $\check{\mathbf{H}} = (0, H_\theta, 0)$.

Assume that the right-hand side is axisymmetric and orthoradial

Then, $\mathbf{H}_{(\delta)}$ is axisymmetric and orthoradial

$$\check{\mathbf{H}}_{(\delta)}(r, \theta, z) = (0, h_{\theta(\delta)}(r, z), 0).$$

Configurations chosen for computations

Configuration A (Cylinder)

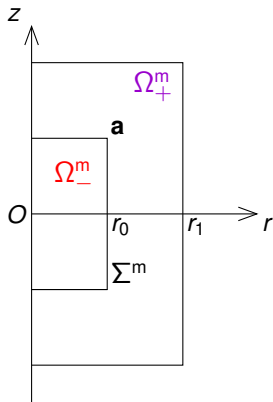


Figure: The meridian domain Ω^m in configuration A

Meshes

Configuration A

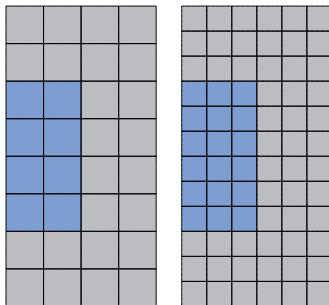


Figure: Meshes \mathfrak{M}_2 , and \mathfrak{M}_3 in configuration A

Configurations chosen for computations

Configuration B (Spheroid)

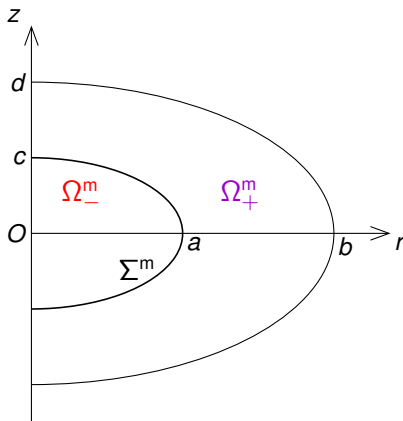


Figure: The meridian domain Ω^m in configuration B

Meshes

Configuration B

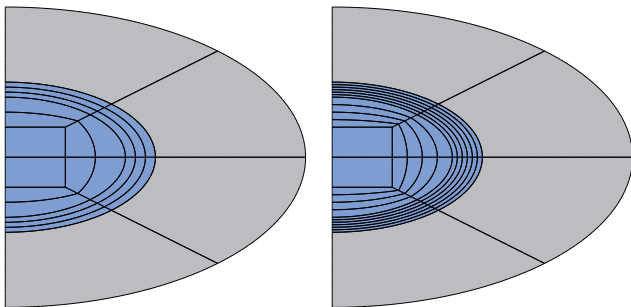


Figure: The meshes \mathcal{M}_3 and \mathcal{M}_6

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Finite Element Method

In FEM computations, we use

- 1 the angular frequency $\omega = 3 \cdot 10^7$.
- 2 the rhs $g = r$ (trace on Γ^m). It is real.
- 3 the high order quadrangular elements available in the finite element library MÉLINA

We compute $h_\theta(\delta)$. Denote the discrete solution by

$$\tilde{h}_{\theta(\delta)} =: \tilde{h}_{\theta,\sigma} \quad \text{with} \quad \delta = \sqrt{\omega \varepsilon_0 / \sigma}.$$

We note that

- 1 The first term $h_{\theta,0}^+$ of the asymptotics of $h_\theta(\delta)$ is real.
- 2 Hence, the imaginary part $\text{Im} h_\theta(\delta)$ is $\mathcal{O}(\delta)$ in the dielectric Ω_+^m .
- 3 Therefore the imaginary part of the computed field is expected to be larger in the conductor and to show the skin effect.

Skin effect in configuration B

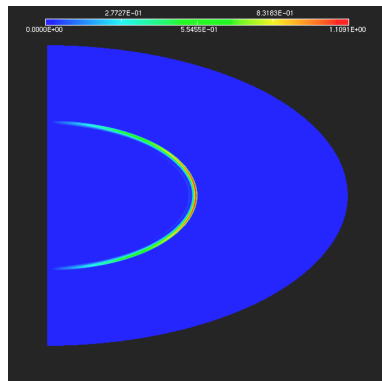
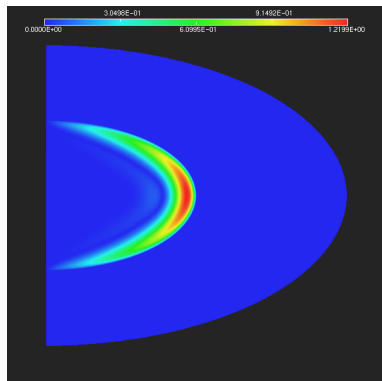


Figure: Configuration B. $|\operatorname{Im} \tilde{h}_{\theta, \sigma}|$ when $\sigma = 5$ and $\sigma = 80$

Skin effect in configuration A

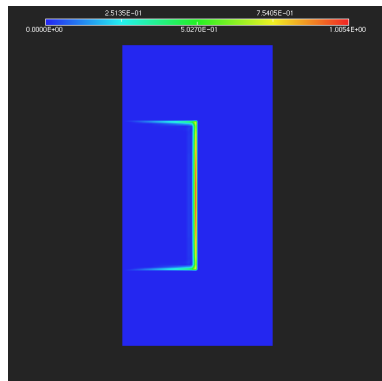
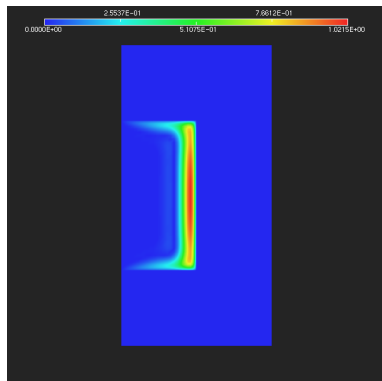


Figure: Configuration A. $|\text{Im } \tilde{h}_{\theta, \sigma}|$ when $\sigma = 5$ and $\sigma = 80$

Influence of the geometry on the skin effect

Configuration B and swaped configuration B

$\mathcal{H} > 0$ on the left, and $\mathcal{H} < 0$ on the right

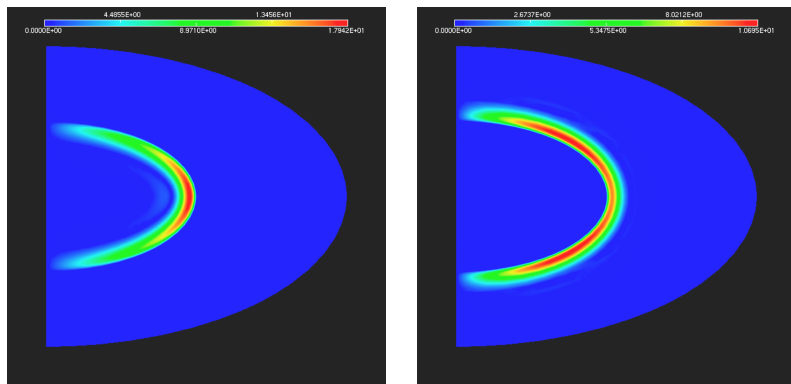


Figure: $|\text{Im } \tilde{h}_{\theta, \sigma}|, \sigma = 5$

Influence of the geometry on the skin effect

Configuration B2 and swaped configuration B2

$\mathcal{H} > 0$ on the left, and $\mathcal{H} < 0$ on the right, and **more prolate ellipsoids**

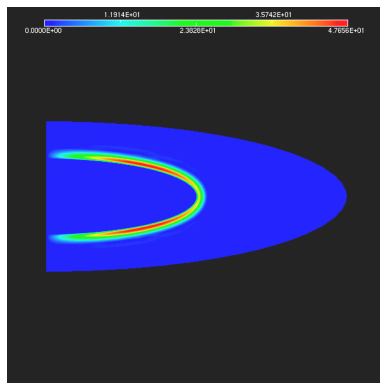
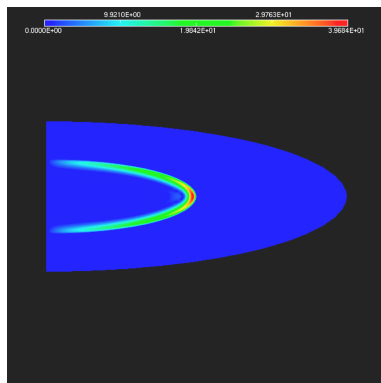


Figure: $|\text{Im } \tilde{h}_{\theta, \sigma}|, \sigma = 5$

Influence of the geometry on the skin effect

Configuration B2 and swaped configuration B2

Zoom of the revious figures.

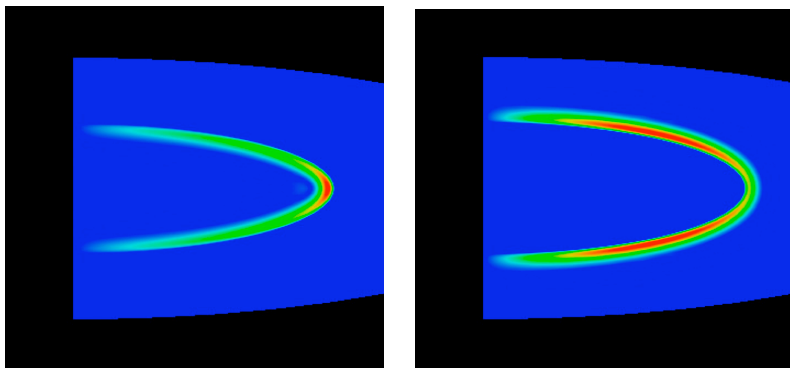


Figure: $|\operatorname{Im} \tilde{h}_{\theta, \sigma}|, \sigma = 5$

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Configuration B

We extract values of $\log_{10} |\tilde{h}_{\theta,\sigma}|$ in Ω_-^m along the axis $z = 0 : y_3 = 2 - r$.

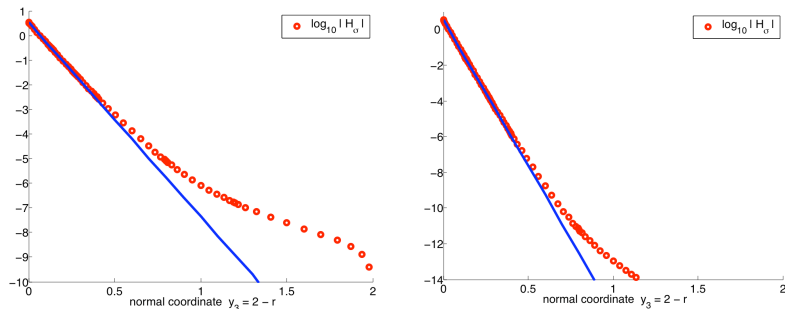


Figure: On the left $\sigma = 20$. On the right, $\sigma = 80$.

The curves exactly behave like lines: the exponential decay shows up.

Configuration A

We extract values of $\log_{10} |\tilde{h}_{\theta,\sigma}|$ in Ω_-^m along the diagonal axis $r = z$

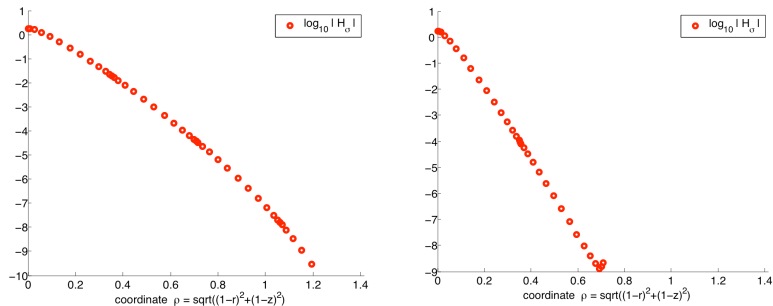
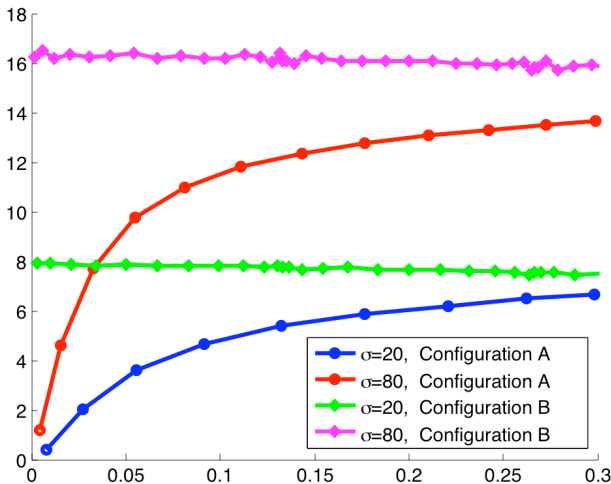


Figure: On the left $\sigma = 20$. On the right, $\sigma = 80$.

The exponential decay is less obvious.

Rates of exponential decay

We plot the *slopes* in the 4 previous figures.



Conclusion

- In config. B, slopes tend to positive limits as $y_3 \rightarrow 0$ (exponential decay).
- The values of the slopes are very close to theoretical ones.
- In config. A, slopes tend to 0 as $\rho \rightarrow 0$ (no exponential decay at corner **a**).
- But exponential decay is restored further away from **a**.
- The principal asymptotic contribution inside the conductor is a profile v_0 globally defined on a sector \mathcal{S} solving the model Dirichlet pb

$$\begin{cases} (\partial_X^2 + \partial_Y^2)v_0 - \lambda^2 v_0 = 0 & \text{in } \mathcal{S}, \\ v_0 = h_0^+(\mathbf{a}) & \text{on } \partial\mathcal{S}, \end{cases}$$

instead the 1D problem in configuration B

$$\begin{cases} \partial_Y^2 v_0 - \lambda^2 v_0 = 0 & \text{for } 0 < Y < +\infty, \\ v_0 = h_0^+ & \text{for } Y = 0. \end{cases}$$