Solution of Dual-Mixed Elasticity Equations Using Arnold–Falk–Winther Element and Discontinuous Petrov–Galerkin Method, A Comparison

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Outline of Presentation

- Variational formulations for elasticity

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- Arnold–Falk–Winther element

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- Numerical experiments with DPG method

Variational Formulations for Elasticity

Linear Elasticity

$$\begin{cases} -\rho\omega^2 u_i - \sigma_{ij,j} = f_i & \text{balance of momentum} \\ \sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij} & \text{Hooke's law and geometry combined} \end{cases}$$

where

- u_i are the displacement components,
- σ_{ij} is the stress tensor,
- ρ is the density,
- μ, λ are the Lame constants,
- ω is the angular frequency.

Linear Elasticity: Formulation in Terms of u_i

Relaxing the momentum equations,

$$\int_\Omega (\sigma_{ij} v_{i,j} - \rho u_i v_i)\, dx + \int_\Gamma \underbrace{\sigma_{ij} n_j}_{t_i} v_i \, dS = \int_\Omega f_i v_i dx$$

Using the remaining equations to eliminate σ_{ij} and building in boundary conditions:

- kinematic BC's: $u_i = u_i^0$ on Γ_u,
- traction BC's $t_i = g_i$ on Γ_t,

Linear Elasticity: Formulation in Terms of u_i

we obtain the variational formulation in terms of displacement (Principle of Virtual Work)

$$\begin{cases} u_i = u_i^0 \text{ on } \Gamma_u \\ \int_\Omega (\mu(u_{i,j} + u_{j,i})v_{i,j} + \lambda u_{k,k} v_{k,k} - \rho u_i v_i)\, dx \\ \qquad = \int_\Omega f_i v_i dx + \int_{\Gamma_t} g_i v_i \, dS \\ \qquad \forall v_i \; : \; v_i = 0 \text{ on } \Gamma_u \end{cases}$$

The resulting energy space:

$$u_i \in H^1(\Omega) := \{u_i \; : \; u_i, u_{i,j} \in L^2(\Omega)\}$$

Linear Elasticity: Dual Formulation

Introducing compliance tensor C_{klij},

$$C_{klij}\sigma_{ij} = \epsilon_{kl}$$

Relaxing the constitutive equation,

$$\int_\Omega C_{klij}\sigma_{ij}\tau_{kl} = \int_\Omega u_{k,l}\tau_{kl} = -\int_\Omega u_k \tau_{kl,l} + \int_{\partial\Omega} u_k \tau_{kl} n_l$$

Eliminating displacement u_k,

$$\begin{aligned} -u_k &= \tfrac{1}{\rho\omega^2}(\sigma_{kj,j} + f_k) && \text{in } \Omega \\ u_k &= u_k^0 && \text{on } \Gamma_u \end{aligned}$$

Linear Elasticity: Dual Formulation

we obtain the dual formulation,

$$\left\{ \begin{array}{l} \sigma_{kl} n_l = g_k \text{ on } \Gamma_t \\ \int_\Omega C_{klij} \sigma_{ij} \tau_{kl} - \int_\Omega \frac{1}{\rho\omega^2} \sigma_{kj,j} \tau_{kl,l} + \int_{\Gamma_3} \beta_{kl}^{-1} \sigma_{kj} n_j \tau_{li} n_i \\ \qquad = \int_\Omega \frac{1}{\rho\omega^2} f_k \tau_{kl,l} + \int_{\Gamma_u} u_k^0 \tau_{kl} n_l \\ \qquad \forall \tau_{ij} = \tau_{ji} \; \tau_{ij} n_j = 0 \text{ on } \Gamma_t \end{array} \right.$$

Energy space:

$$\boldsymbol{H}(\mathbf{div}, \Omega, \text{sym}) := \{\sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \,:\, \sigma_{ij,j} \in L^2(\Omega)\}$$

Troubles: Degeneration at $\omega = 0$, symmetry.

Linear Elasticity: Dual-Mixed Formulation

Keeping displacement u_i as an additional unknown, we obtain

$$\boldsymbol{\sigma} \in \boldsymbol{H}(\mathbf{div}, \Omega, \mathsf{sym}), \sigma_{kl} n_l = g_k \text{ on } \Gamma_t, \qquad \boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$$

$$\left\{ \begin{array}{llll} \int_\Omega C_{klij}\sigma_{ij}\tau_{kl} & + \int_\Omega u_k \tau_{kl,l} & = \int_{\Gamma_u} u_k^0 \tau_{kl} n_l & \\ & & \forall \tau_{ij} = \tau_{ji} \ : \ \tau_{ij} n_j = 0 \text{ on } \Gamma_t & \\ - \int_\Omega \sigma_{ij,j} v_i & - \int_\Omega \rho\omega^2 u_i v_i & = \int_\Omega f_i v_i & \\ & & & \forall v_i \end{array} \right.$$

Trouble: symmetry.

Linear Elasticity: Dual-Mixed Formulation with Weakly Imposed Symmetry

Adding infinitesimal rotations ω_{ij} as an additional unknown, we can remove the assumption on symmetry,

$$\boldsymbol{\sigma} \in \boldsymbol{H}(\mathbf{div}, \Omega), \sigma_{kl} n_l = g_k \text{ on } \Gamma_t, \quad \boldsymbol{u} \in \boldsymbol{H}^1(\Omega) \quad \boldsymbol{\omega} \in \boldsymbol{L}^2(\Omega, \text{asym})$$

$$\begin{cases} \int_\Omega C_{klij}\sigma_{ij}\tau_{kl} + \int_\Omega u_k \tau_{kl,l} + \int_\Omega \omega_{kl}\tau_{kl} = \int_{\Gamma_u} u_k^0 \tau_{kl} n_l \\ \qquad \forall \tau_{ij} \; : \; \tau_{ij} n_j = 0 \text{ on } \Gamma_t \\ -\int_\Omega \sigma_{ij,j} v_i - \int_\Omega \rho\omega^2 u_i v_i = \int_\Omega f_i v_i \\ \qquad \forall v_i \\ \int_\Omega \sigma_{ij} w_{ij} = 0 \\ \qquad \forall w_{ij} = -w_{ji} \end{cases}$$

Linear Elasticity: Ultra–Weak Formulation

Relaxing *both* momentum and constitutive/geometry relations, we get,

$$
\begin{cases}
\sigma_{ij} = \sigma_{ji}, u_i, \omega_{ij} = -\omega_{ji} \in L^2(\Omega), \hat{u}_i \in H^{1/2}(\Gamma_t), \hat{t}_i \in \tilde{H}^{-1/2}(\Gamma_u) \\
\int_\Omega C_{ijkl}\sigma_{ij}\psi_l + \int_\Omega u_k \psi_{l,l} - \int_{\Gamma_t} \hat{u}_k \psi_l n_l + \int_\Omega \omega_{kl}\psi_l \\
\qquad = \int_{\Gamma_u} u_k^0 \psi_l n_l \quad \forall \psi \in H(\mathrm{div}, \Omega),\ k = 1,2,3 \\
\int_\Omega \sigma_{ij} v_{,j} - \int \Gamma_u \hat{t}_i v - \int_\Omega \rho\omega^2 u_i v \\
\qquad = \int_\Omega f_i v + \int_{\Gamma_t} t_i^0 v \quad \forall v \in H^1(\Omega),\ i = 1,2,3
\end{cases}
$$

Linear Elasticity: DPG Formulation
Bottasso, Micheletti, Sacco, Causin 2002–

Same as the ultra-weak formulation but applied element–wise,

$$
\begin{cases}
\sigma_{ij} = \sigma_{ji}, u_i, \omega_{ij} \in L^2(K), \hat{u}_i \in H^{1/2}(\Gamma_t \cup \Gamma_0), \hat{t}_i \in \tilde{H}^{-1/2}(\Gamma_u \cup \Gamma_0) \\
\int_K C_{ijkl}\sigma_{ij}\psi_l + \int_K u_k \psi_{l,l} - \int_{(\Gamma_t \cup \Gamma_0)\cap \partial K} \hat{u}_k \psi_l n_l + \int_K \omega_{kl}\psi_l \\
\qquad \int_{\Gamma_u \cap \partial K} u_k^0 \psi_l n_l \quad \forall \psi \in H(\text{div}, K),\ k = 1,2,3 \\
\int_K \sigma_{ij} v_{,j} - \int (\Gamma_u \cup \Gamma_0) \cap \partial K \hat{t}_i v - \int_K \rho\omega^2 u_i v \\
\qquad = \int_K f_i v + \int_{\Gamma_t \cap \partial K} t_i^0 v \quad \forall v \in H^1(K),\ i = 1,2,3
\end{cases}
$$

Arnold–Falk–Winther Element

Arnold–Falk–Winther Element

Based on a part of the exact sequence for tetrahedral element

$$
\begin{array}{ccccccc}
H^1 & \xrightarrow{\textbf{grad}} & \boldsymbol{H}(\textbf{curl}) & \xrightarrow{\textbf{curl}} & \boldsymbol{H}(\text{div}) & \xrightarrow{\text{div}} & L^2 \\
\mathcal{P}^{p+3} & \xrightarrow{\textbf{grad}} & \boldsymbol{P}^{p+2} & \xrightarrow{\textbf{curl}} & \boldsymbol{P}^{p+1} & \xrightarrow{\text{div}} & \mathcal{P}^{p}
\end{array}
$$

Discretization:

$$
\begin{array}{ll}
(\sigma_{i1}, \sigma_{i2}, \sigma_{i3}) \in \boldsymbol{P}^{p+1} & i = 1, 2, 3 \\
u_i \in \mathcal{P}^p & i = 1, 2, 3 \\
\omega_{ij} \in \mathcal{P}^p & ij = 12, 13, 23
\end{array}
$$

Arnold–Falk–Winther Element

Based on a part of the exact sequence for tetrahedral element

$$\begin{array}{ccccccc} H^1 & \xrightarrow{\textbf{grad}} & \boldsymbol{H}(\textbf{curl}) & \xrightarrow{\textbf{curl}} & \boldsymbol{H}(\text{div}) & \xrightarrow{\text{div}} & L^2 \\ \mathcal{P}_{p+3}\Lambda^0(\mathcal{T}_h) & \xrightarrow{\text{grad}} & \mathcal{P}_{p+2}\Lambda^1(\mathcal{T}_h) & \xrightarrow{\text{curl}} & \mathcal{P}_{p+1}\Lambda^2(\mathcal{T}_h) & \xrightarrow{\text{div}} & \mathcal{P}_{p}\Lambda^3(\mathcal{T}_h) \end{array}$$

Discretization:

$$\begin{array}{ll} (\sigma_{i1}, \sigma_{i2}, \sigma_{i3}) \in \boldsymbol{P}^{p+1} & i = 1, 2, 3 \\ u_i \in \mathcal{P}^p & i = 1, 2, 3 \\ \omega_{ij} \in \mathcal{P}^p & ij = 12, 13, 23 \end{array}$$

AFW Element

Stability proof involves also Nédélec's tetrahedron of the first type,

$$\mathcal{P}_{p+1}\Lambda^0(\mathcal{T}_h) \xrightarrow{\text{grad}} \mathcal{P}^-_{p+1}\Lambda^1(\mathcal{T}_h) \xrightarrow{\text{curl}} \mathcal{P}^-_{p+1}\Lambda^2(\mathcal{T}_h) \xrightarrow{\text{div}} \mathcal{P}_p\Lambda^3(\mathcal{T}_h)$$

where

$$\mathcal{P}^-_p(T;\mathbb{V}) := \mathcal{P}_{p-1}(T;\mathbb{V}) + \boldsymbol{x} \times \mathcal{P}_{p-1}(T;\mathbb{V})$$

$$\mathcal{P}^-_p(T;\mathbb{V}) := \mathcal{P}_{p-1}(T;\mathbb{V}) + \boldsymbol{x}\mathcal{P}_{p-1}(T)$$

AFW Element. Stability Proof

The following three diagrams need to commute,

$$\begin{array}{ccc} H^1(\Omega;\mathbb{M}) & \xrightarrow{\text{div}} & L^2(\Omega;\mathbb{V}) \\ \Pi^2_{p,h}\downarrow & & \Pi^3_{p,h}\downarrow \\ \mathcal{P}_{p+1}\Lambda^2(\mathcal{T}_h;\mathbb{V}) & \xrightarrow{\text{div}} & \mathcal{P}_p\Lambda^3(\mathcal{T}_h;\mathbb{V}) \end{array} \quad \begin{array}{ccc} H^1(\Omega;\mathbb{M}) & \xrightarrow{\text{div}} & L^2(\Omega;\mathbb{V}) \\ \Pi^{2,-}_{p,h}\downarrow & & \Pi^3_{p,h}\downarrow \\ \mathcal{P}^-_{p+1}\Lambda^2(\mathcal{T}_h;\mathbb{V}) & \xrightarrow{\Pi^3_{p,h}\circ\text{div}} & \mathcal{P}_p\Lambda^3(\mathcal{T}_h;\mathbb{V}) \end{array}$$

$$\begin{array}{ccc} H^1(\Omega;\mathbb{M}) & \xrightarrow{S_1} & H^1(\Omega;\mathbb{M}) \\ \overline{\Pi}^{1,-}_{p,h}\downarrow & & \Pi^{2,-}_{p,h}\downarrow \\ \mathcal{P}^-_{p+2}\Lambda^1(\mathcal{T}_h;\mathbb{V}) & \xrightarrow{\Pi^{2,-}_{p,h}\circ S_1} & \mathcal{P}^-_{p+1}\Lambda^2(\mathcal{T}_h;\mathbb{V}) \end{array}$$

where

$$S_1 W = W^\top - \text{tr}(W) I$$

AFW Element. Stability Proof, Frederick's Idea

- Introduce a one-parameter family of PB-like operators $\Pi^{2,-}_{T,t}$ onto $\mathcal{P}^-_{p+1}\Lambda^2(\mathcal{T}_h;\mathbb{V})$, and a corresponding one-parameter family of operators $\overline{\Pi}^{1,-}_{T,t}$ into $\mathcal{P}^-_{p+2}\Lambda^1(\mathcal{T}_h;\mathbb{V})$, with $t \in [0,1]$.

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- We design $\Pi^{2,-}_{T,t}$ and $\overline{\Pi}^{1,-}_{T,t}$ in such a way that $\Pi^{2,-}_{T,0}$ coincides with Pojection–Based operator $\Pi^{2,-}_{h}$ (and, therefore, it is well-defined) and that we can show that $\overline{\Pi}^{1,-}_{T,1}$ is well-defined as well.

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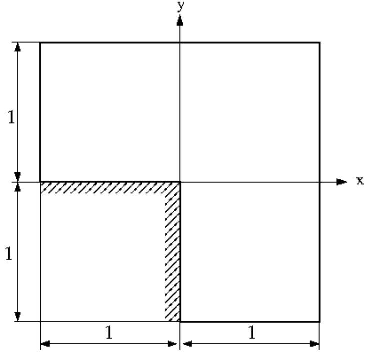
- The non-singularity of both operators translates into non-zero determinants of the corresponding matrix representations with respect to specific bases. Both determinants are polynomials in t and, since they are non-zero at $t=0$ or $t=1$, there must not not be identically zero.

AFW Element. Stability Proof, Frederick's Idea

- Introduce a one-parameter family of PB-like operators $\Pi^{2,-}_{T,t}$ onto $\mathcal{P}^{-}_{p+1}\Lambda^2(\mathcal{T}_h;\mathbb{V})$, and a corresponding one-parameter family of operators $\overline{\Pi}^{1,-}_{T,t}$ into $\mathcal{P}^{-}_{p+2}\Lambda^1(\mathcal{T}_h;\mathbb{V})$, with $t \in [0,1]$.
- We design $\Pi^{2,-}_{T,t}$ and $\overline{\Pi}^{1,-}_{T,t}$ in such a way that $\Pi^{2,-}_{T,0}$ coincides with Pojection–Based operator $\Pi^{2,-}_{h}$ (and, therefore, it is well-defined) and that we can show that $\overline{\Pi}^{1,-}_{T,1}$ is well-defined as well.
- The non-singularity of both operators translates into non-zero determinants of the corresponding matrix representations with respect to specific bases. Both determinants are polynomials in t and, since they are non-zero at $t=0$ or $t=1$, there must not not be identically zero.
- By the Fundamental Theorem of Algebra, the determinants can have only a finite number of roots. This proves that both operators are well-defined for $t \in [0,1]$, except for a finite number of values.

Numerical experiments with AFW Element

L-shape Domain Problem

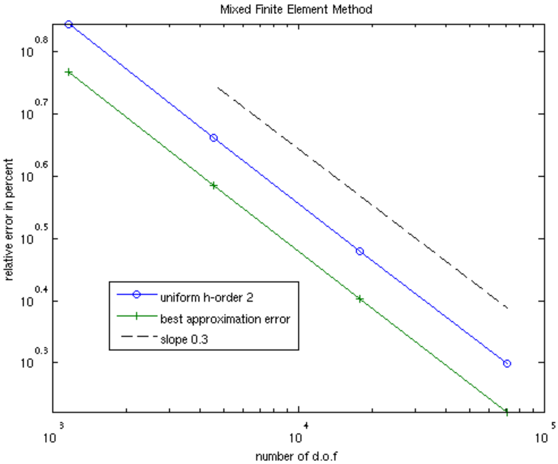


$$\sigma_x, \sigma_{xy}, \sigma_y, \omega \approx r^{-0.39596} \implies \sigma_x, \sigma_{xy}, \sigma_y, \omega \in H^{1-0.39596-\epsilon}(\Omega)$$

Expected convergence rates:

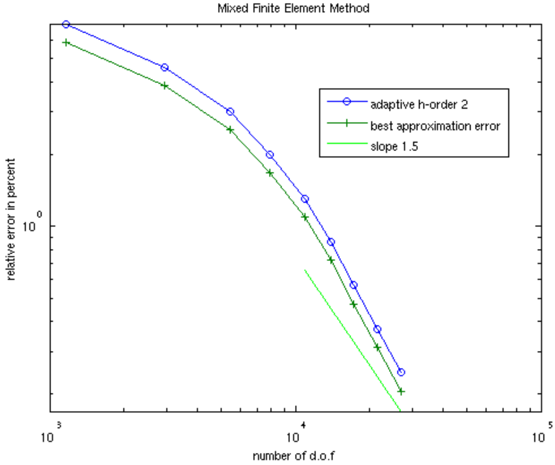
- Uniform h-refinements: $h^{0.60404} \approx N^{-0.30202}$
- Adaptive h-refinements: $N^{-1.5}$ (p=2)
- Adaptive hp-refinements: better (exponential ?)

L-shape Domain Problem



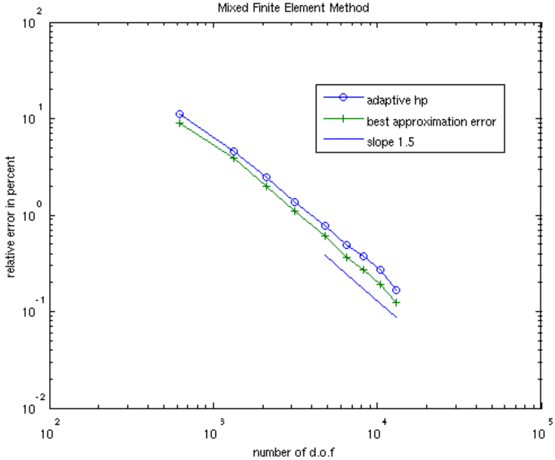
Uniform h-refinements. Comparison of the actual approximation and the best approximation errors.

L-shape Domain Problem



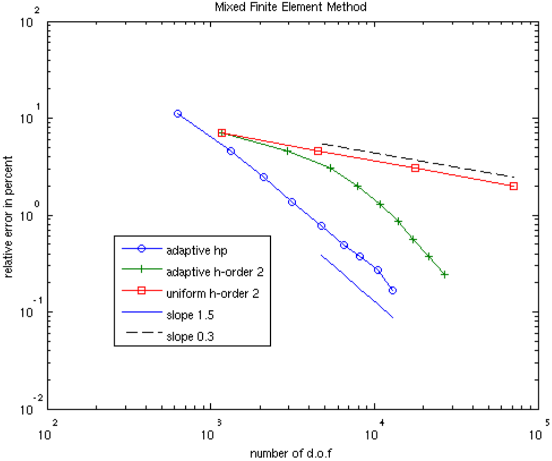
h-adaptive refinements. Comparison of the actual approximation and the best approximation errors.

L-shape Domain Problem



hp-adaptive refinements. Comparison of the actual approximation and the best approximation errors.

L-shape Domain Problem



Comparison of different refinement strategies for the mixed method. Reative energy (L^2-) error vs. number of d.o.f.

Petrov–Galerkin Method with Optimal Test Functions Abstract B^3 Framework

Abstract Variational Problem

$$\left\{ \begin{array}{l} u \in U \\ b(u,v) = l(v) \quad \forall v \in V \end{array} \right. \quad \Leftrightarrow \quad \begin{array}{l} Bu = l \quad B : U \to V' \\ < Bu, v >= b(u,v) \, \forall v \in V \end{array}$$

where

- U, V are Hilbert spaces,
- $b(u,v)$ is a continuous bilinear form on $U \times V$,

$$|b(u,v)| \leq M \|u\|_U \, \|v\|_V$$

  that satisfies the inf-sup condition ($\Leftrightarrow B$ is bounded below),

$$\inf_{\|u\|_U = 1} \sup_{\|v\|_V = 1} |b(u,v)| =: \gamma > 0$$

- $l \in V'$ represents the load and satisfies the compatibility condition $l(v) = 0, \forall v \in V_0$ where

$$V_0 := \{ v \in V \; : \; b(u,v) = 0 \quad \forall u \in U \}$$

Energy Norm

Banach Closed Range Theorem implies that there exists a unique solution u that depends continuously upon the data, $\|u\| \leq \frac{1}{\gamma}\|l\|_{V'}$. The supremum in the inf-sup condition defines an equivalent, problem-dependent *energy (residual) norm*,

$$\|u\|_E := \sup_{\|v\|=1} |b(u,v)| = \|Bu\|_{V'}$$

For the energy norm, $M = \gamma = 1$. Recalling that the Riesz operator is an isometry form V into V', we may characterize the energy norm in an equivalent way as

$$\|u\|_E = \|v_u\|_V$$

where v_u is the solution of the variational problem,

$$\begin{cases} v_u \in V \\ (v_u, \delta v)_V = b(u, \delta v) \quad \forall \delta v \in V \end{cases}$$

Optimal Test Functions

Select your favorite trial basis functions: $e_j,\ j = 1, \ldots, N$. For each function e_j, introduce a corresponding *optimal test (basis) function* $\bar{e}_j \in V$ that realizes the supremum,

$$|b(e_j, \bar{e}_j)| = \sup_{\|v\|_V = 1} |b(e_j, v)|$$

i.e. it solves the variational problem,

$$\begin{cases} \bar{e}_j \in V \\ (\bar{e}_j, \delta v)_V = b(e_j, \delta v) \quad \forall \delta v \in V \end{cases}$$

Define the discrete test space as $\bar{V}_{hp} := \text{span}\{\bar{e}_j,\ j = 1, \ldots, N\} \subset V$. It follows from the construction of the optimal test functions that the *discrete* inf-sup constant

$$\inf_{\|u_{hp}\|_E = 1} \sup_{\|v_{hp}\| = 1} |b(u_{hp}, v_{hp})| = 1$$

The Best Approximation

Consequently, Babuška's Theorem

$$\|u - u_{hp}\|_E \leq \frac{M}{\gamma_{hp}} \inf_{w_{hp} \in U_{hp}} \|u - w_{hp}\|_E$$

implies that

$$\|u - u_{hp}\|_E \leq \inf_{w_{hp} \in U_{hp}} \|u - w_{hp}\|_E$$

i.e., the method delivers the *best approximation error* in the energy norm.

Stiffness Matrix Is Symmetric and Positive Definite

$$b(e_i, \bar{e}_j) = (\bar{e}_i, \bar{e}_j)_V = (\bar{e}_j, \bar{e}_i)_V = b(e_j, \bar{e}_i)$$

Energy Norm of FE Error $e_{hp} = u - u_{hp}$

can be computed *without* knowing the exact solution.

$$\begin{cases} v_{e_{hp}} \in V \\ (v_{e_{hp}}, \delta v)_V = b(u - u_{hp}, \delta v) = l(\delta v) - b(u_{hp}, \delta v) \quad \forall \delta v \in V \end{cases}$$

We have then

$$\|e_{hp}\|_E = \|v_{e_{hp}}\|_V$$

We shall call $v_{e_{hp}}$ *the error representation function*

Note: No need for an a-posteriori error estimation.

Approximate Optimal Test Functions

- In practice, the optimal test functions and the error representation functions are computed using an "enriched" test space V_{hp}. For instance, if trial functions $e_i \in \mathcal{P}^p$, we may compute the approximate optimal test functions in space $\mathcal{P}^{p+\Delta p}$. If the discretization error corresponding to the resolution of the Riesz operator is under control, the method retains its stability properties. We have then

$$\bar{V}_{hp} \subset V_{hp}$$

- In the DG setting, inversion of the Riesz operator is done element-wise which makes the method practical. Crucial fact is that the test functions are discontinuous (trial need not...).

Optimal Test Norm

Q: Can we select the norm in the test space in such a way that the corresponding energy norm coincides with the original norm (of choice) in U ?

Optimal test norm

A: Yes! Choose:

$$\|v\|_V = \operatorname{sup}_{u \in U} \frac{|(u,v)|}{\|u\|_U}$$

(under assumption that

$$V_0 = \{v \in V \ : \ b(u,v) = 0 \quad \forall u \in U\}$$

is trivial)

Relation with an Old Idea [D+Oden 1985]

Given:

- A variational problem,

$$\left\{\begin{array}{l} u \in U \\ b(u,v) = l(v) \quad \forall v \in V \end{array}\right.$$

- inner product $(\cdot,\cdot)_{opt}$ generating an optimal norm $\|\cdot\|_{opt}$ equivalent to norm in U,
- trial functions $e_i \in U_h \subset U$.

Construct optimal test functions by solving the variational problem:

$$\left\{\begin{array}{l} \hat{e}_i \in U \\ (\delta u, \hat{e}_i)_{opt} = b(\delta u, e_i) \quad \forall \delta u \in U \end{array}\right.$$

Relation with an Old Idea [D+Oden 1985]

Then, error orthogonality relation

$$b(u - u_h, \hat{e}_j) = 0$$

translates into

$$(u - u_h, e_j)_{opt} = 0$$

which implies that the Petrov–Galerkin method delivers the best approximation error in the optimal norm.
The optimal test functions are solutions of the adjoint equation with the adjoint defined with respect to the optimal inner product.

DPG Method for Elasticity

Discretization

triangles:

$$\sigma_x, \sigma_{xy}, \sigma_y, \omega, u.v \in \mathcal{P}^p(K), \quad \hat{u}, \hat{v}, \hat{t}_x, \hat{t}_y \in \mathcal{P}^{p+\Delta p_f}(e)$$

quadrilaterals:

$$\sigma_x, \sigma_{xy}, \sigma_y, \omega, u.v \in \mathcal{Q}^{(p,q)}(K) := \mathcal{P}^p(I) \otimes \mathcal{P}^q(I), K = I^2 \quad \hat{u}, \hat{v}, \hat{t}_x, \hat{t}_y \in$$

enriched spaces for approximate test functions: $\Delta p = 3$

Max rule for determining approximation for fluxes:

triangles: $p_e = \max\{p_1, p_2(, p_3)\} + 1$

quadrilaterals: $p_e = \max\{q_1, q_2(, q_3)\}$ (accounting for directionality)

(piecewise polynomials used for 2-1 edges)

Test Norms

Standard norm:

$$\|v\|^2 = \sum_{i=1}^{3} \int_K \{|\mathrm{div} q^i|^2 + |q^i|^2\} + \sum_{i=1}^{3} \int_K \{|\boldsymbol{\nabla} v^i|^2 + |v^i|^2\}$$

Optimal norm:

$$\begin{aligned}\|v\|^2 \; &= \textstyle\sum_{k=1}^{3} \sum_{l=1}^{3} \|\{C_{ijkl} q^i_j + v^k_{,l}\}_s\|^2_{L^2(\Omega)} + \sum_{i=1}^{3} \|q^i_{j,j}\|^2_{L^2(\Omega)} \\ &+ \textstyle\sum_{k=1}^{3} \sum_{l=1}^{3} \|\{q^i_j\}_a\|^2_{L^2(\Omega)} \\ &+ \textstyle\sum_{i=1}^{3} \|[q^i_n]\|^2_{?} + \|[v^i]\|^2_{?}\end{aligned}$$

Test Norms

Standard norm:

$$\|v\|^2 = \sum_{i=1}^{3} \int_K \{|\mathrm{div} q^i|^2 + |q^i|^2\} + \sum_{i=1}^{3} \int_K \{|\nabla v^i|^2 + |v^i|^2\}$$

Approximate but localizable optimal norm:

$$\begin{aligned} \|v\|^2 &= \sum_{k=1}^{3} \sum_{l=1}^{3} \|\{C_{ijkl} q_j^i + v_{,l}^k\}_s\|_{L^2(\Omega)}^2 + \sum_{i=1}^{3} \|q_{j,j}^i\|_{L^2(\Omega)}^2 \\ &+ \sum_{k=1}^{3} \sum_{l=1}^{3} \|\{q_j^i\}_a\|_{L^2(\Omega)}^2 \\ &+ \beta \sum_{i=1}^{3} \{\|q^i\|_{L^2(\Omega)}^2 + \|v^i\|_{L^2(\Omega)}^2\} \end{aligned}$$

Computation of Anisotropy Factor

Computation of error function

$$\begin{cases} (\boldsymbol{\tau}, \boldsymbol{v}) \in V_K \\ ((\boldsymbol{\tau}, \boldsymbol{v}), (\delta\boldsymbol{\tau}, \delta\boldsymbol{v}))_K = b_K(U_{hp}, (\delta\boldsymbol{\tau}, \delta\boldsymbol{v})) - l_K((\delta\boldsymbol{\tau}, \delta\boldsymbol{v})) \\ \hfill \forall(\delta\boldsymbol{\tau}, \delta\boldsymbol{v}) \in V_K \end{cases}$$

$$c_1 = \sum_i \int_K (|\tau_{i,1}|^2 + |\frac{\partial v_i}{\partial x_1}|^2)\, dx \quad c_2 = \sum_i \int_K (|\tau_{i,2}|^2 + |\frac{\partial v_i}{\partial x_2}|^2)\, dx$$

$$\text{Refinement flag} = \begin{cases} 10 & \text{if } c_1 \geq 10 c_2 \\ 01 & \text{if } c_2 \geq 10 c_1 \\ 11 & \text{otherwise} \end{cases}$$

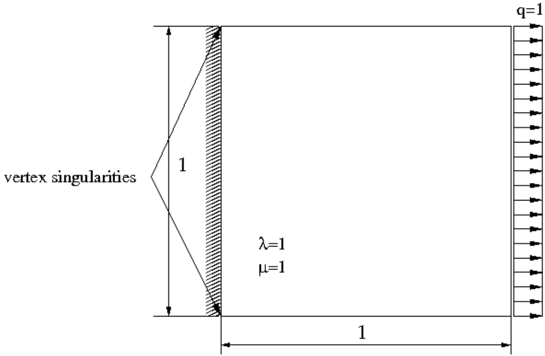
Greedy h Algorithm

Set $\alpha = 0.7$
For $i = 1, \ldots, i_{max}$
  Solve the problem on the current mesh
  For each element K in the mesh
    Compute element error contribution e_K and isotropy flag
  end of loop through elements
  For each element K in the mesh
    if $e_K > \alpha^2 \max_K e_K$ then
      h-refine the element
    endif
  end of loop through elements
end of loop through mesh refinements

Ainsworth Mark(ing) hp Algorithm

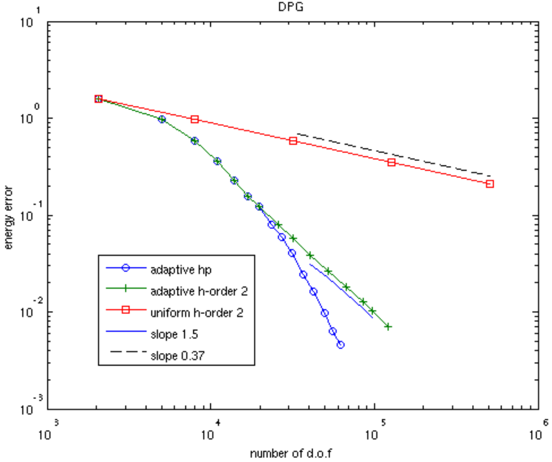
Mark singular vertices
Set $\alpha = 0.7$
For $i = 1, \ldots, i_{max}$
    Solve the problem on the current mesh
    For each element K in the mesh
        Compute element error contribution e_K and isotropy flag
    end of loop through elements
    For each element K in the mesh
        if $e_K > \alpha^2 \max_K e_K$ then
            if K contains a singular vertex then
                h-refine the element
            else
                p-refine the element
            endif
        endif
    end of loop through elements
end of loop through mesh refinements

Plate Problem



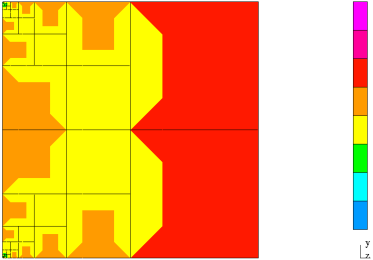
$$\mu = \lambda = 1, q = 1$$

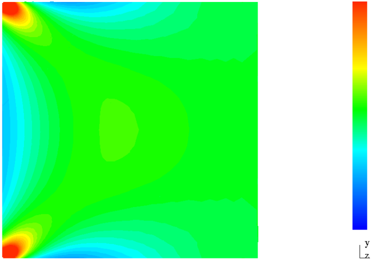
Plate Problem



Convergence history for uniform, h-adaptive and hp-adaptive refinements. Energy norm of the error vs. number of d.o.f.

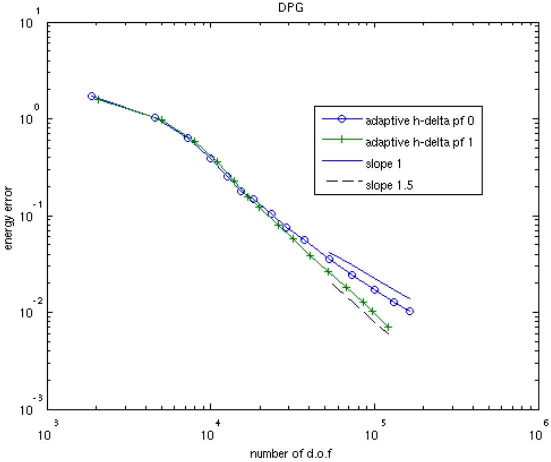
Plate Problem





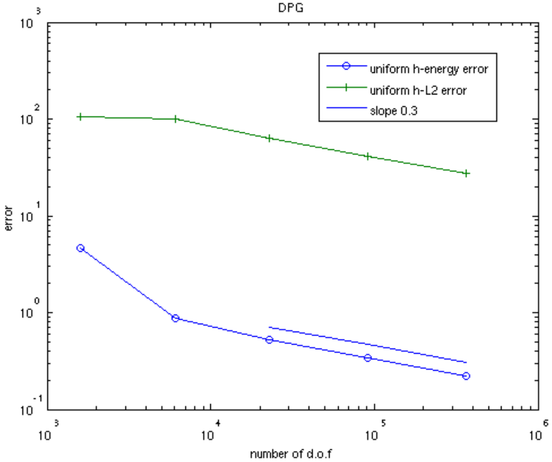
Optimal hp mesh after 15 refinements and the corresponding distribution of σ_x.

Plate Problem



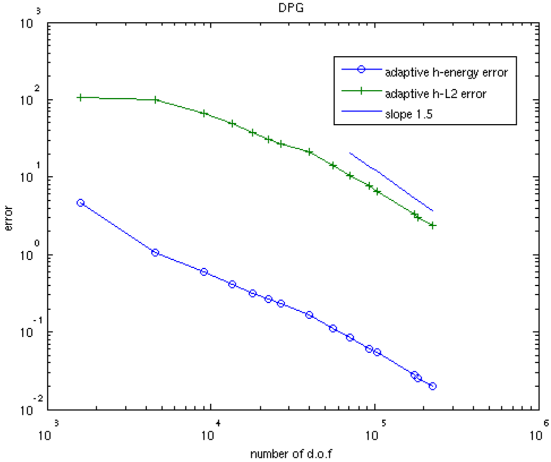
Convergence history for h-adaptive refinements using fluxes of order $p+1$ and order p. Decreasing the order for fluxes results in a loss of the optimal convergence rate.

L-shape Domain Problem. Standard Norm



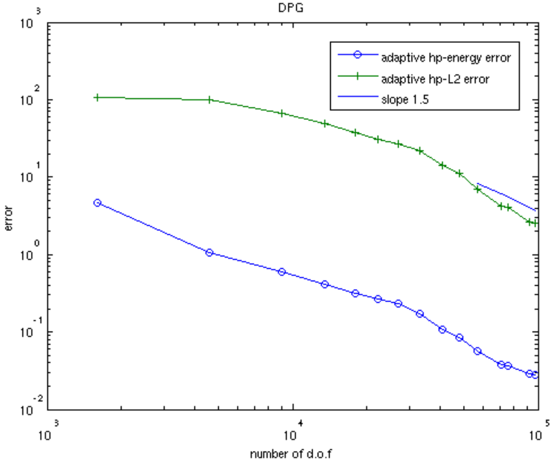
Uniform h-refinements. Energy error vs. L^2-error.

L-shape Domain Problem. Standard Norm



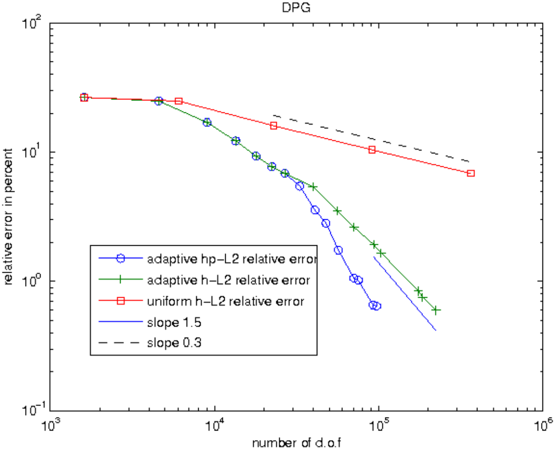
h-adaptive refinements. Energy error vs. L^2-error.

L-shape Domain Problem. Standard Norm



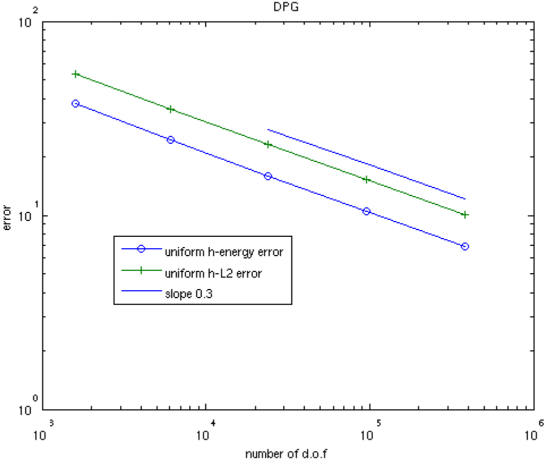
hp-adaptive refinements. Energy error vs. L^2-error.

L-shape Domain Problem. Standard Norm



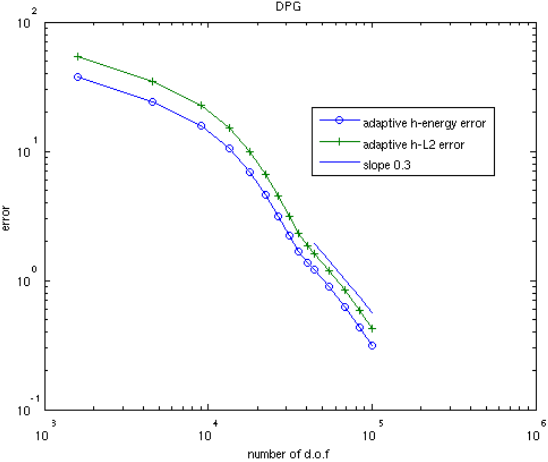
Comparison of different refinement strategies. Relative L^2-error vs. number of d.o.f.

L-shape Domain Problem. Optimal Norm



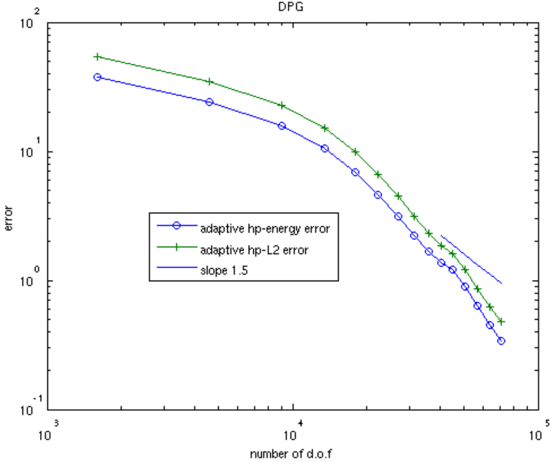
Uniform h-refinements. Energy error vs. L^2-error.

L-shape Domain Problem. Optimal Norm



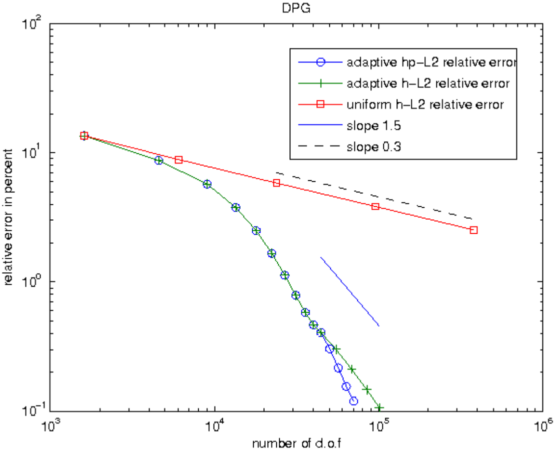
Adaptive h-refinements. Energy error vs. L^2-error.

L-shape Domain Problem. Optimal Norm



Adaptive h-refinements. Energy error vs. L^2-error.

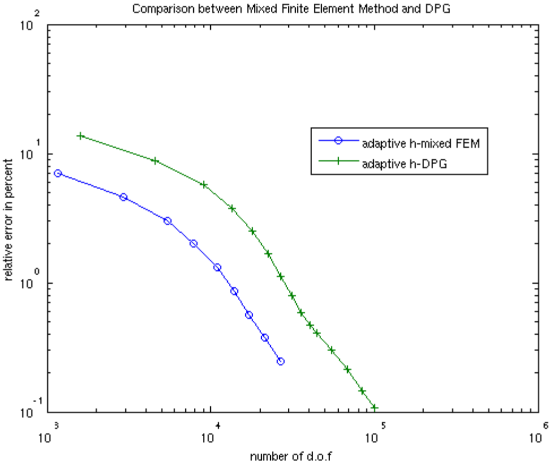
L-shape Domain Problem. Optimal Norm



Comparison of refinement strategies.

And the Winner Is...

And the Winner Is...



Comparison of mixed and DPG methods using relative L^2-norm.

Conclusions

- Both AFW and DPG methods deliver discretization errors of the same order as the best approximation error.

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Conclusions

- Both AFW and DPG methods deliver discretization errors of the same order as the best approximation error.
- DPG is more expensive but offers great perspectives for additional speed up: hybrid medthod, SPD stiffness matrix.
- The main advantage of DPG is its universality. In principle, it applies to any linear problem.
- Convergence analysis for DPG in multidimensions is a virgin teritory.

Thank You !

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