Canonical Families of Finite Element Differential Forms and Their Properties

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Outline Of Talk

1. Define two complexes of polynomial differential forms and discuss their relationships.
2. Define finite element differential forms and their degrees of freedom.
3. Discuss unisolvence.

Differential forms

$\omega \in \Lambda^k(\Omega)$, differential k-forms on $\Omega \subset \mathbb{R}^n$:

$$\omega_x(v_1, \dots, v_k) \in \mathbb{R}, \quad x \in \Omega, \; v_i \in \mathbb{R}^n,$$

k-linear and alternating in the v_i.

E.g., $dx^i(v) = v^i$, $dx^i \wedge dx^j(v, w) = v^i w^j - v^j w^i$.

$$\omega \in \Lambda^k(\Omega) \iff \omega_x = \sum_{1 \le \sigma_1 < \cdots < \sigma_k \le n} a_\sigma(x)\, dx_{\sigma_1} \wedge \cdots \wedge dx_{\sigma_k}$$

- Wedge product: $\omega \in \Lambda^k$, $\mu \in \Lambda^j \implies \omega \wedge \mu \in \Lambda^{k+j}$
- Exterior derivative: $\omega \in \Lambda^k \implies d\omega \in \Lambda^{k+1}$
- A k-form can be integrated over a k-dimensional set, $\int_S \omega$
- Stokes theorem: $\int_S d\mu = \int_{\partial S} \operatorname{tr} \mu \quad (\mu \in \Lambda^k, \dim S = k - 1)$

Polynomial differential forms

$\omega \in \mathcal{P}_r\Lambda^k$ or $\mathcal{H}_r\Lambda^k$, polynomial or homogenous polynomial differential k-forms of degree r:

$$\omega_x = \sum a_\sigma(x)\, dx_{\sigma_1} \wedge \cdots \wedge dx_{\sigma_k}, \qquad a_\sigma \in \mathcal{P}_r \text{ or } \mathcal{H}_r$$

For each polynomial degree $r \geq 0$, get homogeneous polynomial subcomplex of de Rham complex:

$$0 \to \mathcal{H}_r\Lambda^0 \xrightarrow{d} \mathcal{H}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{H}_{r-n}\Lambda^n \to 0$$

Taking direct sum, get polynomial de Rham complex:

$$0 \to \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n \to 0$$

Koszul complex

Koszul differential: κ: $\Lambda^k(\mathbb{R}^n) \mapsto \Lambda^{k-1}(\mathbb{R}^n)$

$$(\kappa\omega)_x(v_1, \dots, v_{k-1}) = \omega_x\big(x, v_1, \dots, v_{k-1}\big), \quad v_i \in \mathbb{R}^n,$$

$$\kappa(dx^i) = x_i, \quad \kappa(dx^i \wedge dx^j) = x^i dx^j - x^j dx^i, \dots$$

Clearly $\kappa \circ \kappa = 0$ and $\kappa : \mathcal{H}_r\Lambda^k \mapsto \mathcal{H}_{r+1}\Lambda^{k-1}$, i.e., κ increases polynomial degree and decreases form degree; exact opposite of exterior derivative d.

Homogeneous *Koszul complex*

$$0 \to \mathcal{H}_{r-n}\Lambda^n \xrightarrow{\kappa} \mathcal{H}_{r-n+1}\Lambda^{n-1} \xrightarrow{\kappa} \cdots \xrightarrow{\kappa} \mathcal{H}_r\Lambda^0 \to 0$$

Adding over polynomial degrees, get Koszul complex,

$$0 \to \mathcal{P}_{r-n}\Lambda^n \xrightarrow{\kappa} \mathcal{P}_{r-n+1}\Lambda^{n-1} \xrightarrow{\kappa} \cdots \xrightarrow{\kappa} \mathcal{P}_r\Lambda^0 \to 0$$

For $\Omega \subset \mathbb{R}^3$, the Koszul complex is:

$$0 \to \mathcal{P}_{r-3}(\Omega) \xrightarrow{x} \mathcal{P}_{r-2}(\Omega; \mathbb{R}^3) \xrightarrow{\times x} \mathcal{P}_{r-1}(\Omega; \mathbb{R}^3) \xrightarrow{\cdot x} \mathcal{P}_r(\Omega) \to 0$$

Direct sum decomposition of $\mathcal{H}_r\Lambda^k$

The homotopy relation (derivable from Cartan's magic formula) says:

$$(d\kappa + \kappa d)\omega = (r+k)\omega, \quad \omega \in \mathcal{H}_r\Lambda^k.$$

Using this relation, can show

$$\boxed{\mathcal{H}_r\Lambda^k = \kappa\mathcal{H}_{r-1}\Lambda^{k+1} \oplus d\mathcal{H}_{r+1}\Lambda^{k-1}.}$$

Proof: If $\omega \in \mathcal{H}_r\Lambda^k$, then $\eta = d\omega \in \mathcal{H}_{r-1}\Lambda^{k+1}$ and $\mu = \kappa\omega \in \mathcal{H}_{r+1}\Lambda^{k-1}$, and by homotopy formula: $\omega = c(\kappa\eta + d\mu)$, $(c = (r+k)^{-1})$. So, $\mathcal{H}_r\Lambda^k = \kappa\mathcal{H}_{r-1}\Lambda^{k+1} + d\mathcal{H}_{r+1}\Lambda^{k-1}$.

To show sum is direct, suppose $\omega \in \kappa\mathcal{H}_{r-1}\Lambda^{k+1} \cap d\mathcal{H}_{r+1}\Lambda^{k-1}$. Then $d\omega = 0$ and $\kappa\omega = 0$ (since $d \circ d = \kappa \circ \kappa = 0$), and again by homotopy formula, $\omega = 0$.

The space $\mathcal{P}_r^-\Lambda^k(\mathbb{R}^n)$

$$\mathcal{P}_r\Lambda^k = \mathcal{P}_{r-1}\Lambda^k \oplus \mathcal{H}_r\Lambda^k = \mathcal{P}_{r-1}\Lambda^k \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1} \oplus d\mathcal{H}_{r+1}\Lambda^{k-1}.$$

Use above to define space of k-forms intermediate between $\mathcal{P}_{r-1}\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ (generalizations of Raviart-Thomas-Nedéléc spaces):

$$\boxed{\mathcal{P}_r^-\Lambda^k = \mathcal{P}_{r-1}\Lambda^k \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}}$$

$$\boxed{\mathcal{P}_r\Lambda^k = \mathcal{P}_r^-\Lambda^k \oplus d\mathcal{H}_{r+1}\Lambda^{k-1}.}$$

Lemma: If $\omega \in \mathcal{P}_r^-\Lambda^k$ and $d\omega = 0$, then $\omega \in \mathcal{P}_{r-1}\Lambda^k$.

Proof: $\omega = \mu + \kappa\nu \in \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$

Since $d\omega = 0$, $d\kappa\nu = 0$.

But $d\kappa\nu = 0 \implies \kappa\nu = 0$: $\kappa\nu = c(d\kappa + \kappa d)\kappa\nu = 0$ by the homotopy formula.

2^{n-1} exact sequences

Have exact sequences for $\mathcal{P}_r^-$ and $\mathcal{P}_r$ spaces:

$$\mathbb{R} \hookrightarrow \mathcal{P}_r^-\Lambda^0 \xrightarrow{d} \mathcal{P}_r^-\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n \to 0,$$

$$\mathbb{R} \hookrightarrow \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n \to 0.$$

Can combine these two sequences to get 2^{n-1} exact sequences.

Since $\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0$ and $\mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n$, when $n = 3$, four sequences are:

$$\mathbb{R} \hookrightarrow \mathcal{P}_r\Lambda^0 \xrightarrow{\quad d \quad} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{\quad d \quad} \mathcal{P}_{r-2}\Lambda^2 \xrightarrow{\quad d \quad} \mathcal{P}_{r-3}\Lambda^3 \to 0,$$

$$\mathbb{R} \hookrightarrow \mathcal{P}_r\Lambda^0 \xrightarrow{\quad d \quad} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{\quad d \quad} \mathcal{P}_{r-1}^-\Lambda^2 \xrightarrow{\quad d \quad} \mathcal{P}_{r-2}\Lambda^3 \to 0,$$

$$\mathbb{R} \hookrightarrow \mathcal{P}_r\Lambda^0 \xrightarrow{\quad d \quad} \mathcal{P}_r^-\Lambda^1 \xrightarrow{\quad d \quad} \mathcal{P}_{r-1}\Lambda^2 \xrightarrow{\quad d \quad} \mathcal{P}_{r-2}\Lambda^3 \to 0,$$

$$\mathbb{R} \hookrightarrow \mathcal{P}_r\Lambda^0 \xrightarrow{\quad d \quad} \mathcal{P}_r^-\Lambda^1 \xrightarrow{\quad d \quad} \mathcal{P}_r^-\Lambda^2 \xrightarrow{\quad d \quad} \mathcal{P}_{r-1}\Lambda^3 \to 0.$$

Finite element differential forms: shape functions

Ω – bounded polyhedral domain $\subset \mathbb{R}^n$ triangulated into finite set $\mathcal{T}$ of n-simplices satisfying:

Union of elements of $\mathcal{T}$ is closure of Ω and intersection of any two is either empty or a common subsimplex of each.

Define two families of spaces of finite element differential forms with respect to $\mathcal{T}$: $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$.

$\mathcal{P}_1^-\Lambda^k(\mathcal{T})$ are classical Whitney forms.

Shape functions: First choose for each $T \in \mathcal{T}$ corresponding polynomial space $\mathcal{P}_r\Lambda^k(T)$ or $\mathcal{P}_r^-\Lambda^k(T)$.

Degrees of freedom

Next choose DOFs, i.e., basis for dual space with each element associated to a particular subsimplex $f \in \Delta(T)$ (set of all subsimplices of simplex T). DOFs determine interelement continuity: if a subsimplex is shared by two simplices, DOFs are the same.

For a d-dimensional subsimplex f of T, our DOFs have form

$$\omega \mapsto \int_f \operatorname{tr}_f \omega \wedge \eta, \qquad \eta \in \Lambda^{d-k}(f), \quad d \geq k.$$

DOFs for $\mathcal{P}_r\Lambda^k(T)$: $\int_f \operatorname{tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f - k}(f)$

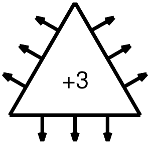
DOFs for $\mathcal{P}^-_r\Lambda^k(T)$: $\int_f \operatorname{tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f - k}(f)$

Note: DOFs for each family defined with weights in the other family.

Example: BDM$_2$ in 2D

DOFs for $\mathcal{P}_r\Lambda^k(T)$: $\int_f \operatorname{tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f - k}(f)$

BDM$_2$: $r = 2$, $k = 1$, $n = 2$. $\dim \mathcal{P}_2\Lambda^1(T) = 12$



vertices	no DOFs
edges	$\eta \in \mathcal{P}_2^-\Lambda^0(f) = \mathcal{P}_2\Lambda^0(f)$
triangle	$\eta \in \mathcal{P}_1^-\Lambda^1(T)$

Unisolvence of degrees of freedom of $\mathcal{P}_r\Lambda^k(T)$

Theorem: Let $0 \le k \le n$ and $r \ge 1$. If $\omega \in \mathcal{P}_r\Lambda^k(T)$ satisfies

$$\int_f \operatorname{tr}_f \omega \wedge \eta = 0, \qquad \eta \in \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f - k}(f), \quad f \in \Delta(T),$$

then $\omega = 0$.

Proof: Key step: prove following lemma.

Lemma: Let $0 \le k \le d$, $r \ge 1$, and f a subsimplex of T with $\dim f = d$. If $\omega \in \mathring{\mathcal{P}}_r\Lambda^k(f) = \{\omega \in \mathcal{P}_r\Lambda^k(f) : \operatorname{tr}_{\partial f} \omega = 0\}$ satisfies

$$\int_f \omega \wedge \eta = 0, \qquad \eta \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f), \qquad \text{then} \quad \omega = 0.$$

Supposing Lemma, prove Theorem by showing $\operatorname{tr}_f \omega = 0$ on all subsimplices f of dim k, then $k+1$, ... If dim $f = k$, hypotheses of Theorem says $\int_f \operatorname{tr}_f \omega \wedge \eta = 0$, $\forall \eta \in \mathcal{P}^-_r\Lambda^0(f) = \mathcal{P}_r\Lambda^0(f)$. Since $\operatorname{tr}_f \omega \in \mathcal{P}_r\Lambda^k(f)$, $\operatorname{tr}_f \omega = 0$. If dim $f = k+1$, then $\operatorname{tr}_f \omega \in \mathring{\mathcal{P}}_r\Lambda^k(f)$. Can apply Lemma. Continue to prove Theorem.

Rough idea of proof of Lemma

Characterize $\mathring{\mathcal{P}}_r\Lambda^k(f)$ and make explicit choice of η to show if $\phi \in \mathring{\mathcal{P}}_r\Lambda^k(f)$ satisfies

$$(*) \qquad \int_f \phi \wedge \eta = 0, \qquad \eta \in \mathcal{P}_{r+k-d}\Lambda^{d-k}(f), \qquad \text{then} \quad \phi = 0.$$

Note: stronger hypothesis then in Lemma, since now wedge product is zero $\forall \eta \in \mathcal{P}_{r+k-d}\Lambda^{d-k}(f)$, instead of $\mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f)$.

Using decomposition

$$\boxed{\mathcal{P}_{r+k-d}\Lambda^{d-k} = \mathcal{P}^-_{r+k-d}\Lambda^{d-k} \oplus d\mathcal{H}_{r+1+k-d}\Lambda^{d-k-1}}$$

and hypotheses of Lemma, result follows from (*) if we show

$$\int_f \omega \wedge d\mu = 0, \qquad \mu \in \mathcal{H}_{r+1+k-d}\Lambda^{d-k-1}.$$

In fact, show

$$\int_f \omega \wedge d\mu = 0, \qquad \mu \in \Lambda^{d-k-1}.$$

Rough idea of proof of lemma – continued

If $\omega \in \mathring{\mathcal{P}}_r\Lambda^k(f)$, then $d\omega \in \mathring{\mathcal{P}}_{r-1}\Lambda^{k+1}(f)$, so $\operatorname{tr}_{\partial f} d\omega = 0$.

$$\Longrightarrow \qquad \int_f \omega \wedge d\mu = \pm \int_f d\omega \wedge \mu, \quad \mu \in \Lambda^{d-k-1}(f).$$

Hence, enough to show $d\omega = 0$.

For $\mu \in \mathcal{P}_{r+k-d}\Lambda^{d-k-1}(f)$,

$$d\mu \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f) \subset \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f).$$

Conclude from hypothesis of Lemma:

$$\int_f d\omega \wedge \mu = \int_f \omega \wedge d\mu = 0, \qquad \mu \in \mathcal{P}_{r+k-d}\Lambda^{d-k-1}(f).$$

Applying (*) with $r \to r-1$ and $k \to k+1$, get $d\omega = 0$.

Unisolvence of degrees of freedom of $\mathcal{P}_r^-\Lambda^k(T)$

Theorem: Let $0 \le k \le n$ and $r \ge 1$. If $\omega \in \mathcal{P}_r^-\Lambda^k(T)$ satisfies

$$\int_f \operatorname{tr}_f \omega \wedge \eta = 0, \qquad \eta \in \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f-k}(f), \quad f \in \Delta(T),$$

then $\omega = 0$.

Proof: Key step: prove following lemma.

Lemma: Let $0 \le k \le d$, $r \ge 1$, and f a subsimplex of T with $\dim f = d$. If

$$\omega \in \mathring{\mathcal{P}}_r^-\Lambda^k(f) = \{\omega \in \mathcal{P}_r^-\Lambda^k(f) : \operatorname{tr}_{\partial f} \omega = 0\}$$

satisfies

$$\int_f \omega \wedge \eta = 0, \qquad \eta \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f), \qquad \text{then} \quad \omega = 0.$$

Rough idea of proof of lemma

Note if $\omega \in \mathring{\mathcal{P}}_{r-1}\Lambda^k(f)$, instead of $\mathring{\mathcal{P}}_r^-\Lambda^k(f)$, then $\omega = 0$ by (*).

However, for $\omega \in \mathring{\mathcal{P}}_r^-\Lambda^k(f)$, if $d\omega = 0$, then $\omega \in \mathring{\mathcal{P}}_{r-1}\Lambda^k(f)$.

Hence, enough to show that $d\omega = 0$.

For $\omega \in \mathring{\mathcal{P}}_r^-\Lambda^k(f)$, $d\omega \in \mathring{\mathcal{P}}_{r-1}\Lambda^{k+1}(f)$, so $\operatorname{tr}_{\partial f} d\omega = 0$. Integrating by parts:

$$\int_f d\omega \wedge \mu = \pm \int_f \omega \wedge d\mu, \quad \mu \in \Lambda^{d-k-1}(f).$$

For $\mu \in \mathcal{P}_{r+k-d}\Lambda^{d-k-1}(f)$, $d\mu \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f)$, so for such μ, $\int_f \omega \wedge d\mu = 0$ and hence $\int_f d\omega \wedge \mu = 0$.

Applying (*) with $r \to r-1$ and $k \to k+1$, conclude that $d\omega = 0$.

Summary: Key properties of F.E. differential forms

1. Two canonical families: $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$, $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$.
2. Uniform treatment of shape functions, DOFs, unisolvence, ...
3. DOFs for each family defined with weights in the other family.
4. Combine spaces to form 2^{n-1} complexes for each r.
5. $\{\omega \in L^2\Lambda^k : \omega|_T \in \mathcal{P}_r\Lambda^k(T),\ \forall T \in \mathcal{T}_h,$ DOFs single-valued on subsimplices of $\mathcal{T}_h\,\}$

$$= \{\omega \in H\Lambda^k : \omega|_T \in \mathcal{P}_r\Lambda^k(T),\ \forall T \in \mathcal{T}_h\}.$$

(Similarly for $\mathcal{P}_r^-$.)