Mimetic Finite Differences for Eigenvalue Problems

Francesca Gardini

Department of Mathematics, University of Pavia (Italy)

in collaboration with A. Cangiani, G. Manzini

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Outline

1. The Eigenvalue Problem
2. Mimetic Finite Difference Discretization
3. Spectral Approximation of Compact Operators
4. Main Results
5. Numerical Results

The Eigenvalue Problem

$$\begin{cases} \vec{F} = -\mathbb{K}\nabla p & \text{in } \Omega \\ \operatorname{div} \vec{F} = \lambda p & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega \end{cases} \qquad (\vec{F} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega)$$

$\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ Lipschitz bounded domain

Variational Form

find $\lambda \in \mathbb{R}$ s.t. there exists $(\vec{F}, p) \in H(\operatorname{div}, \Omega) \times L^2(\Omega)$, with $p \neq 0$:

$$\begin{cases} (\mathbb{K}^{-1}\vec{F}, \vec{G}) - (p, \operatorname{div} \vec{G}) = 0 & \forall\, \vec{G} \in H(\operatorname{div}, \Omega) \\ (\operatorname{div} \vec{F}, q) = \lambda(p, q) & \forall q \in L^2(\Omega) \end{cases}$$

$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_i \leq \ldots \nearrow +\infty$

$(\vec{F}, p) \in H^\sigma(\Omega)^d \times H^{1+\sigma}(\Omega)$ for some $\sigma > 1/2$ (depending on Ω)

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The Mimetic Finite Difference Method

- Main idea: mimic fundamental properties of the continuous problem through the definition of discrete operators that satisfy discrete analogs of the fundamental relations of vector and tensor calculus (e.g., Gauss-Green's identities)
- Advantages: the MFD method allows for general polyhedral meshes with non-matching and non-convex elements. This simplify adaptive mesh refinement/de-refinement. Moreover, polyhedral meshes naturally arises in the treatment of complex solution domains and heterogeneous materials.

The MFD method has been successfully employed for solving

- diffusion and convection-diffusion problems
- electromagnetic problems
- linear elasticity models
- fluid flows problem

Higher-order methods, a posteriori estimators for the diffusion problem and post-processing technique have also been developed.

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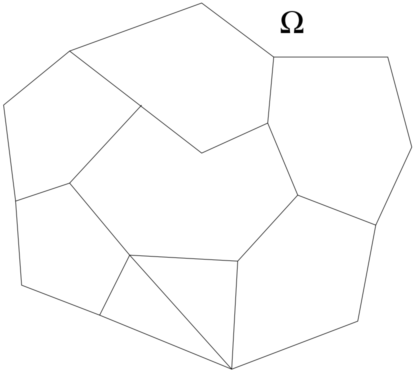

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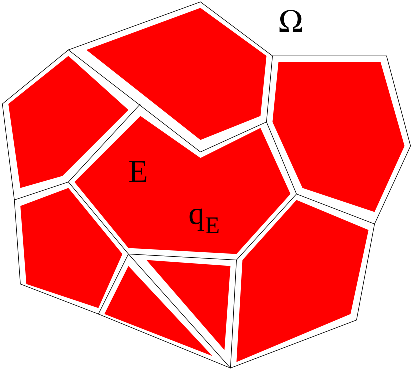

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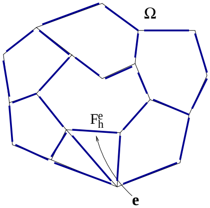

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- *Interpolation operators*:

$$(\mathbf{q}^{\mathrm{I}})_E = \frac{1}{|E|} \int_E q \quad \forall E \in \mathscr{T}_h$$

$$(\vec{G}^{\mathrm{I}})^e = \frac{1}{|e|} \int_e \vec{G} \cdot \mathbf{n}^e \quad \text{for any face } e \text{ of } \mathscr{T}_h$$

Scalar product in Q_h (L^2-type)

$$[\mathbf{p}, \mathbf{q}]_{Q_h} = \sum_E |E| p_E q_E$$

Scalar product in X_h

$$[\mathbf{F}, \mathbf{G}]_{X_h} = \sum_E [\mathbf{F}, \mathbf{G}]_E$$

$[\cdot, \cdot]_E$ local scalar product which defines the method

Discrete operators

- $\mathcal{DIV}_h : X_h \to Q_h$

$$(\mathcal{DIV}_h \mathbf{G})_E = \frac{1}{|E|} \sum_{e \in \partial E} G_E^e |e|$$

- $\mathcal{G}_h$ ($\approx -\mathbb{K}\nabla$) : $Q_h \to X_h$

$$[\mathbf{G}, \mathcal{G}_h \mathbf{q}]_{X_h} = [\mathbf{q}, \mathcal{DIV}_h \mathbf{G}]_{Q_h}$$

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$G_E^e = \pm G^e$ (depending on the orientation of $\mathbf{n}^e$ with respect to ∂E)

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Matrix form

$$\begin{pmatrix} A & -B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{F}_h \\ \mathbf{p}_h \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \mathbf{F}_h \\ \mathbf{p}_h \end{pmatrix}$$

Source Problem and its MFD Discretization

Source Problem

Given $b \in L^2(\Omega)$, find $(\vec{F}^s, p^s) \in H(\mathrm{div}, \Omega) \times L^2(\Omega)$ such that

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A priori error estimate
[Brezzi-Lipnikov-Shashkov, SINUM 2005]

$$\|p^s - \mathbf{p}_h^s\|_0 \leq Ch\left(\|p^s\|_2 + \|b\|_1\right)$$

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Assumptions on the local scalar product $[\cdot,\cdot]_E$

- stability condition

$$s_* \sum_{e\in\partial E} (G_E^e)^2|E| \le [\mathbf{G},\mathbf{G}]_E \le S^* \sum_{e\in\partial E} (G_E^e)^2|E|$$

- consistency condition

$$[(\mathbb{K}\nabla q^1)^{\mathrm{I}},\mathbf{G}]_E = -\int_E q^1(\mathcal{DIV}_h\mathbf{G})_E + \sum_{e\in\partial E} G_E^e \int_e q^1$$

with q^1 linear function on E

These conditions are fullfilled if

$$[\mathbf{F},\mathbf{G}]_{X_h} = \sum_E \int_E \mathbb{K}^{-1}\mathcal{R}_E(\mathbf{F})\cdot\mathcal{R}_E(\mathbf{G})$$

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Lifting Operator

For every element $E \in \mathscr{T}_h$ there exists a lifting operator $\mathcal{R}_E$ s.t.

$$\mathcal{R}_E(\mathbf{G}_E)|_e \cdot \mathbf{n}_E^e = G_E^e \qquad \forall e \in \partial E$$
$$\operatorname{div} \mathcal{R}_E(\mathbf{G}_E) = (\mathcal{DIV}_h \mathbf{G})_E \quad \text{in } E$$

for all $\mathbf{G} \in X_h$, and

$$\mathcal{R}_E(\vec{G}_E^{\mathrm{I}}) = \vec{G} \qquad (\mathbb{P}_0 - \textit{compatibility})$$

for all vector-valued functions $\vec{G}$ constant on E

Remark

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Spectral Approximation for Compact Operators

- $T : L^2(\Omega) \rightarrow L^2(\Omega)$ *second component* solution operator

$$\begin{cases} (\mathbb{K}^{-1}\vec{F}^s, \mathbf{G}) - (Tb, \operatorname{div} \vec{G}) & = & 0 & \forall \, \vec{G} \in H(\operatorname{div}, \Omega) \\ (\operatorname{div} \vec{F}^s, q) & = & (b, q) & \forall q \in L^2(\Omega) \end{cases}$$

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Convergence of Eigensolutions

Some notation

- λ_i, $\lambda_{i,h}$ eigenvalues of T and T_h (repeated according to their multiplicities)
- $\mathcal{E}_i$, $\mathcal{E}_{i,h}$ eigenspaces corresponding to λ_i and $\lambda_{i,h}$
- $m : \mathbb{N} \to \mathbb{N}$ such that for $N \in \mathbb{N}$

  $m(N)$ denotes the number of eigenvalues less than or equal to the Nth distinct eigenvalue

  $\lambda_{m(1)} < \lambda_{m(2)} < \ldots < \lambda_{m(N)} < \ldots$

- $\hat{\delta}(V, W) = \max\{\delta(V, W), \delta(W, V)\}$, V, W subspaces of $L^2(\Omega)$

  $$\delta(V, W) = \sup_{\substack{v \in V,\\ \|v\|_0 = 1}} \inf_{w \in W} \|v - w\|_0$$

Definition of convergence

$\forall \varepsilon > 0,\ \forall N \in \mathbb{N}\ \ \exists h_0 > 0$ such that $\forall h \leq h_0$

$$\max_{i=1,\dots,m(N)} |\lambda_i - \lambda_{i,h}| \leq \varepsilon$$

$$\hat{\delta}\left(\bigoplus_{i=1}^{m(N)} \mathcal{E}_i, \bigoplus_{i=1}^{m(N)} \mathcal{E}_{i,h}\right) \leq \varepsilon$$

Then no spurious eigenvalues pollute the spectrum, indeed

- each continuous eigenvalue is approximated by a number of discrete eigenvalues (counted with their multiplicity) that corresponds exactly to its multiplicity
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Convergence of Eigenmodes

A sufficient condition for the correct spectral approximation of a compact operator T is the uniform convergence to T of the family of discrete operators $\{T_h\}$, *i.e.*,

$$\|T - T_h\|_{\mathscr{L}(L^2(\Omega))} \to 0, \quad \textit{as } h \to 0$$

or, equivalently,

$$\|Tb - T_h b\|_0 = \|p^s - \mathbf{p}_h^s\|_0 \leq C\rho(h)\|b\|_0 \quad \forall b \in L^2(\Omega),$$

with $\rho(h)$ tending to zero as h goes to zero.

Convergence of Eigenmodes

A sufficient condition for the correct spectral approximation of a compact operator T is the uniform convergence to T of the family of discrete operators $\{T_h\}$, *i.e.*,

$$\|T - T_h\|_{\mathscr{L}(L^2(\Omega))} \to 0, \quad \text{as } h \to 0$$

or, equivalently,

$$\|Tb - T_h b\|_0 = \|p^s - \mathbf{p}_h^s\|_0 \leq C\rho(h)\|b\|_0 \quad \forall b \in L^2(\Omega),$$

with $\rho(h)$ *tending to zero as* h *goes to zero.*

Main Results

[Cangiani-G.-Manzini, CMAME (published online)]

New a priori error estimate

$$\|p^s - \mathbf{p}_h^s\|_0 \leq Ch^t \|b\|_0 \qquad t = \min\{1, \sigma\}$$

Convergence of eigensolutions

The family of operators $\{T_h\}$ converges uniformly to the operator T, that is,

$$\|T - T_h\|_{\mathscr{L}(L^2(\Omega))} \to 0 \quad \text{as } h \to 0$$

Erorr estimates

$$|\lambda - \lambda_h| \leq Ch^{2t}$$

$$\hat{\delta}(\mathcal{E}_\lambda, \mathcal{E}_{\lambda_h}) \leq Ch^t \qquad t = \min\{1, \sigma\}$$

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Starting Point

The MFD discretization of the source problem can be rewritten as

Given $b \in L^2(\Omega)$, find $(\mathbf{F}_h^s, \mathbf{p}_h^s) \in X_h \times Q_h$ such that

$$\begin{cases} (\mathbb{K}^{-1}\mathcal{R}\mathbf{F}_h^s, \mathcal{R}\mathbf{G}) - (\mathbf{p}_h^s, \operatorname{div}\mathcal{R}\mathbf{G}) = 0 & \forall\, \mathbf{G} \in X_h \\ (\operatorname{div}\mathcal{R}\mathbf{F}_h^s, \mathbf{q}) = (\mathbf{b}^{\mathrm{I}}, \mathbf{q}) & \forall\, \mathbf{q} \in Q_h \end{cases}$$

Error Equations:

$$\begin{cases} \left(\mathbb{K}^{-1}(\mathbf{F}^s - \mathcal{R}\mathbf{F}_h^s), \mathcal{R}\mathbf{G}\right) - (p^s - \mathbf{p}_h^s, \operatorname{div}\mathcal{R}\mathbf{G}) = 0 & \forall\, \mathbf{G} \in X_h \\ (\operatorname{div}(\mathbf{F}^s - \mathcal{R}\mathbf{F}_d^s), \mathbf{q}) = 0 & \forall\, \mathbf{q} \in Q_h \end{cases}$$

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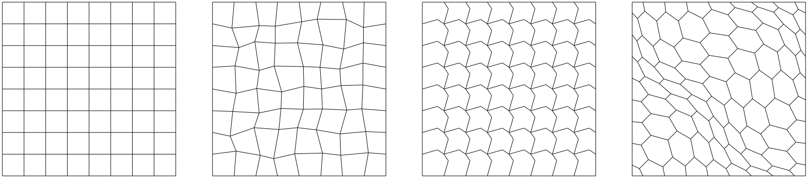
Test 1 (*Square domain, Dirichlet b.c., constant* $\mathbb{K}$)

$$\begin{array}{l}\Omega = (0,\pi)^2 \\ \mathbb{K} = \mathbb{I}\end{array} \qquad \Rightarrow \qquad \lambda = m_x^2 + m_y^2, \quad m_x, m_y \in \mathbb{N}\backslash\{0\}$$



uniform | random-quadrilateral | non-convex | mainly-hexagonal

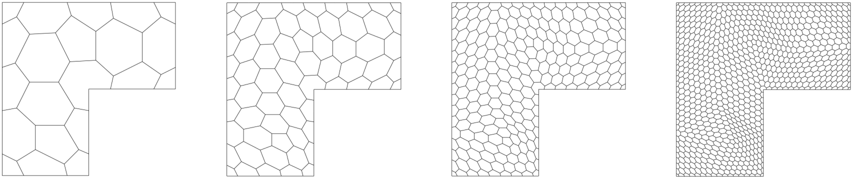
Exact			Computed		Slope
			Square mesh		
	$n =$ 4	8	16	32	
2	1.89928	1.97443	1.99358	1.99839	1.998
5	4.19192	4.78578	4.94565	4.98636	1.995
5	4.19192	4.78578	4.94565	4.98636	1.995
8	6.48456	7.59713	7.89772	7.97433	1.994
10	6.48456	8.99326	9.73955	9.93433	1.987
10	6.48456	8.99326	9.73955	9.93433	1.987
d.o.f	40	144	544	2112	
			Random-quadrilateral mesh		
	$n =$ 4	8	16	32	
2	1.87630	1.97711	1.99415	1.99850	1.96
5	4.10410	4.75143	4.94793	4.98717	2.06
5	4.35797	4.80898	4.95231	4.98751	1.91
8	6.40334	7.60310	7.90181	7.97588	2.02
10	6.74350	8.89931	9.73645	9.93790	2.08
10	7.21066	9.00442	9.77049	9.93909	1.91
d.o.f	40	144	544	2112	

Exact	Computed				Slope
Non-convex mesh					
	$n =$ 4	8	16	32	
2	1.72600	1.92149	1.97880	1.99452	1.95
5	3.05997	4.38210	4.82570	4.95451	1.94
5	3.19712	4.45325	4.84997	4.96115	1.95
8	3.99035	6.84007	7.67071	7.91307	1.92
10	4.11101	7.58900	9.27929	9.80818	1.91
10	4.36713	7.61811	9.28185	9.80837	1.91
d.o.f	64	256	1024	4096	
Mainly-hexagonal mesh					
	$n =$ 4	8	16	32	
2	1.67604	1.89449	1.96938	1.99174	1.89
5	3.22581	4.33719	4.80181	4.94669	1.89
5	3.39888	4.43402	4.83809	4.95716	1.92
8	4.70540	6.49496	7.50945	7.86235	1.83
10	4.73793	7.81226	9.34452	9.82605	1.91
10	5.00997	7.91864	9.37567	9.83337	1.91
d.o.f	76	244	868	3268	

Test 2 (*L-shaped domain, Neumann b.c., constant* $\mathbb{K}$)

Ω = L-shaped domain
$\mathbb{K} = \mathbb{I}$

$\Rightarrow$ no analytical expression for the eigenvalues
(reference values from M. Dauge's benchmark)



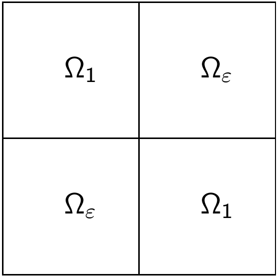
Exact	Computed				Slope
1.47562	1.33921	1.41169	1.45043	1.46608	1.40
3.53403	3.20231	3.42695	3.50297	3.52569	1.90
9.86960	8.08447	9.23274	9.67060	9.81403	1.84
d.o.f	76	224	736	2624	

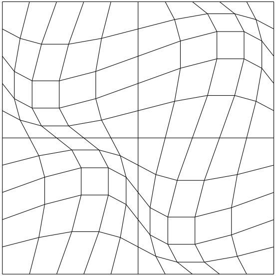
Test 3 (*Square domain, Neumann b.c., discontinuous* $\mathbb{K}$)

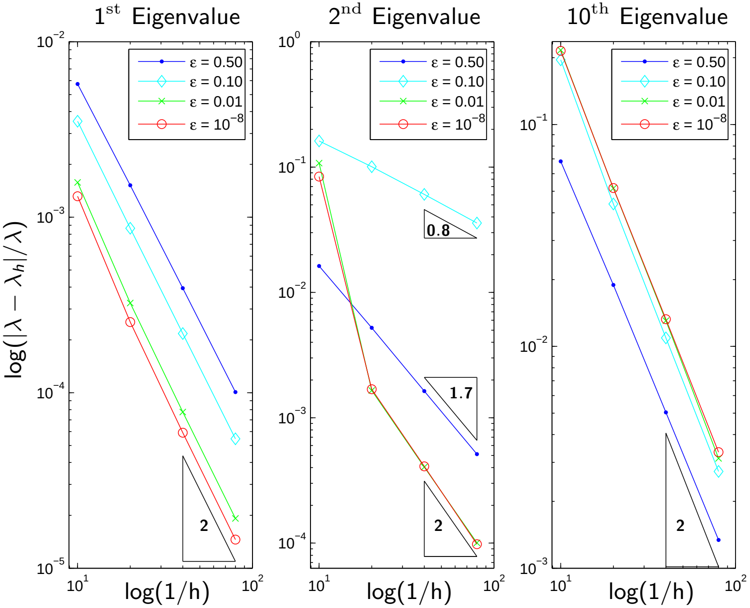
$\Omega = (0,\pi)^2$
$\mathbb{K}|_{\Omega_1} = \mathbb{I}$
$\mathbb{K}|_{\Omega_\varepsilon} = \varepsilon^{-1}\mathbb{I}$

$\Rightarrow$

no analytical expression for the eigenvalues
(reference values from M. Dauge's benchmark)



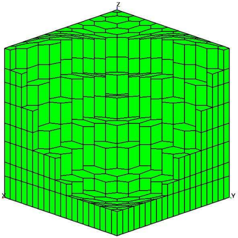




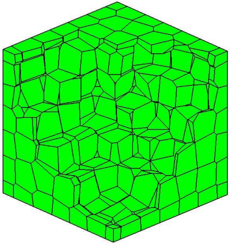
1st Eigenvalue
2nd Eigenvalue
10th Eigenvalue
ε = 0.50
ε = 0.10
ε = 0.01
ε = 10−8
log(|λ − λh|/λ)
log(1/h)
0.8
1.7
2

Test 4 (*Cube domain, Dirichlet b.c., constant* $\mathbb{K}$)

$$\begin{array}{l}\Omega = (0,\pi)^3 \\ \mathbb{K} = \mathbb{I}\end{array} \qquad \Rightarrow \qquad \lambda = m_x^2 + m_y^2 + m_z^2, \quad m_x, m_y, m_z \in \mathbb{N}\backslash\{0\}$$

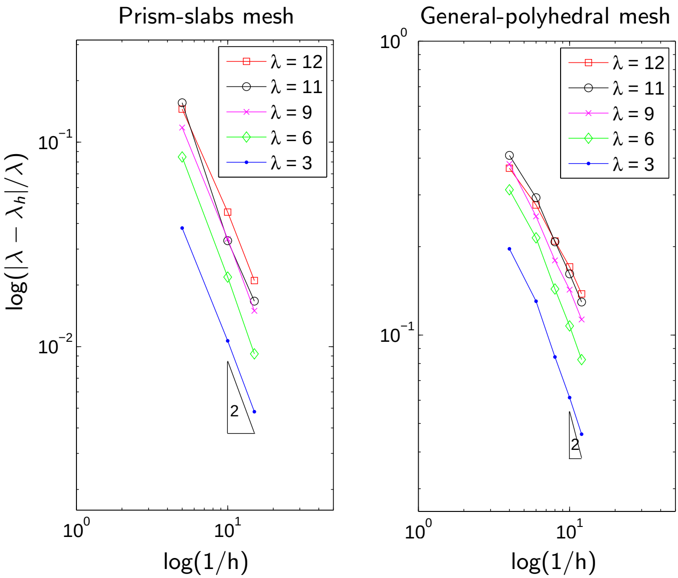


prism-slabs mesh



general-polyhedral

Error Plots


Prism-slabs mesh
General-polyhedral mesh
λ = 12
λ = 11
λ = 9
λ = 6
λ = 3
2
log(|λ − λ_h|/λ)
log(1/h)
10^0
10^1
10^-1
10^-2

Summary

We stated the convergence of the MFD method applied to eigenvalue problems.

We proved

- a new a priori error estimate for the source problem, which improves the original error bound as it does not require the H^1-regularity of the datum
- optimal convergence rates for the numerical approximation of eigenvalues and associated eigenspaces

Numerical evidence confirming the optimality of the method has also been given.