hp-finite elements for modeling bone conduction of sound in the human head

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Objective:
Develop a multi-physics *hp* FE code to investigate bone conduction of sound in the human head.

Contents:

1 Part: Air Force project
- coupling acoustics with elasticity;
- variational formulation;
- results for a simplified head model;

2 II Part: overview of FE code
- shape functions;
- element refinements;

3 III Part: geometry modeling
- surface reconstruction.
Air Force project
### Acoustics

**continuity eq:** \( \frac{d \rho}{dt} + \rho \text{div} \mathbf{v} = 0 \)

**linearize** \( \frac{\partial \rho}{\partial t} + \rho_0 \text{div} \mathbf{v} = 0 \)

**momentum eq:** \( \rho \frac{d \mathbf{v}}{dt} + \nabla p = 0 \)

**linearize** \( \rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0 \)

**state eq:** \( p = p(\rho) \)

**linearize** \( p = c^2 \rho \)

\[ \implies \frac{i \omega}{c^2} p + \rho_0 \text{div} \mathbf{v} = 0 \quad ; \quad i \omega \rho_0 \mathbf{v} + \nabla p = 0 \]
### Acoustics

<table>
<thead>
<tr>
<th>Equation Type</th>
<th>Original Equation</th>
<th>Linearized Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuity Eq:</td>
<td>( \frac{d \rho}{dt} + \rho \text{div} \mathbf{v} = 0 )</td>
<td>( \frac{\partial \rho}{\partial t} + \rho_0 \text{div} \mathbf{v} = 0 )</td>
</tr>
<tr>
<td>Momentum Eq:</td>
<td>( \rho \frac{d \mathbf{v}}{dt} + \nabla p = 0 )</td>
<td>( \rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla p = 0 )</td>
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\[
\Rightarrow \quad \frac{i \omega}{c^2} p + \rho_0 \text{div} \mathbf{v} = 0 ; \quad i \omega \rho_0 \mathbf{v} + \nabla p = 0
\]

### Elasticity

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</thead>
<tbody>
<tr>
<td>Momentum Eq:</td>
<td>( \rho \frac{d \mathbf{v}}{dt} = \text{div} \sigma + \mathbf{f} )</td>
<td>( \rho \frac{\partial \mathbf{v}}{\partial t} = \text{div} \sigma + \mathbf{f} )</td>
</tr>
<tr>
<td>Strain Def:</td>
<td>( \varepsilon = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) )</td>
<td>( \varepsilon = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) )</td>
</tr>
<tr>
<td>Constitutive Rel:</td>
<td>( \sigma_{ij} = E_{ijkl} \varepsilon_{kl} )</td>
<td>( \sigma_{ij} = E_{ijkl} u_{k,l} )</td>
</tr>
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</table>

\[
\Rightarrow \quad -\omega^2 \rho \mathbf{u} - \text{div} \sigma = \mathbf{f} ; \quad \sigma_{ij} = E_{ijkl} u_{k,l}
\]
Acoustics coupled with elasticity

Primary: $p$
Secondary: $v$

Primary: $u$
Secondary: $\sigma$

$\Omega_a$
$\Gamma_I$
$\Omega_e$
Acoustics coupled with elasticity

primary: $p$ \leftrightarrow secondary: $\sigma$

secondary: $\nu$ \leftrightarrow primary: $u$

$\Omega_a$, $\Omega_e$, $\Gamma_I$
Acoustics coupled with elasticity

primary: \( p \) \(-p n = \sigma n\) secondary: \( \sigma \)

secondary: \( v \) \(v \cdot n = i \omega u \cdot n\) primary: \( u \)
Acoustics coupled with elasticity

\[ -\rho n = \sigma n \]
\[ \mathbf{v} \cdot n = i\omega \mathbf{u} \cdot n \]
Acoustics coupled with elasticity

\[ p = p_D \quad \text{on} \quad \Gamma_{D,a} \]
\[ u = u_D \quad \text{on} \quad \Gamma_{D,e} \]
\[ -p n = \sigma n \quad \text{on} \quad \Gamma_{N,a} \]
\[ v \cdot n = i\omega u \cdot n \quad \text{on} \quad \Gamma_{C,e} \]
\[ v \cdot n = v_0 \quad \text{on} \quad \Omega_a \]
\[ v \cdot n = dp + v_0 \quad \text{on} \quad \Gamma_{C,a} \]
\[ \sigma n + i\omega \beta u = g \quad \text{on} \quad \Omega_e \]
Acoustics coupled with elasticity

\[ p = p_D \]

\[-p \mathbf{n} = \sigma \mathbf{n} \]

\[ \mathbf{v} \cdot \mathbf{n} = i\omega \mathbf{u} \cdot \mathbf{n} \]

\[ \int_{\Omega_a} \left( \frac{i\omega}{c^2} \mathbf{p}_q - \rho_0 \mathbf{v} \cdot \nabla q \right) + \]

\[ + \rho_0 \int_{\partial\Omega_a} \mathbf{v} \cdot \mathbf{n} q = 0 \quad \forall q \]

\[ \mathbf{v} \cdot \mathbf{n} = v_0 \]

\[ \mathbf{v} \cdot \mathbf{n} = d \mathbf{p} + v_0 \]
Acoustics coupled with elasticity

\[ p = p_D \]
\[ -p \mathbf{n} = \sigma \mathbf{n} \]
\[ \mathbf{v} \cdot \mathbf{n} = i\omega \mathbf{u} \cdot \mathbf{n} \]

\[ \int_{\Omega_a} \left( \frac{i\omega}{c^2} pq + \frac{1}{i\omega} \nabla p \cdot \nabla q \right) + \theta_0 \int_{\partial\Omega_a} \mathbf{v} \cdot \mathbf{n} q = 0 \quad \forall q \]

\[ \mathbf{v} \cdot \mathbf{n} = v_0 \]

\[ \mathbf{v} \cdot \mathbf{n} = dp + v_0 \]
Acoustics coupled with elasticity

\[ -p \mathbf{n} = \mathbf{\sigma n} \]

\[ \mathbf{v} \cdot \mathbf{n} = i\omega \mathbf{u} \cdot \mathbf{n} \]

\[ \int_{\Omega_a} \left( \frac{i\omega}{c^2} p q + \frac{1}{i\omega} \nabla p \cdot \nabla q \right) + \theta_0 \int_{\partial\Omega_a} \mathbf{v} \cdot \mathbf{n} q = 0 \quad \forall q \]

\[ \int_{\Omega_e} \left( -\omega^2 \rho_s \mathbf{u} \cdot \mathbf{v} + \mathbf{\sigma} : \nabla \mathbf{v} \right) + \int_{\partial\Omega_e} (\mathbf{\sigma n}) \cdot \mathbf{v} = \int_{\Omega_e} \mathbf{f} \cdot \mathbf{v} \quad \forall \mathbf{v} \]

\[ \mathbf{u} = \mathbf{u_D} \]

\[ \Gamma_{D,e} \]

\[ \Gamma_{N,e} \]

\[ \mathbf{\sigma n} = \mathbf{g} \]

\[ \Omega_e \]

\[ \Omega_a \]

\[ \Gamma_I \]

\[ \Gamma_{C,e} \]

\[ \mathbf{\sigma n} + i\omega \beta \mathbf{u} = \mathbf{g} \]
Acoustics coupled with elasticity

\[
\begin{align*}
\int_{\Omega_e} \left( -\omega^2 \rho_s u \cdot v + (E \nabla u) : \nabla v \right) &+ \int_{\Omega_a} \left( \frac{i\omega}{c^2} pq + \frac{1}{i\omega} \nabla p \cdot \nabla q \right) + \\
&+ \theta_0 \int_{\partial\Omega_a} v \cdot n q = 0 \quad \forall q
\end{align*}
\]

\[
\sigma n + i\omega \beta u = g
\]
Find \( u \in \tilde{u}_D + V, \ p \in \tilde{p}_D + V \) such that:

\[
\begin{align*}
\text{b}_{ee}(u, v) + \text{b}_{ae}(p, v) &= l_e(v) \quad \forall v \in V \\
\text{b}_{ea}(u, q) + \text{b}_{aa}(p, q) &= l_a(q) \quad \forall q \in V
\end{align*}
\]

where:

\[
\begin{align*}
V &= \{ v \in H^1(\Omega_e)^3 : v = 0 \text{ on } \Gamma_{D,e} \} ; \quad V = \{ q \in H^1(\Omega_a) : q = 0 \text{ on } \Gamma_{D,a} \} \\
\text{b}_{ee}(u, v) &= \int_{\Omega_e} (E_{ijkl} u_{k,l} v_{i,j} - \varrho_s \omega^2 u_i v_i) \, dx + i\omega \int_{\Gamma_{C,e}} \beta_{ij} u_j v_i \, dS \\
\text{b}_{ae}(p, v) &= \int_{\Gamma_I} p v_i n_i \, dS \\
\text{b}_{ea}(u, q) &= -\omega^2 \varrho_f \int_{\Gamma_I} u_i n_i q \, dS \\
\text{b}_{aa}(p, q) &= \int_{\Omega_a} (\nabla p \cdot \nabla q - k^2 pq) \, dx + i\omega \int_{\Gamma_{C,a}} \varrho_f dq \, dS \\
l_e(v) &= \int_{\Omega_e} f_i v_i \, dx + \int_{\Gamma_{N,e} \cup \Gamma_{C,e}} g_i v_i \, dS ; \quad l_a(q) = i\omega \varrho_f \int_{\Gamma_{N,a} \cup \Gamma_{C,a}} v_0 q \, dS
\end{align*}
\]
By introducing product space $W = V \times V$ we obtain:

Find $(u, q) \in W$ such that:

$$
\int_{\Omega_e} E_{ijkl} u_{k,l} v_{i,j} + \int_{\Omega_a} \nabla p \cdot \nabla q + \\
\left. a((u,p);(v,q)) \right|_{\partial \Omega}
$$

$$
- \int_{\Omega_e} \rho_s \omega^2 u_i v_i - \int_{\Omega_a} k^2 p q + i\omega \int_{\Gamma_{C,e}} \beta_{ij} u_j v_i + i\omega \int_{\Gamma_{C,a}} \varphi_f dp q + \int_{\Gamma_l} p v_i n_i - \omega^2 \varphi_f \int_{\Gamma_l} u_i n_i q
$$

$$
c((u,p);(v,q))
$$

$$
= \int_{\Omega_e} f_i v_i + \int_{\Gamma_{N,e} \cup \Gamma_{C,e}} g_i v_i + i\omega \varphi_f \int_{\Gamma_{N,a} \cup \Gamma_{C,a}} v_0 q \quad \forall (v, q)
$$

$$
l((v,q))
$$

Equivalently, if $w = (u, p)$, then

Find $w \in W$ such that $(I + A^{-1} C) w = A^{-1} l$ in $W$

where

$A : W \to W^* \text{ is coercive ; } C : W \to L^2 \text{ is compact}$

Fredholm alternative guarantees existence and uniqueness of the solution for a smooth and bounded domain.
Simplified Head Model

P. Gatto, K. Kim, M. Paszynski, W. Rachowicz, and L. Demkowicz

$hp$-finite elements
Simplified Head Model

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hp-finite elements
Simplified Head Model

Ossicles
Cochlea
Eustachian Tube
Tympanic Membrane
Middle Ear
Oval Window

Ossicles
Oval window
Cochlea
Tympanic membrane
Basilar membrane
Model is generated by NETGEN and our Geometry Modeling Package:
- 16,004 tetrahedra
- 3,228 prisms
- 56 pyramids
- 1,070,190 dof's for $p = 5$
Pressure distributions for reference values $p_{ref} = 100 \text{[Pa]}$, $h_{ref} = 0.01 \text{[m]}$, $\omega_{ref} = 400\pi$:

Total acoustic pressure (incident and scattered) on skull, $\omega = 400\pi$

Total pressure (elastic and acoustic) on upper half of cochlea, $\omega = 400\pi$

<table>
<thead>
<tr>
<th>material</th>
<th>$E$ [MPa]</th>
<th>$\nu$</th>
<th>$\rho$ [kg/m$^3$]</th>
<th>$c_p$ [m/s]</th>
<th>$c_s$ [m/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>tissue (brain)</td>
<td>0.67</td>
<td>0.48</td>
<td>1040</td>
<td>75</td>
<td>15</td>
</tr>
<tr>
<td>skull (bone)</td>
<td>6500</td>
<td>0.22</td>
<td>1412</td>
<td>2293</td>
<td>1374</td>
</tr>
<tr>
<td>cochlea (water)</td>
<td>1000</td>
<td>1.2</td>
<td>1500</td>
<td></td>
<td></td>
</tr>
<tr>
<td>air</td>
<td></td>
<td></td>
<td></td>
<td>344</td>
<td></td>
</tr>
</tbody>
</table>

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FE code: $hp3d$
FE code: hp3d

Supports elements of all shapes: hexahedra, prisms, tetrahedra, and pyramids;

anisotropic: $\mathcal{P}^p(\xi_1) \otimes \mathcal{P}^q(\xi_2) \otimes \mathcal{P}^r(\xi_3)$

isotropic: $\mathcal{P}^p(\xi_1, \xi_2, \xi_3)$
Hierarchical shape functions: vertex shape function, edge bubbles, face bubbles, and interior bubbles;

see: P. Gatto and L. Demkowicz, “Construction of \(H^1\)-Conforming Shape Functions for Elements of All Types and Transfinite Interpolation”, Finite Elements in Design and Analysis, March 2010.
Orientations taken into account in the shape functions routine;

old approach

new approach
Orientations taken into account in the shape functions routine;

call shape3H(Type,Xi,Norder,Nedge_orient,Nface_orient, Nrdof,ShapH,DshapH)
Orientations taken into account in the shape functions routine;

old approach

new approach

call shape3H(Type, Xi, Norder, Nedge_orient, Nface_orient, Nr dof, ShapH, DshapH)

Isotropic – tetrahedron and pyramid – and anisotropic – prism – refinements;

isotropic ref.

cut off corners

shortest diag.

isotropic ref.

anisotropic ref.

anisotropic ref.
Geometry modeling
Parametric element

Build exact geometry map $\mathbf{x} = \mathbf{x}(\xi)$ through Transfinite Interpolation;

Generate geometry dof’s through Projection Based Interpolation:

$$
\mathbf{x}_{hp} = \mathbf{x}_v + \mathbf{x}_e + \mathbf{x}_f + \mathbf{x}_m
$$

$$
\mathbf{x}_v = \mathbf{x} \quad \forall \mathbf{v}
$$

$$
\int_e \partial_s (\mathbf{x} - \mathbf{x}_v - \mathbf{x}_e) \partial_s \varphi = 0 \quad \forall \varphi
$$

$$
\int_f \nabla_f (\mathbf{x} - \mathbf{x}_v - \mathbf{x}_e - \mathbf{x}_f) \cdot \nabla_f \varphi = 0 \quad \forall \varphi
$$

$$
\int_K \nabla (\mathbf{x} - \mathbf{x}_v - \mathbf{x}_e - \mathbf{x}_f - \mathbf{x}_m) \cdot \nabla \varphi = 0 \quad \forall \varphi
$$

Geometry dof’s are updated after every $hp$ refinement:
Parametric element

Build exact geometry map \( x = x(\xi) \) through Transfinite Interpolation;

Generate geometry dof’s through Projection Based Interpolation:

\[
\begin{align*}
x_{hp} &= x_v + x_e + x_f + x_m \\
x_v &= x \\
\int_{e} \partial_s (x - x_v - x_e) \partial_s \varphi &= 0 \\
\int_{f} \nabla_f( x - x_v - x_e - x_f ) \cdot \nabla_f \varphi &= 0 \\
\int_{K} \nabla(x - x_v - x_e - x_f - x_m) \cdot \nabla \varphi &= 0
\end{align*}
\]

Geometry dof’s are updated after every \( hp \) refinement:
Build exact geometry map \( x = x(\xi) \) through Transfinite Interpolation;

Generate geometry dof’s through Projection Based Interpolation:

\[
x_{hp} = x_v + x_e + x_f + x_m
\]

\[
x_v = x
\]

\[
\int_e \partial_s (x - x_v - x_e) \partial_s \varphi = 0
\]

\[
\int_f \nabla_f (x - x_v - x_e - x_f) \cdot \nabla_f \varphi = 0
\]

\[
\int_K \nabla (x - x_v - x_e - x_f - x_m) \cdot \nabla \varphi = 0
\]

Geometry dof’s are updated after every \( hp \) refinement:
Towards a more realistic model

Data coming from an MRI scan, i.e. points laying on iso-surfaces, are available for real objects;

How should the exact geometry map be built?
Towards a more realistic model

Data coming from and MRI scan, i.e. points laying on iso-surfaces, are available for real objects;

How should the exact geometry map be built?

Reconstruct smooth $G^1$ surfaces by joining polynomial patches;
use transfinite interpolation to obtain conforming blocks.
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A reconstruction scheme for unstructured quadrilateral grids:

Demkowicz et al, $G^1$-Interpolation and geometry reconstruction for higher order finite elements, CMAME, June 2008.
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What about triangular grids?

Gregory patch, i.e NURBS of deg. 7

Piecewise quartic Bezier patch

Quartic Bezier patch

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Towards a more realistic model

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What about triangular grids?

*We developed a fully local interpolation scheme of degree 7 for triangular grids.*
Surface reconstruction

Fine grid

NETGEN mesh

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hp -finite elements
Let $\mathbf{r} = \mathbf{r}(x, y)$ be the surface parameterization, then:

\[
\begin{align*}
    n_i \frac{\partial^2 r_i}{\partial s} &= -a_{ij} \frac{\partial r_i}{\partial s} \frac{\partial r_j}{\partial s} \\
    n_i \frac{\partial^2 r_i}{\partial x \partial y} &= -a_{ij} \frac{\partial r_i}{\partial x} \frac{\partial r_j}{\partial y}
\end{align*}
\]

reconstruct normal $\mathbf{n}$ and curvature \{a_{ij}\} at each point;

for each point determine tangent directions;

reconstruct curves as polynomials of degree 5 by minimizing $\int |\ddot{\mathbf{r}}|^2 ds$. 

Wireframe
Surface reconstruction

Fine linear mesh and reconstructed surface

14,316 surface triangles

538 surface triangles

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$hp$-finite elements
Summary

- **Overview of Air Force project:**
  - acoustics coupled with elasticity;
  - simplified model (algebraic surfaces);
  - preliminary results;

- **Geometry Modeling Package:** transfinite interpolation for algebraic surfaces;
- overview of FE code `hp3d`:
  - elements of all shapes (isotropic and anisotropic);
  - shape functions and orientations;
  - projection based interpolation;
  - refinements;
- surface reconstruction: smooth interpolation on triangular grids.

Thank you for your attention!

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