

A non-standard Finite Element Method based on boundary integral operators

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Motivation

Let $\Omega \subset \mathbb{R}^d$ a domain. We want to solve the PDE

$$-\operatorname{div}(a(x)\nabla u(x)) = f(x), \quad x \in \Omega$$

plus suitable boundary conditions on $\Gamma = \partial\Omega$.

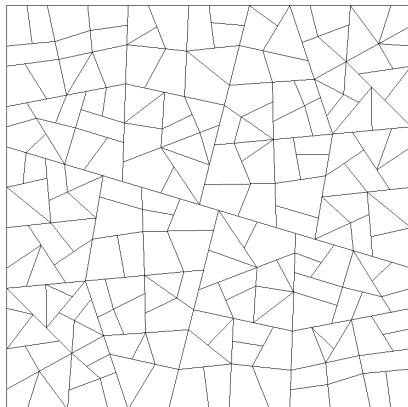
Discretize the domain by a mesh consisting of N polygonal/polyhedral elements T_i :

$$\bar{\Omega} = \bigcup_{i=1}^N \bar{T}_i$$

Meshes

We want to treat general polygonal/polyhedral meshes:

- arbitrary and mixed element shapes
- hanging nodes supported naturally



Motivation

These kinds of meshes arise, e.g., in

- reservoir simulation,

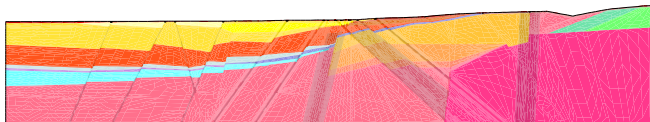
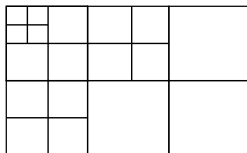


Image: rock permeability $\alpha(x)$ at Sellafield, UK. (c) NIREX UK Ltd.

- adaptive mesh refinement.



Another advantage: automatic mesh manipulation (join, split, manipulate elements)

Other possible approaches

Mimetic finite difference method

- Kuznetsov, Lipnikov, Shashkov:
The mimetic finite difference method on polygonal meshes for diffusion-type problems, 2004
- Brezzi, Lipnikov, Shashkov:
Convergence of Mimetic Finite Difference Method for Diffusion Problems on Polyhedral Meshes, 2004

Discontinuous Galerkin method

- Dolejsi, Feistauer, Sobotikova:
Analysis of the discontinuous Galerkin method for nonlinear convection-diffusion problems, 2005

Our approach

Has its roots in symmetric BEM Domain Decomposition method.

- 1 Use piecewise PDE-harmonic ansatz functions in every element (local Trefftz method).
- 2 Reduce the variational formulation to the *skeleton* of the mesh.
- 3 Use boundary integral operators to solve local Dirichlet-to-Neumann problems.

Prior work

A selection of previous work related to this approach:

- Hsiao, Wendland:
Domain Decomposition in Boundary Element Methods, 1991.
- Hsiao, Steinbach, Wendland:
Domain decomposition methods via boundary integral equations, 2000.
- Copeland, Langer, Pusch:
From the boundary element method to local Trefftz finite element methods on polyhedral meshes, 2009.

Model problem

Our method is designed to treat piecewise constant coefficients:

$$a(x) \equiv a_i \quad \text{for } x \in T_i$$

For simplicity, we assume $a(x) \equiv 1$, $f \equiv 0$, i.e. Laplace equation:

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega, \\ u &= g && \text{on } \Gamma = \partial\Omega. \end{aligned}$$

Variational formulation

For all $v \in H_0^1(\Omega)$,

$$0 = \int_{\Omega} \nabla u \cdot \nabla v$$

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Variational formulation

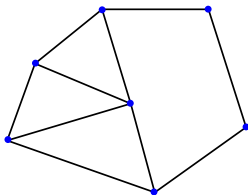
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Is defined purely on the element boundaries $(\Gamma_i)_{i=1,\dots,N}$.

Function spaces on the skeleton

Let $\Gamma_S := \bigcup_i \Gamma_i$ denote the *skeleton* of the mesh.



Let $H^{1/2}(\Gamma_S)$ be the trace space of $H^1(\Omega)$ onto Γ_S .

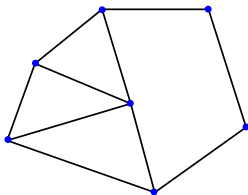
Let

$$W = \{v \in H^{1/2}(\Gamma_S) : v|_{\Gamma} = 0\}$$

be the space of skeletal functions which are zero on the boundary of Ω .

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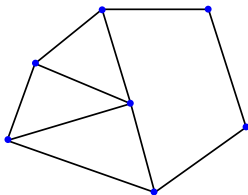
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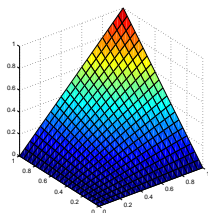
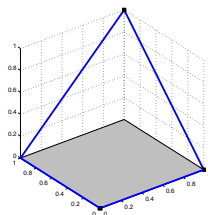
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Harmonic extension

For any (sufficiently regular) function on the boundary of a domain, there is a unique way to extend it to a harmonic function ($\Delta u = 0$) in the domain:

$$\mathcal{H}_i : H^{1/2}(\Gamma_i) \rightarrow H^1(T_i)$$



We also define $\mathcal{H}_S : H^{1/2}(\Gamma_S) \rightarrow H^1(\Omega)$ by piecing together the local harmonic extensions.

Skeletal variational formulation

Recall:

$$\sum_i \int_{\Gamma_i} \frac{\partial u}{\partial n_i} v = 0 \quad \forall v \in H_0^1(\Omega). \quad (1)$$

Making use of the bijection

$$u_i = \gamma_{T_i} u \quad \longleftrightarrow \quad u = \mathcal{H}_i u_i \text{ on } T_i,$$

we can rewrite (1) as

$$\sum_i \int_{\Gamma_i} \frac{\partial}{\partial n_i} (\mathcal{H}_i u_i) v_i = 0 \quad \forall v \in W. \quad (2)$$

With the *Dirichlet-to-Neumann map* $S_i u_i := \frac{\partial}{\partial n_i} (\mathcal{H}_i u_i)$, we get

$$\sum_i \int_{\Gamma_i} S_i u_i v_i = 0 \quad \forall v \in W. \quad (3)$$

Relation to standard domain VF

Standard variational formulation

Find $u_\Omega \in H^1(\Omega)$ with $u_\Omega|_\Gamma = g$ such that for all $v_\Omega \in H_0^1(\Omega)$,

$$\int_{\Omega} \nabla u_\Omega \cdot \nabla v_\Omega \, dx = 0.$$

Skeletal variational formulation

Find $u \in H^{1/2}(\Gamma_S)$ with $u|_\Gamma = g$ such that for all $v \in W$,

$$\sum_{i=1}^N \langle S_i u_i, v_i \rangle_{\Gamma_i} = 0.$$

The formulations are **equivalent**: $u_\Omega = \mathcal{H}_i u_i$ on T_i .

Evaluating the Dirichlet-to-Neumann map

Evaluating $S_i u_i = \frac{\partial}{\partial n_i}(\mathcal{H}_i u_i)$:

- 1 solve a Laplace problem on T_i with Dirichlet data u_i ,
- 2 obtain Neumann data of solution.

More directly:

$$S_i = D_i + \left(\frac{1}{2}I + K_i'\right) V_i^{-1} \left(\frac{1}{2}I + K_i\right)$$

with the boundary integral operators

- V_i , single layer potential operator,
- K_i , double layer potential operator,
- D_i , hypersingular operator.

Defined in terms of the *fundamental solution* of the differential operator.

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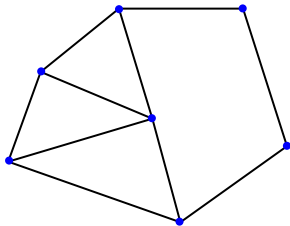
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Discretization

A Galerkin method is obtained by choosing a finite-dimensional subspace $W_h \subset W$.

E.g.: piecewise linear, continuous functions (on the skeleton).
Degrees of freedom are chosen in the vertices of the elements.



Discretizing the Dirichlet-to-Neumann map

Symmetric representation of **Steklov-Poincaré operator**:

$$S_i u_i = D_i u_i + \left(\frac{1}{2}I + K_i'\right) w_i, \quad \text{where}$$
$$V_i w_i = \left(\frac{1}{2}I + K_i\right) u_i$$

Problem: We cannot compute $w_i \in H^{-1/2}(\Gamma_i)$ exactly. Instead, perform a local Galerkin projection $w_{h,i} \approx w_i$ to some finite-dimensional space and compute

$$\tilde{S}_i u_i = D_i u_i + \left(\frac{1}{2}I + K_i'\right) w_{h,i}.$$

Natural choice: space of piecewise constant boundary functions, since w_i represents a normal derivative.

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Discretization of the variational problem

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This creates a **s.p.d.** and **sparse** stiffness matrix.
 For purely tetrahedral mesh: **identical** to FEM.

Skeletal energy norm

We introduce the skeletal energy norm on $H^{1/2}(\Gamma_S)$,

$$\|v\|_S := \left(\sum_{i=1}^N \langle S_i v_i, v_i \rangle \right)^{1/2} = \left(\sum_{i=1}^N |\mathcal{H}_i v_i|_{H^1(T_i)}^2 \right)^{1/2} .$$

Céa-type error estimate

Theorem

$$|u_\Omega - \mathcal{H}_S u_h|_{H^1(\Omega)} \leq C_S \left(\inf_{v_h \in W_h} \|u - v_h\|_S + \sqrt{\sum_{i=1}^N \inf_{z_{h,i} \in Z_{h,i}} \left\| \frac{\partial u_\Omega}{\partial n_i} - z_{h,i} \right\|_{V_i}^2} \right),$$

where C_S is a mesh-dependent constant.

Proof: Using the first lemma of Strang.

For a proper error estimate, we have to:

- bound the mesh constant C_S ,
- estimate the Dirichlet approximation error,
- estimate the Neumann approximation error.

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Mesh constants

The constant C_S depends only on

$$c_{0,i} = \inf_{v \in H_*^{1/2}(\Gamma_i)} \frac{\langle D_i v, v \rangle}{\langle V_i^{-1} v, v \rangle} \in (0, \frac{1}{4}) \quad \forall i \in \{1, \dots, N\}.$$

Recent results by C. Pechstein:

$c_{0,i}$ may be bounded away from zero solely in terms of

- the Jones parameter $C_U(T_i)$,
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Estimating the approximation errors

Dirichlet approximation error:

Introduce a regular local triangulation of T_i and use standard FEM approximation results.

Neumann approximation error:

- recent results by C. Pechstein:
bound $\| \cdot \|_{V_i}$ by $C(C_U(T_i), C_P(T_i)) \| \cdot \|_{H_*^{-1/2}(\Gamma_i)}$
- again use a local triangulation of T_i composed of regular tetrahedra
- estimate error piecewise on each triangle on the surface of these tetrahedra

Main result

Theorem

If every polyhedral element T_i

- has a regular tetrahedral triangulation,
- has a uniform upper bound on the number of its surface triangles,
- has a uniform upper bound on its Jones and Poincar e constants,

and if $u_\Omega \in H^2(\Omega)$ is the exact solution, we have

$$|u_\Omega - \mathcal{H}_S u_h|_{H^1(\Omega)} \leq C \left(\sum_i h_i^2 |u_\Omega|_{H^2(T_i)}^2 \right)^{\frac{1}{2}} \leq Ch |u_\Omega|_{H^2(\Omega)}.$$

A 3D numerical example

- implementation in C++ based on PARMAX framework by Pechstein & Copeland
- generation of BEM matrix entries: OSTBEM (Steinbach et al.)
- solution of linear system by non-preconditioned CG method

Problem specification

Solve the pure Dirichlet Laplace equation,

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega, \\ u(x) &= g(x) && \forall x \in \partial\Omega, \end{aligned}$$

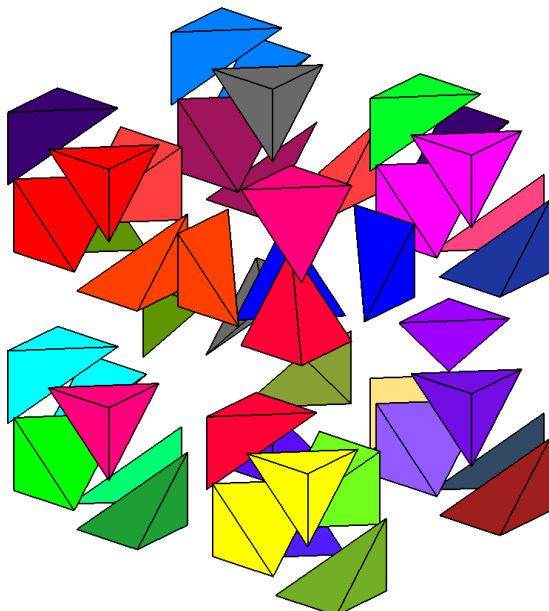
on the unit cube, $\Omega = (0, 1)^3$.

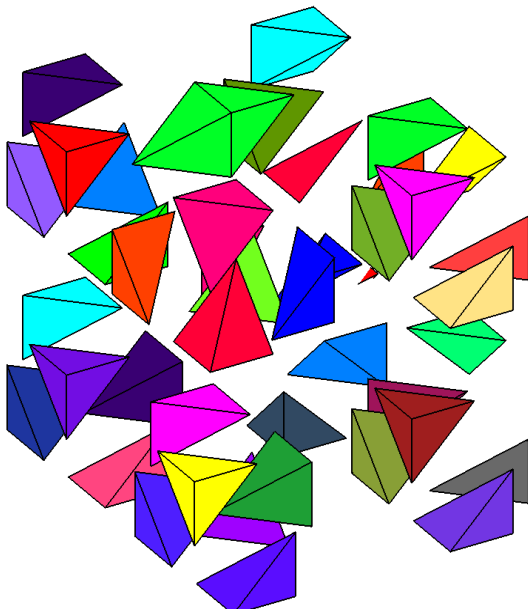
Exact solution: $u(x, y, z) = \exp(x) \cos(y)(1 + z)$

Two sets of meshes:

- a purely tetrahedral mesh,
- a mixed mesh consisting of tetrahedra and polyhedra

Number of degrees of freedom is identical.



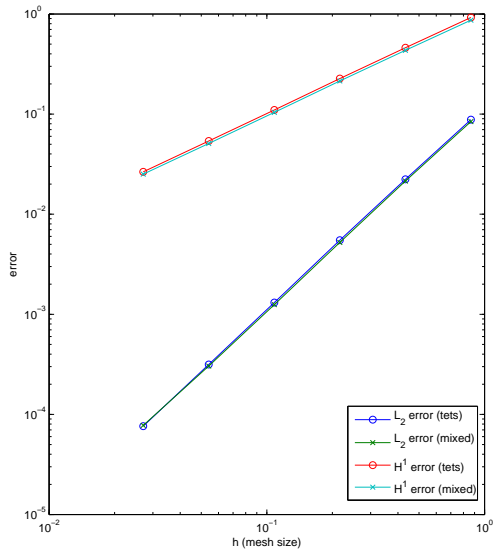


Tetrahedral mesh:

mesh size h	H^1 error	L^2 error	#tets
0.866025	0.923507	0.0879679	48
0.433013	0.459565	0.0223147	384
0.216506	0.226186	0.00549834	3.072
0.108253	0.109806	0.00131165	24.576
0.0541266	0.0537825	0.000315016	196.608
0.0270633	0.0264988	7.62441e-05	1.572.864

Mixed mesh:

mesh size h	H^1 error	L^2 error	#tets	#polys
0.866025	0.867685	0.0842554	40	4
0.433013	0.433557	0.0214242	258	63
0.216506	0.214188	0.00522372	2.044	514
0.108253	0.103955	0.00124863	15.822	4.377
0.0541266	0.0508436	0.000304395	125.350	35.629
0.0270633	0.0251327	7.76704e-05	996.390	288.237



Outlook

- treat convection-diffusion-reaction problems:

$$-\operatorname{div}(a(x)\nabla u(x)) + b(x) \cdot \nabla u(x) + c(x)u(x) = f(x), \quad x \in \Omega$$

- DG discretization
- develop optimal solvers
- higher order schemes: \rightarrow Levy functions (Wendland)

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G. C. Hsiao, O. Steinbach, and W. L. Wendland.

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G. C. Hsiao and W. L. Wendland.

Domain decomposition in boundary element methods.

In R. Glowinski, Y. A. Kuznetsov, G. Meurant, J. Périaux, and O. B. Widlund, editors, *Proceedings of the Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations, Moscow, May 21-25, 1990*, pages 41–49, Philadelphia, 1991. SIAM.



C. Pechstein.

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DK-Report 09-11, Doctoral Program in Computational Mathematics, Johannes Kepler University Linz, Austria, 2009. Submitted.

Boundary integral operators

Fundamental solution of 3D Laplace operator: $U^*(x, y) = \frac{1}{4\pi|x-y|}$.

$$V_i : H^{-1/2}(\Gamma_i) \rightarrow H^{1/2}(\Gamma_i) :$$

$$(V_i v)(y) = \int_{\Gamma_i} U^*(x, y) v(x) ds_x$$

$$K_i : H^{1/2}(\Gamma_i) \rightarrow H^{1/2}(\Gamma_i) :$$

$$(K_i u)(y) = \int_{\Gamma_i} \frac{\partial U^*}{\partial n_x}(x, y) u(x) ds_x$$

$$K'_i : H^{-1/2}(\Gamma_i) \rightarrow H^{-1/2}(\Gamma_i) :$$

$$(K'_i v)(y) = \int_{\Gamma_i} \frac{\partial U^*}{\partial n_y}(x, y) v(x) ds_x$$

$$D_i : H^{1/2}(\Gamma_i) \rightarrow H^{-1/2}(\Gamma_i) :$$

$$(D_i u)(y) = -\frac{\partial}{\partial n_y} \int_{\Gamma_i} \frac{\partial U^*}{\partial n_x}(x, y) (u(x) - u(y)) ds_x$$

Norms for error estimates

We use harmonic extensions \mathcal{H}_i heavily.

Natural norms to work in:

$$|v_i|_{H^{1/2}(\Gamma_i)} := |\mathcal{H}_i v_i|_{H^1(T_i)} = \inf_{\substack{\phi \in H^1(T_i) \\ \phi|_{\Gamma_i} = v_i}} |\phi|_{H^1(T_i)},$$

$$\|v_i\|_{H^{1/2}(\Gamma_i)}^2 := \frac{1}{\text{diam}(T_i)^2} \|\mathcal{H}_i v_i\|_{L_2(T_i)}^2 + |\mathcal{H}_i v_i|_{H^1(T_i)}^2.$$

Easily extended to the skeleton Γ_S :

$$\|v\|_S := \left(\sum_{i=1}^N |v_i|_{H^{1/2}(\Gamma_i)}^2 \right)^{1/2} = |\mathcal{H}_S v|_{H^1(\Omega)}.$$

Jones parameter

Definition (Uniform domain)

A bounded and connected set $D \subset \mathbb{R}^d$ is called a *uniform domain* if there exists a constant $C_U(D)$ such that any two points $x_1, x_2 \in D$ can be joined by a rectifiable curve $\gamma(t) : [0, 1] \rightarrow D$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$ such that

$$\begin{aligned} \ell(\gamma) &\leq C_U(D) |x_1 - x_2|, \\ \min_{i=1,2} |x_i - \gamma(t)| &\leq C_U(D) \operatorname{dist}(\gamma(t), \partial D) \quad \forall t \in [0, 1]. \end{aligned}$$

We call the smallest such $C_U(D)$ the *Jones parameter* of D .
Any Lipschitz domain is a uniform domain.

Poincaré constant

Definition (Poincaré constant)

Let D be a uniform domain. The Poincaré constant $C_P(D)$ is the smallest constant such that

$$\inf_{c \in \mathbb{R}} \|u - c\|_{L_2(D)} \leq C_P(D) \operatorname{diam}(D) |u|_{H^1(D)} \quad \forall u \in H^1(D).$$

$C_P(D)$ can be bounded using the constant in an isoperimetric inequality.

For convex domains, $C_P(D) \leq \frac{1}{\pi}$ (Payne & Weinberger 1960; Bebendorf 2003).

Mesh constants

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$$c_{0,i} = \inf_{v \in H_*^{1/2}(\Gamma_i)} \frac{\langle D_i v, v \rangle}{\langle V_i^{-1} v, v \rangle} \in (0, \frac{1}{4}) \quad \forall i \in \{1, \dots, N\}.$$

Recent results by C. Pechstein:

For each element, fix a ball B_i enclosing it:

$$B_i \supset \bar{T}_i, \quad \text{dist}(\partial B_i, \partial T_i) \geq \frac{1}{2} \text{diam}(T_i).$$

$c_{0,i}$ may be bounded away from zero solely in terms of

- the Jones parameters $C_U(T_i)$ and $C_U(B_i \setminus \bar{T}_i)$,
- the Poincaré constants $C_P(T_i)$ and $C_P(B_i \setminus \bar{T}_i)$.

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Mesh regularity (3D)

Standard regularity assumptions on every tetrahedron $\tau \in \Xi_j$:
with J_τ Jacobian of affine mapping from unit tetrahedron to τ ,

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This implies regularity of the boundary faces:
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Estimating the Dirichlet approximation error

- Conforming global triangulation $\Xi = \bigcup_i \Xi_i$.
- Introduce globally cont., pw. linear FE space V_h over Ξ .
- Choose suitable FE interpolant $I_{V_h} u_\Omega \in V_h$.
- Skeletal interpolant $I_{W_h} u_\Omega := \gamma_S(V_h u_\Omega) \in W_h$.
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Estimating the Neumann approximation error

Steps in this estimation are quite standard, but:

- choice of norms is delicate,
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On the Neumann approximation theorem

Standard approximation on pw. constant boundary function space (e.g. Steinbach):

$$\|v - Q_{h,i}v\|_{L_2(\Gamma_i)} \leq C (\text{diam } T_i)^{1/2} |v|_{H_{\sim\text{pw}}^{1/2}(\Gamma_i)}.$$

We use the norm relations (for $v \in H^{1/2}(\Gamma_i)$)

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