A 3D plane wave basis for elastic wave problems

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Non-Standard Numerical Methods for PDE's

Non-polynomial basis methods

- The partition of unity finite element method (PUFEM) by Babuška and Melenk (1997).
- Least squares method (LSM) by Monk and Wang (1999).
- Discontinuous enrichment method (DEM) by Farhat et al. (2001).
- Discontinuous Galerkin method (DGM) by Farhat et al. (2003), Gittelson, Hiptmair and Perugia (2007).
- Discontinuous Petrov-Galerkin method (DPGM) by Demkowicz et al. (2009)
- Non-polynomial FEM by Barnett and Betcke (2009)
- The ultra-weak variational formulation (UWVF) by Després (1994), Cessenat and Després (1998).

The UWVF

- Special form of the DGM, Huttunen, Malinen and Monk (2006), Gabard (2007)
- Originally plane wave basis functions, (in 2D Bessel basis possible choice)
- Uses FE meshes
- Number of basis functions can vary from element to element
- Matrices resulting in the UWVF are sparse

Navier equation

Let Ω be a computational domain with the boundary $\Gamma = \partial\Omega$ and let Ω consists of non-overlapping elements, i.e. $\Omega = \cup_{k=1}^{N} \Omega_k$ where N is the number of elements. For each Ω_k the Navier equation is

$$\mu \Delta \mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 \quad \text{in } \Omega_k \tag{1}$$

where ω is the angular frequency of the field, $\mathbf{u}$ is the time-harmonic displacement vector, λ and μ are the Lamé constants and ρ is the density of the medium.

Lamé constants and wave speeds

The Lamé constants can be expressed as

$$\mu = \frac{E}{2(1-\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \tag{2}$$

where E is the Young's modulus and ν is the Poisson ratio. The wave speeds for the P-wave and S-wave are,

$$c_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad c_S = \sqrt{\frac{\mu}{\rho}}. \tag{3}$$

Traction operator

Traction operator $\mathbf{T}^{(\mathbf{n})}(\mathbf{u})$ maps local displacements to local tractions on any closed surface S and it is defined as

$$\mathbf{T}^{(\mathbf{n})}(\mathbf{u}) = 2\mu\frac{\partial \mathbf{u}}{\partial \mathbf{n}} + \lambda\mathbf{n}\nabla\cdot\mathbf{u} + \mu\mathbf{n}\times\nabla\times\mathbf{u}. \tag{4}$$

where $\mathbf{n}$ is an outward unit normal to the surface S.

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where $\mathbf{n}$ is an outward unit normal to the surface S. In addition, the complex conjugate of the traction operator $\mathbf{T}$ is

$$\overline{\mathbf{T}^{(\mathbf{n})}}(\mathbf{u}) = 2\overline{\mu}\frac{\partial \mathbf{u}}{\partial \mathbf{n}} + \overline{\lambda}\mathbf{n}\nabla\cdot\mathbf{u} + \overline{\mu}\mathbf{n}\times\nabla\times\mathbf{u} \tag{5}$$

and $\overline{\overline{\mathbf{T}^{(\mathbf{n})}}(\mathbf{u})} = \mathbf{T}^{(\mathbf{n})}(\overline{\mathbf{u}})$.

Faces and exterior boundary

Let Ω_k and Ω_j be neighboring elements and share a common face. The interface between Ω_k and Ω_j is denoted by $\sum_{k,j}$. Therefore on $\sum_{k,j}$ the following conditions hold

$$\mathbf{u}|_{\Omega_k} = \mathbf{u}|_{\Omega_j} \tag{6}$$

$$\mathbf{T}^{(\mathbf{n}|_{\Omega_k})}(\mathbf{u}|_{\Omega_k}) = -\mathbf{T}^{(\mathbf{n}|_{\Omega_j})}(\mathbf{u}|_{\Omega_j}) \tag{7}$$

where $\mathbf{n}|_{\Omega_k}$ is an outward normal to Ω_k and similarly $\mathbf{n}|_{\Omega_j}$ to Ω_j (note that $\mathbf{n}|_{\Omega_k} = -\mathbf{n}|_{\Omega_j}$).

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$$\mathbf{T}^{(\mathbf{n})}(\mathbf{u}) - i\sigma\mathbf{u} = Q(-\mathbf{T}^{(\mathbf{n})}(\mathbf{u}) - i\sigma\mathbf{u}) + g \quad \text{on } \Gamma \tag{8}$$

where g is the source term, Q specifies the boundary conditions and σ is a coupling parameter (flux parameter).

Isometry Lemma

It can be shown that

$$\begin{aligned}
&\sum_k \int_{\partial\Omega_k} \sigma^{-1} \left(-\mathbf{T}^{(\mathbf{n}_k)}(\mathbf{u}_k) - i\sigma\mathbf{u}_k\right) \cdot \overline{\left(-\overline{\mathbf{T}^{(\mathbf{n}_k)}}(\mathbf{e}_k) - i\sigma\mathbf{e}_k\right)} \\
&= \sum_k \int_{\partial\Omega_k} \sigma^{-1} \left(\mathbf{T}^{(\mathbf{n}_k)}(\mathbf{u}_k) - i\sigma\mathbf{u}_k\right) \cdot \overline{\left(\overline{\mathbf{T}^{(\mathbf{n}_k)}}(\mathbf{e}_k) - i\sigma\mathbf{e}_k\right)}
\end{aligned} \tag{9}$$

where $\mathbf{u}_k$ is the solution of the Navier equation (1) and $\mathbf{e}_k$ is the test function that satisfies the adjoint Navier's equation.

The UWVF

Using the “Isometry Lemma” and boundary conditions we obtain the UWVF as

$$\sum_k \int_{\partial\Omega_{\boldsymbol{k}}} \sigma^{-1}\mathcal{X}_k\cdot\overline{(-\overline{\mathbf{T}^{(\mathbf{n}_k)}}(\mathbf{e}_k) - i\sigma\mathbf{e}_k)} - \sum_k\sum_j \int_{\sum_{\boldsymbol{k},\boldsymbol{j}}} \sigma^{-1}\mathcal{X}_j\cdot\overline{(\overline{\mathbf{T}^{(\mathbf{n}_k)}}(\mathbf{e}_k) - i\sigma\mathbf{e}_k)}$$

$$-\sum_k \int_{\Gamma_{\boldsymbol{k}}} Q\sigma^{-1}\mathcal{X}_k\cdot\overline{(\overline{\mathbf{T}^{(\mathbf{n}_k)}}(\mathbf{e}_k) - i\sigma\mathbf{e}_k)} = \sum_k \int_{\Gamma_{\boldsymbol{k}}} \sigma^{-1} g\cdot\overline{(\overline{\mathbf{T}^{(\mathbf{n}_k)}}(\mathbf{e}_k) - i\sigma\mathbf{e}_k)} \tag{10}$$

where $\mathcal{X}_k = \mathbf{T}^{(\mathbf{n}_k)}(\mathbf{u}_k) - i\sigma\mathbf{u}_k$ on $\partial\Omega_k$.

Discretization

The solution of the adjoint Navier equation is separated into three components (Helmholtz decomposition): P-wave, SH-wave and SV-wave. Therefore

$$\mathbf{e}_k = \mathbf{e}_{k,P} + \mathbf{e}_{k,SH} + \mathbf{e}_{k,SV} \tag{11}$$

which satisfy $\nabla \times \mathbf{e}_P = 0$ and $\nabla \cdot \mathbf{e}_{SH} = \nabla \cdot \mathbf{e}_{SV} = 0$.

Similarly, the approximation for $\mathcal{X}_k$ is

$$\mathcal{X}_k \approx \sum_{\ell=1}^{p_P^k} \left[x_{k,\ell}^P \left(-\mathbf{T}^{(\mathbf{n}_k)}(\mathbf{e}_{k,\ell}^P) - i\sigma \mathbf{e}_{k,\ell}^P \right) \right] + \sum_{\ell=1}^{p_S^k} \left[x_{k,\ell}^{SH} \left(-\mathbf{T}^{(\mathbf{n}_k)}(\mathbf{e}_{k,\ell}^{SH}) - i\sigma \mathbf{e}_{k,\ell}^{SH} \right) \right] + \sum_{\ell=1}^{p_S^k} \left[x_{k,\ell}^{SV} \left(-\mathbf{T}^{(\mathbf{n}_k)}(\mathbf{e}_{k,\ell}^{SV}) - i\sigma \mathbf{e}_{k,\ell}^{SV} \right) \right].$$

where

$$\mathbf{e}_{k,\ell}^P = \begin{cases} \mathbf{a}_{k,\ell} \exp(i\overline{\kappa}_P \mathbf{a}_{k,\ell} \cdot \mathbf{x}) & \text{in } \Omega_k \\ 0 & \text{elsewhere} \end{cases}$$

$$\mathbf{e}_{k,\ell}^{SH} = \begin{cases} \mathbf{a}_{k,\ell}^{\perp} \exp(i\overline{\kappa}_{SH} \mathbf{a}_{k,\ell} \cdot \mathbf{x}) & \text{in } \Omega_k \\ 0 & \text{elsewhere} \end{cases}$$

$$\mathbf{e}_{k,\ell}^{SV} = \begin{cases} \mathbf{a}_{k,\ell}^{\perp} \times \mathbf{a}_{k,\ell} \exp(i\overline{\kappa}_{SV} \mathbf{a}_{k,\ell} \cdot \mathbf{x}) & \text{in } \Omega_k \\ 0 & \text{elsewhere} \end{cases}$$

where $\mathbf{a}_{k,\ell}$ is the direction of propagation.

Discrete UWVF

Find $\mathcal{X}_{h,k} \in V_{h,k}, k = 1, 2, \dots, N$ such that

$$\sum_k \int_{\partial\Omega_k} \sigma^{-1}\mathcal{X}_{h,k} \cdot \overline{\mathcal{Y}_{h,k}} - \sum_k \sum_j \int_{\Sigma_{k,j}} \sigma^{-1}\mathcal{X}_{h,j} \cdot \overline{F_k(\mathcal{Y}_{h,k})}$$
$$- \sum_k \int_{\Gamma_k} Q\sigma^{-1}\mathcal{X}_{h,k} \cdot \overline{F_k(\mathcal{Y}_{h,k})} = \sum_k \int_{\Gamma_k} \sigma^{-1} g \cdot \overline{F_k(\mathcal{Y}_{h,k})}$$

for all $\mathcal{Y}_{h,k} \in V_{h,k}, k = 1, 2, \dots, N$ where

$$F_k(\mathcal{Y}_{h,k}) \approx \sum_{\ell=1}^{p_P^k} \left[y_{k,\ell}^P \left(\mathbf{T}^{(\mathbf{n}_k)}(\mathbf{e}_{k,\ell}^P) - i\sigma \mathbf{e}_{k,\ell}^P \right) \right]$$
$$+ \sum_{\ell=1}^{p_S^k} \left[y_{k,\ell}^{SH} \left(\mathbf{T}^{(\mathbf{n}_k)}(\mathbf{e}_{k,\ell}^{SH}) - i\sigma \mathbf{e}_{k,\ell}^{SH} \right) \right]$$
$$+ \sum_{\ell=1}^{p_S^k} \left[y_{k,\ell}^{SV} \left(\mathbf{T}^{(\mathbf{n}_k)}(\mathbf{e}_{k,\ell}^{SV}) - i\sigma \mathbf{e}_{k,\ell}^{SV} \right) \right]. \qquad (12)$$

Matrices

The discrete UWVF can be written in a matrix form as

$$(D - C)X = b \quad \Rightarrow \quad (I - D^{-1}C)X = D^{-1}b \tag{13}$$

where D is a sparse block diagonal matrix $D = \operatorname{diag}(D^1, D^2, \ldots, D^k, \ldots, D^N)$ so that

$$D^k = \begin{pmatrix} D^k_{P,P,\ell,m} & D^k_{SH,P,\ell,m} & D^k_{SV,P,\ell,m} \\ D^k_{P,SH,\ell,m} & D^k_{SH,SH,\ell,m} & D^k_{SV,SH,\ell,m} \\ D^k_{P,SV,\ell,m} & D^k_{SH,SV,\ell,m} & D^k_{SV,SV,\ell,m} \end{pmatrix}. \tag{14}$$

where, for example,

$$D^k_{P,SH,\ell,m} = \int_{\partial\Omega_{\boldsymbol{k}}} \sigma^{-1} \left(-\mathbf{T}^{(\mathbf{n}_k)}(\mathbf{e}^P_{k,m}) - i\sigma \mathbf{e}^P_{k,m} \right) \cdot \overline{\left(-\overline{\mathbf{T}^{(\mathbf{n}_k)}}(\mathbf{e}^{SH}_{k,\ell}) - i\sigma \mathbf{e}^{SH}_{k,\ell} \right)}. \tag{15}$$

Matrices

Sparse matrix C consists of blocks C^k and $C^{k,j}$. Matrix blocks C^k are on the diagonal and $C^{k,j}$ are on the off-diagonal of matrix C. Matrix block C^k can be written as follows

$$C^k = \begin{pmatrix} C^k_{P,P,\ell,m} & C^k_{SH,P,\ell,m} & C^k_{SV,P,\ell,m} \\ C^k_{P,SH,\ell,m} & C^k_{SH,SH,\ell,m} & C^k_{SV,SH,\ell,m} \\ C^k_{P,SV,\ell,m} & C^k_{SH,SV,\ell,m} & C^k_{SV,SV,\ell,m} \end{pmatrix} \qquad (16)$$

where, for example, $C^k_{P,SH,\ell,m}$ is of the form

$$C^k_{P,SH,\ell,m} = \int_{\Gamma_k} Q\sigma^{-1}\left(-\mathbf{T}^{(\mathbf{n}_k)}(\mathbf{e}^P_{k,m}) - i\sigma\mathbf{e}^P_{k,m}\right)\cdot\overline{\left(\overline{\mathbf{T}^{(\mathbf{n}_k)}}(\mathbf{e}^{SH}_{k,\ell}) - i\sigma\mathbf{e}^{SH}_{k,\ell}\right)}, \qquad (17)$$

similarly others.

Matrices

The off-diagonal block matrix $C^{k,j}$ is as follows

$$C^{k,j} = \begin{pmatrix} C^{k,j}_{P,P,\ell,m} & C^{k,j}_{SH,P,\ell,m} & C^{k,j}_{SV,P,\ell,m} \\ C^{k,j}_{P,SH,\ell,m} & C^{k,j}_{SH,SH,\ell,m} & C^{k,j}_{SV,SH,\ell,m} \\ C^{k,j}_{P,SV,\ell,m} & C^{k,j}_{SH,SV,\ell,m} & C^{k,j}_{SV,SV,\ell,m} \end{pmatrix} \tag{18}$$

where, for example, $C^{k,j}_{P,SH,\ell,m}$ is of the form

$$C^{k,j}_{P,SH,\ell,m} = \int_{\Sigma_{k,j}} \sigma^{-1} \left(\mathbf{T}^{(\mathbf{n}_k)}(\mathbf{e}^{P}_{j,m}) - i\sigma\mathbf{e}^{P}_{j,m}\right) \cdot \overline{\left(\overline{\mathbf{T}^{(\mathbf{n}_k)}}(\mathbf{e}^{SH}_{k,\ell}) - i\sigma\mathbf{e}^{SH}_{k,\ell}\right)}, \tag{19}$$

others can be derived in a similar manner.

Plane wave propagation in a unit cube

The exact solution is of the form

$$\begin{aligned}\mathbf{u} = {} & A_1 \mathbf{d} \exp(i\kappa_P \mathbf{x} \cdot \mathbf{d}) + A_2 \mathbf{d}_{SH} \exp(i\kappa_S \mathbf{x} \cdot \mathbf{d}) \\ & + A_3 \mathbf{d}_{SV} \exp(i\kappa_S \mathbf{x} \cdot \mathbf{d})\end{aligned}$$

where the wave numbers are $\kappa_P = \omega / c_P$, $\kappa_S = \omega / c_S$, the direction $\mathbf{d} \approx [-0.73 \quad 0.45 \quad 0.51]$, $\mathbf{d}_{SH} = \mathbf{d}^{\perp}$, $\mathbf{d}_{SV} = \mathbf{d}^{\perp} \times \mathbf{d}$ and the amplitudes $A_1 = A_2 = A_3 = 1$. In addition, $\nabla \times \mathbf{u}_P = 0$ and $\nabla \cdot \mathbf{u}_{SH} = \nabla \cdot \mathbf{u}_{SV} = 0$. As a boundary condition we choose $Q = 0$.

Flux parameter

In numerical simulations we use an ad hoc choice for coupling parameter (flux parameter) that is

$$\sigma = \omega\rho\mathbb{R}\{c_P\}I \tag{20}$$

where I is the unit matrix.
More investigations of the optimal flux parameter will be investigated in (near) future.

Mesh 1

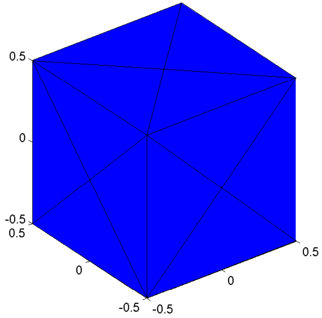

Figure: The mesh. The maximum centroid-vertex distance (element diameter) for element $h = 0.4979$. Number of tetrahedra 24, faces 60 and vertices 14.

Results for p-convergence

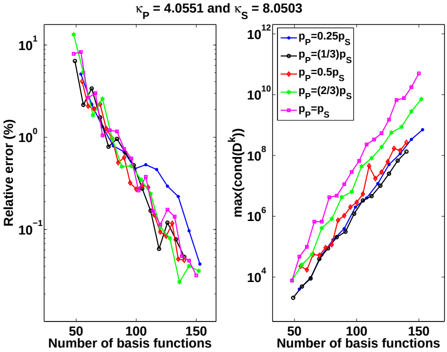

Figure: Results when $\kappa_P = 4.0551$, $\kappa_{SH} = \kappa_{SV} = 8.0503$ with different ratios between p_P/p_S and mesh size is fixed.

Coarsest and densest mesh

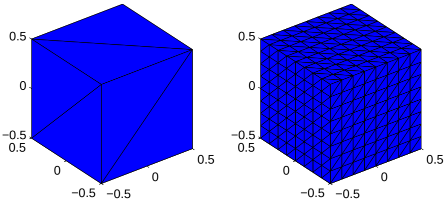

Figure: The coarsest $h_{max} = 0.7395$ and densest meshes $h_{max} = 0.1269$.

Results for h-convergence

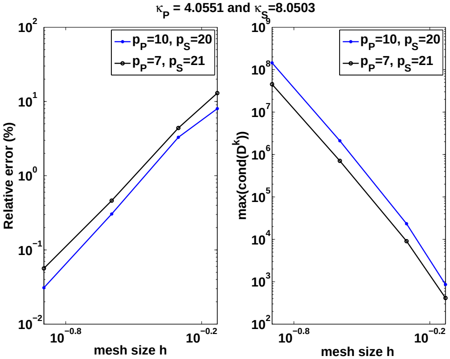

Figure: Results when $\kappa_P = 4.0551$, $\kappa_{SH} = \kappa_{SV} = 8.0503$ with different ratios between p_P/p_S. Number of basis functions per element blue line $p_{tot} = 50$ and black line $p_{tot} = 49$.

Results for h-convergence

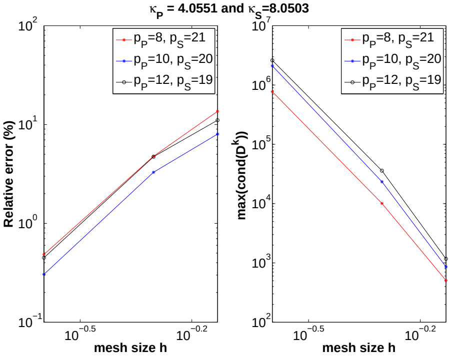

Figure: Results when $\kappa_P = 4.0551$, $\kappa_{SH} = \kappa_{SV} = 8.0503$ with different ratios between p_P/p_S. Number of basis functions per element $p_{tot} = 50$.

Mesh

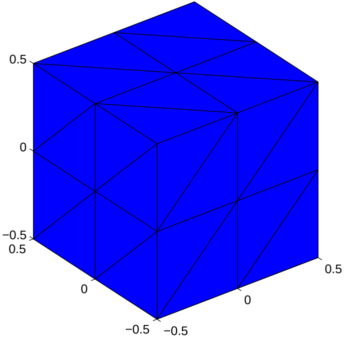

Figure: The mesh when $h_{max} = 0.4978$.

Table: Results when $p_P = 25$ and $p_S = 50$, mesh is fixed and wave number varies.

κ_P	κ_S	error (%)	$\max(\mathrm{cond}(D^k))$
4.0551	8.0503	0.0321	5.8143e8
5.0689	10.0629	0.1319	4.5035e7
6.0826	12.0755	0.4232	5.4503e6
7.0964	14.0881	1.1347	9.2297e5
8.1102	16.1007	1.6142	2.0051e5

- Preliminary results show that the UWVF can be applied to the 3D elastic wave problems,
- Work in progress,
- More investigations needed, especially,
    - finding optimal flux parameter,
    - optimal ratio between the basis functions,
    - problems including surface waves,
    - scattering,
    - HIFU,
    - etc.

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Thank you for your attention!