A 3D plane wave basis for elastic wave problems

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Non-Standard Numerical Methods for PDE’s
Introduction

Navier problem

Derivation of the UWVF

Numerical results

Conclusions
Non-polynomial basis methods

- The partition of unity finite element method (PUFEM) by Babuška and Melenk (1997).
- Discontinuous enrichment method (DEM) by Farhat et al. (2001).
- Discontinuous Galerkin method (DGM) by Farhat et al. (2003), Gittelson, Hiptmair and Perugia (2007).
- Discontinuous Petrov-Galerkin method (DPGM) by Demkowicz et al. (2009)
- Non-polynomial FEM by Barnett and Betcke (2009)
The UWVF

- Special form of the DGM, Huttunen, Malinen and Monk (2006), Gabard (2007)
- Originally plane wave basis functions, (in 2D Bessel basis possible choice)
- Uses FE meshes
- Number of basis functions can vary from element to element
- Matrices resulting in the UWVF are sparse
Navier equation

Let $\Omega$ be a computational domain with the boundary $\Gamma = \partial \Omega$ and let $\Omega$ consists of non-overlapping elements, i.e. $\Omega = \bigcup_{k=1}^{N} \Omega_k$ where $N$ is the number of elements. For each $\Omega_k$ the Navier equation is

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 \quad \text{in } \Omega_k \quad (1)$$

where $\omega$ is the angular frequency of the field, $\mathbf{u}$ is the time-harmonic displacement vector, $\lambda$ and $\mu$ are the Lamé constants and $\rho$ is the density of the medium.
Lamé constants and wave speeds

The Lamé constants can be expressed as

\[ \mu = \frac{E}{2(1 - \nu)}, \quad \lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \]

where \( E \) is the Young's modulus and \( \nu \) is the Poisson ratio. The wave speeds for the P-wave and S-wave are,

\[ c_P = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_S = \sqrt{\frac{\mu}{\rho}}. \]
Traction operator

Traction operator $T^{(n)}(u)$ maps local displacements to local tractions on any closed surface $S$ and it is defined as

$$T^{(n)}(u) = 2\mu \frac{\partial u}{\partial n} + \lambda n \nabla \cdot u + \mu n \times \nabla \times u.$$  (4)

where $n$ is an outward unit normal to the surface $S$.
Traction operator

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$$T^{(n)}(u) = 2\mu \frac{\partial u}{\partial n} + \lambda n \nabla \cdot u + \mu n \times \nabla \times u. \quad (4)$$

where $n$ is an outward unit normal to the surface $S$. In addition, the complex conjugate of the traction operator $T$ is

$$\overline{T^{(n)}(u)} = 2\mu \frac{\partial u}{\partial n} + \overline{\lambda} n \nabla \cdot u + \overline{\mu} n \times \nabla \times u \quad (5)$$

and $\overline{T^{(n)}(u)} = T^{(n)}(\overline{u})$. 

Luostari, Huttunen & Monk
Faces and exterior boundary

Let $\Omega_k$ and $\Omega_j$ be neighboring elements and share a common face. The interface between $\Omega_k$ and $\Omega_j$ is denoted by $\sum_{k,j}$. Therefore on $\sum_{k,j}$ the following conditions hold

$$u|_{\Omega_k} = u|_{\Omega_j}$$  \hspace{1cm} (6)

$$\mathbf{T}^{(n|_{\Omega_k})}(u|_{\Omega_k}) = -\mathbf{T}^{(n|_{\Omega_j})}(u|_{\Omega_j})$$  \hspace{1cm} (7)

where $n|_{\Omega_k}$ is an outward normal to $\Omega_k$ and similarly $n|_{\Omega_j}$ to $\Omega_j$ (note that $n|_{\Omega_k} = -n|_{\Omega_j}$).
Faces and exterior boundary

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where $n|_{\Omega_k}$ is an outward normal to $\Omega_k$ and similarly $n|_{\Omega_j}$ to $\Omega_j$ (note that $n|_{\Omega_k} = -n|_{\Omega_j}$). On the exterior boundary $\Gamma$ we have

$$T^{(n)}(u) - i\sigma u = Q(-T^{(n)}(u) - i\sigma u) + g \quad \text{on } \Gamma \quad (8)$$

where $g$ is the source term, $Q$ specifies the boundary conditions and $\sigma$ is a coupling parameter (flux parameter).
Isometry Lemma

It can be shown that
\[
\sum_k \int_{\partial \Omega_k} \sigma^{-1} \left( -\mathbf{T}^{(n_k)}(u_k) - i\sigma u_k \right) \cdot \left( -\mathbf{T}^{(n_k)}(e_k) - i\sigma e_k \right) = \sum_k \int_{\partial \Omega_k} \sigma^{-1} \left( \mathbf{T}^{(n_k)}(u_k) - i\sigma u_k \right) \cdot \left( \mathbf{T}^{(n_k)}(e_k) - i\sigma e_k \right)
\]

where $u_k$ is the solution of the Navier equation (1) and $e_k$ is the test function that satisfies the adjoint Navier's equation.
The UWVF

Using the “Isometry Lemma” and boundary conditions we obtain the UWVF as

\[
\sum_k \int_{\partial \Omega_k} \sigma^{-1} \mathcal{X}_k \cdot (\overline{T(n_k)(e_k)} - i\sigma e_k) - \sum_k \sum_j \int_{\sum_k} \sigma^{-1} \mathcal{X}_j \cdot (\overline{T(n_k)(e_k)} - i\sigma e_k)
\]

\[
- \sum_k \int_{\Gamma_k} Q \sigma^{-1} \mathcal{X}_k \cdot (\overline{T(n_k)(e_k)} - i\sigma e_k) = \sum_k \int_{\Gamma_k} \sigma^{-1} g \cdot (\overline{T(n_k)(e_k)} - i\sigma e_k)
\]

(10)

where \( \mathcal{X}_k = T(n_k)(u_k) - i\sigma u_k \) on \( \partial \Omega_k \).
Discretization

The solution of the adjoint Navier equation is separated into three components (Helmholtz decomposition): P-wave, SH-wave and SV-wave. Therefore

$$e_k = e_{k,P} + e_{k,SH} + e_{k,SV}$$

(11)

which satisfy $\nabla \times e_P = 0$ and $\nabla \cdot e_{SH} = \nabla \cdot e_{SV} = 0$. 
Similarly, the approximation for $X_k$ is

$$X_k \approx \sum_{\ell=1}^{p_p^k} x_{k,\ell}^P \left[ -T^{(n_k)}(e_{k,\ell}^P) - i\sigma e_{k,\ell}^P \right]$$

$$+ \sum_{\ell=1}^{p_s^k} x_{k,\ell}^{SH} \left[ -T^{(n_k)}(e_{k,\ell}^{SH}) - i\sigma e_{k,\ell}^{SH} \right]$$

$$+ \sum_{\ell=1}^{p_s^k} x_{k,\ell}^{SV} \left[ -T^{(n_k)}(e_{k,\ell}^{SV}) - i\sigma e_{k,\ell}^{SV} \right].$$

where

$$e_{k,\ell}^P = \begin{cases} a_{k,\ell} \exp(iK_P a_{k,\ell} \cdot x) & \text{in } \Omega_k \\ 0 & \text{elsewhere} \end{cases}$$

$$e_{k,\ell}^{SH} = \begin{cases} a_{k,\ell}^{\perp} \exp(iK_{SH} a_{k,\ell} \cdot x) & \text{in } \Omega_k \\ 0 & \text{elsewhere} \end{cases}$$

$$e_{k,\ell}^{SV} = \begin{cases} a_{k,\ell}^{\perp} \times a_{k,\ell} \exp(iK_{SV} a_{k,\ell} \cdot x) & \text{in } \Omega_k \\ 0 & \text{elsewhere} \end{cases}$$

where $a_{k,\ell}$ is the direction of propagation.
Discrete UWVF

Find $\chi_{h,k} \in V_{h,k}$, $k = 1, 2, \ldots, N$ such that

$$
\sum_k \int_{\partial \Omega_k} \sigma^{-1} \chi_{h,k} \cdot \mathcal{Y}_{h,k} - \sum_k \sum_j \int_{\sum_{k,j}} \sigma^{-1} \chi_{j,k} \cdot F_k(\mathcal{Y}_{h,k}) - \sum_k \int_{\Gamma_k} Q \sigma^{-1} \chi_{h,k} \cdot F_k(\mathcal{Y}_{h,k}) = \sum_k \int_{\Gamma_k} \sigma^{-1} g \cdot F_k(\mathcal{Y}_{h,k})
$$

for all $\mathcal{Y}_{h,k} \in V_{h,k}$, $k = 1, 2, \ldots, N$ where

$$
F_k(\mathcal{Y}_{h,k}) \approx \sum_{\ell=1}^{p_P^k} \left[ y_{k,\ell} \left( T^{(n_k)}(e_{k,\ell}^P) - i \sigma e_{k,\ell}^P \right) \right]
+ \sum_{\ell=1}^{p_S^k} \left[ y_{k,\ell} \left( T^{(n_k)}(e_{k,\ell}^{SH}) - i \sigma e_{k,\ell}^{SH} \right) \right]
+ \sum_{\ell=1}^{p_S^k} \left[ y_{k,\ell} \left( T^{(n_k)}(e_{k,\ell}^{SV}) - i \sigma e_{k,\ell}^{SV} \right) \right].
$$

(12)
Matrices

The discrete UWVF can be written in a matrix form as

\[(D - C)X = b \Rightarrow (I - D^{-1} C)X = D^{-1} b\] (13)

where \(D\) is a sparse block diagonal matrix

\[D = \text{diag}(D^1, D^2, \ldots, D^k, \ldots, D^N)\]

so that

\[
D^k = \begin{pmatrix}
D^k_{P,P,l,m} & D^k_{SH,P,l,m} & D^k_{SV,P,l,m} \\
D^k_{P,SH,l,m} & D^k_{SH,SH,l,m} & D^k_{SV,SH,l,m} \\
D^k_{P,SV,l,m} & D^k_{SH,SV,l,m} & D^k_{SV,SV,l,m}
\end{pmatrix}.
\] (14)

where, for example,

\[
D^k_{P,SH,l,m} = \int_{\partial \Omega_k} \sigma^{-1} \left( -T^{(n_k)}(e^P_{k,m}) - i\sigma e^P_{k,m} \right) \cdot \left( -T^{(n_k)}(e^{SH}_{k,\ell}) - i\sigma e^{SH}_{k,\ell} \right) dA.
\] (15)
Matrices

Sparse matrix $C$ consists of blocks $C^k$ and $C^{k,j}$. Matrix blocks $C^k$ are on the diagonal and $C^{k,j}$ are on the off-diagonal of matrix $C$. Matrix block $C^k$ can be written as follows

$$
C^k = \begin{pmatrix}
C^k_{P,P,\ell,m} & C^k_{SH,P,\ell,m} & C^k_{SV,P,\ell,m} \\
C^k_{P,SH,\ell,m} & C^k_{SH,SH,\ell,m} & C^k_{SV,SH,\ell,m} \\
C^k_{P,SV,\ell,m} & C^k_{SH,SV,\ell,m} & C^k_{SV,SV,\ell,m}
\end{pmatrix}
$$

(16)

where, for example, $C^k_{P,SH,\ell,m}$ is of the form

$$
C^k_{P,SH,\ell,m} = \int_{\Gamma_k} Q\sigma^{-1} \left( -\mathbf{T}^{(n_k)}(e^P_{k,m}) - i\sigma e^P_{k,m} \right) \cdot \overline{\left( \mathbf{T}^{(n_k)}(e^{SH}_{k,\ell}) - i\sigma e^{SH}_{k,\ell} \right)},
$$

(17)

similarly others.
Matrices

The off-diagonal block matrix $C^{k,j}$ is as follows

$$
\begin{pmatrix}
C_{P,P,\ell,m}^{k,j} & C_{SH,P,\ell,m}^{k,j} & C_{SV,P,\ell,m}^{k,j} \\
C_{P,SH,\ell,m}^{k,j} & C_{SH,SH,\ell,m}^{k,j} & C_{SV,SH,\ell,m}^{k,j} \\
C_{P,SV,\ell,m}^{k,j} & C_{SH,SV,\ell,m}^{k,j} & C_{SV,SV,\ell,m}^{k,j}
\end{pmatrix}
$$

(18)

where, for example, $C_{P,SH,\ell,m}^{k,j}$ is of the form

$$
C_{P,SH,\ell,m}^{k,j} = \int \sigma^{-1} \left( T^{(nk)}(e^P_{j,m}) - i\sigma e^P_{j,m} \right) \cdot \frac{\sqrt{T^{(nk)}(e^{SH}_{k,\ell}) - i\sigma e^{SH}_{k,\ell}}}{\sum_{k,j}}
$$

(19)

others can be derived in a similar manner.
Plane wave propagation in a unit cube

The exact solution is of the form

\[ u = A_1 d \exp(i\kappa_P x \cdot d) + A_2 d_{SH} \exp(i\kappa_S x \cdot d) \]
\[ + A_3 d_{SV} \exp(i\kappa_S x \cdot d) \]

where the wave numbers are \( \kappa_P = \omega/c_P \), \( \kappa_S = \omega/c_S \), the direction \( d \approx [-0.73\, 0.45\, 0.51] \), \( d_{SH} = d^\perp \), \( d_{SV} = d^\perp \times d \) and the amplitudes \( A_1 = A_2 = A_3 = 1 \). In addition, \( \nabla \times u_P = 0 \) and \( \nabla \cdot u_{SH} = \nabla \cdot u_{SV} = 0 \). As a boundary condition we choose \( Q = 0 \).
Flux parameter

In numerical simulations we use an ad hoc choice for coupling parameter (flux parameter) that is

$$\sigma = \omega \rho R \{c_P\} I \quad (20)$$

where $I$ is the unit matrix.

More investigations of the optimal flux parameter will be investigated in (near) future.
Figure: The mesh. The maximum centroid-vertex distance (element diameter) for element $h = 0.4979$. Number of tetrahedra 24, faces 60 and vertices 14.
Results for p-convergence

Figure: Results when $\kappa_p = 4.0551$, $\kappa_{SH} = \kappa_{SV} = 8.0503$ with different ratios between $p_P/p_S$ and mesh size is fixed.
Coarsest and densest mesh

Figure: The coarsest $h_{max} = 0.7395$ and densest meshes $h_{max} = 0.1269$. 
Results for h-convergence

Figure: Results when $\kappa_P = 4.0551$, $\kappa_{SH} = \kappa_{SV} = 8.0503$ with different ratios between $p_P/p_S$. Number of basis functions per element blue line $p_{tot} = 50$ and black line $p_{tot} = 49$. 
Results for $h$-convergence

Figure: Results when $\kappa_P = 4.0551$, $\kappa_{SH} = \kappa_{SV} = 8.0503$ with different ratios between $p_P/p_S$. Number of basis functions per element $p_{tot} = 50$. 
Figure: The mesh when $h_{\text{max}} = 0.4978$. 

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Luostari, Huttunen & Monk 

Outline
Introduction
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Table: Results when $p_P = 25$ and $p_S = 50$, mesh is fixed and wave number varies.

<table>
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<th>$\kappa_P$</th>
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<th>error (%)</th>
<th>max($\text{cond}(D^k)$)</th>
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Preliminary results show that the UWVF can be applied to the 3D elastic wave problems,

- Work in progress,
- More investigations needed, especially,
  - finding optimal flux parameter,
  - optimal ratio between the basis functions,
  - problems including surface waves,
  - scattering,
  - HIFU,
  - etc.
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