High-Order Nodal Mimetic Methods

Gianmarco Manzini

Istituto di Matematica Applicata e Tecnologie Informatiche
(IMATI) C.N.R., Pavia, Italy

in collaboration with L. Beirao da Veiga and K. Lipnikov

1 July 2010, Pavia

Outline

1 Low- and high-order formulation of the method:
   - meshes;
   - degrees of freedom;
   - local consistency and stability conditions;
   - approximation of the source term.

2. A convergence result in a discrete H^1-like norm.

3. Numerical experiments.

4. Conclusions (and possible developments).

The linear diffusion problem

- Differential formulation:

$$\begin{aligned} -\mathrm{div}(\mathsf{K}\nabla u) &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma, \end{aligned}$$

- Variational formulation:

Find $u \in H^1_g(\Omega)$ *such that:*

$$\int_\Omega \mathsf{K}\nabla u \cdot \nabla v \, dV = \int_\Omega fv \, dV \qquad \forall v \in H^1_0(\Omega),$$

Scheme construction in five steps

Steps 1 and 2

1. We decompose Ω into a **mesh** Ω_h of polygons (2-D) or polyhedrons (3-D);
    - admissible meshes may contain non-convex cells and “singular” (AMR) cells;
    - we need some regularity assumptions to avoid pathological cases and perform the convergence analysis;
2. **degrees of freedom**: $\mathcal{V}_h$

$$u,v \in H^1_g(\Omega) \cap C^0(\overline{\Omega}) \quad \longrightarrow \quad u_h, v_h \in \mathcal{V}_h;$$

Scheme construction in five steps

Steps 3 and 4

3. **bilinear form**: $\mathcal{A}_h(\cdot,\cdot) \,:\, \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{R}$

$$\mathcal{A}_h(u_h, v_h) \approx \int_\Omega \mathsf{K}\nabla u \cdot \nabla v \, dV,$$

it is built by “mimicking” a fundamental relation of calculus (*integration by parts*);

4. **linear functional**: $(f,\cdot)_h \,:\, \mathcal{V}_h \to \mathbb{R}$

$$(f, v_h)_h \approx \int_\Omega fv \, dV.$$

MFD construction in five steps
Step 5

5. The variational formulation

Find $u \in H^1_g(\Omega)$ *such that:*

$$\int_\Omega \mathsf{K}\nabla u \cdot \nabla v \, dV = \int_\Omega f v \, dV \qquad \forall v \in H^1_0(\Omega),$$

becomes the "mimetic variational" formulation:

Find $u_h \in \mathcal{V}_{h,g}$ *such that:*

$$\mathcal{A}_h\big(u_h, v_h\big) = \big(f, v_h\big)_h \qquad \forall v_h \in \mathcal{V}_{h,0}.$$

Notation and main reference

- We use the parameters:
  - $m \geq 1$, degrees of the polynomials that we use to state the *consistency condition*;
  - $d = 2, 3$ spatial dimensions;

- We will distinguish between
  - **low-order method**, $m = 1$, $d = 2, 3$,
    →*Brezzi, Buffa, Lipnikov, M2AN, 2009*;
  - **high-order method**, $m \geq 1$, $d = 2$,
    →*Beirao da Veiga, Lipnikov, M., in progress*.

1. Meshes: basic assumptions

- There exists a **conforming** sub-decomposition $\mathcal{S}_h$ of Ω_h in triangles (2-D) or tetrahedrons (3-D) such that

  1. for any polygonal cell $\mathrm{P} \in \Omega_h$, the restriction $\mathcal{S}_{h,\mathrm{P}}$ is formed by a uniformly bounded number of simplexes T;

  2. the simplexes $\mathrm{T} \in \mathcal{S}_h$ are shape-regular (Ciarlet);

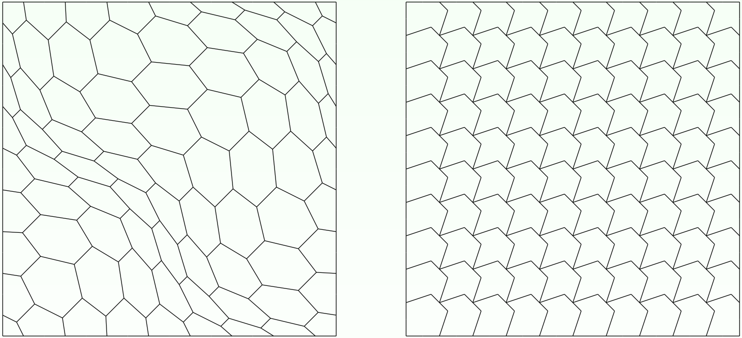
  3. Convergence analysis of high-order setting is more restrictive: 1. and 2. must hold but

     - any polygonal cell $\mathrm{P} \in \Omega_h$ is star-shaped with respect to an internal point $\overline{\mathbf{x}}_{\mathrm{P}}$;
     - $\mathcal{S}_{h,\mathrm{P}}$ is given by connecting $\overline{\mathbf{x}}_{\mathrm{P}}$ to the cell vertices.

- **We do not need to build $\mathcal{S}_h$ in practice!**

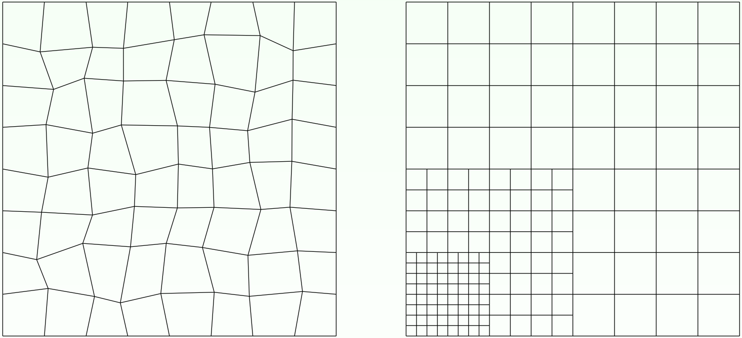
Meshes: 2-D examples

Examples: convex and non-convex polygonal cells



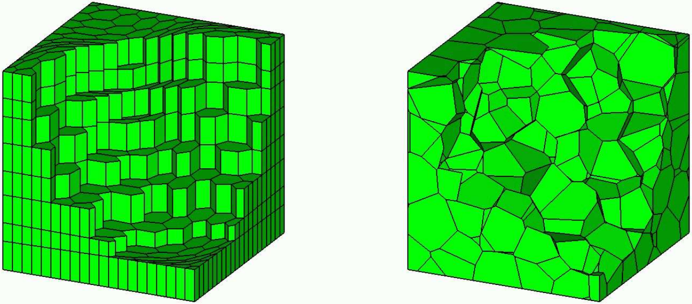
Meshes: 2-D examples

Examples: randomized quads and Adaptive Mesh Refinements (AMR)



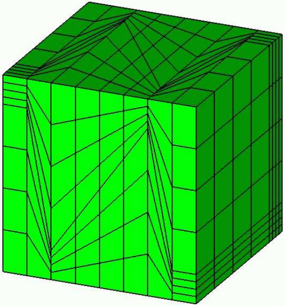
Meshes: 3-D examples

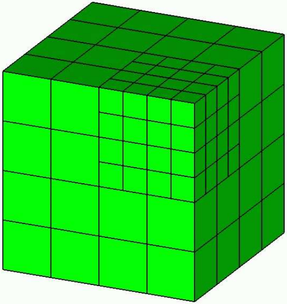
Examples: prismatic and polyhedral cells



Meshes: 3-D examples

Examples: hexahedrons and AMR

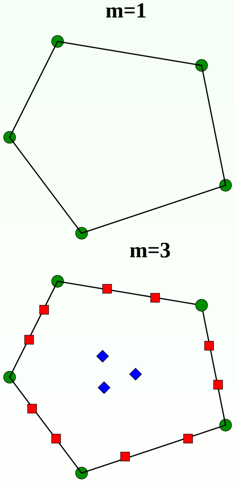




2. Degrees of freedom

A discrete scalar field v_h in $\mathcal{V}_h$ is given by:

- low-order setting ($m = 1$, $d = 2, 3$):
    - one real number per mesh vertex v;
- high-order setting ($m > 1$, $d = 2$):
    - one real number per mesh vertex v;
    - $(m - 1)$ real numbers per mesh face f;
    - $m(m - 1)/2$ real numbers per mesh cell P;



3. Construction of $\mathcal{A}_h(u_h, v_h)$

- $\mathcal{A}_h(u_h, v_h)$ must be
  - **symmetric**, **bounded** and **semi-positive**;
  - **locally defined through an assembly process** (like FEM):

$$\mathcal{A}_h(u_h, v_h) = \sum_{\mathrm{P}} \mathcal{A}_{h,\mathrm{P}}(u_{h,\mathrm{P}}, v_{h,\mathrm{P}})$$

where $u_{h,\mathrm{P}} = u_{h|\mathrm{P}}$, $v_{h,\mathrm{P}} = v_{h|\mathrm{P}}$;

- Any $\mathcal{A}_{h,\mathrm{P}}(u_{h,\mathrm{P}}, v_{h,\mathrm{P}})$ must be a **local approximation**:

$$\forall \mathrm{P} \in \Omega_h \ : \quad \mathcal{A}_{h,\mathrm{P}}(u_{h,\mathrm{P}}, v_{h,\mathrm{P}}) \approx \int_{\mathrm{P}} \mathrm{K}\nabla u \cdot \nabla v \, dV.$$

Towards a local consistency condition

The low-order setting, $m = 1$, $d = 2, 3$

Let K be constant on P.

1. We integrate by parts on every mesh cell P.

   If u is a **linear polynomial** on P:

$$\int_{\mathrm{P}} \mathrm{K}\nabla u \cdot \nabla v \, dV = -\underbrace{\int_{\mathrm{P}} \mathrm{div}(\mathrm{K}\nabla u)\, v \, dV}_{\textbf{equal to zero!}} + \int_{\partial \mathrm{P}} \mathrm{K}\nabla u \cdot \mathbf{n}_{\mathrm{P}} \, v ds;$$

2. we introduce a numerical integration rule for each face $\mathrm{f} = (\mathrm{v}', \mathrm{v}'')$ that is exact for linear polynomials:

$$\sum_{\mathrm{f}\in\partial \mathrm{P}} \underbrace{\mathrm{K}\nabla u \cdot \mathbf{n}_{\mathrm{P,f}}}_{\textbf{constant}} \int_{\mathrm{f}} v ds \approx \sum_{\mathrm{f}\in\partial \mathrm{P}} \mathrm{K}\nabla u \cdot \mathbf{n}_{\mathrm{P,f}} \underbrace{|\mathrm{f}| \frac{v(\mathbf{x}_{\mathrm{v}'}) + v(\mathbf{x}_{\mathrm{v}''})}{2}}_{\substack{\textbf{trapezoidal rule} \\ \text{(for } d = 2)}}.$$

The local consistency condition

The low-order setting, $m = 1$, $d = 2, 3$

HENCE,

$$\forall \mathrm{P} \in \Omega_h \ : \quad \int_{\mathrm{P}} \mathsf{K}\nabla u \cdot \nabla v \, dV \approx \sum_{\mathrm{f}\in\partial\mathrm{P}} \mathsf{K}\nabla u \cdot \mathbf{n}_{\mathrm{P},\mathrm{f}} \, |\mathrm{f}| \, \frac{v(\mathbf{x}_{\mathrm{v}'}) + v(\mathbf{x}_{\mathrm{v}''})}{2}$$

for any linear polynomial u and v enough regular.

THUS, WE REQUIRE THAT:

- the degrees of freedom approximate the vertex values: $v_{h|\mathrm{v}} := v_{\mathrm{v}} \approx v(\mathbf{x}_{\mathrm{v}'})$;
- for every **linear polynomial** u and every discrete field $v_h \in \mathcal{V}_h$ the bilinear form (for $d = 2$) satisfies

$$\mathcal{A}_{h,\mathrm{P}}\big(v_h, u^{\mathrm{I}}\big) := \sum_{\mathrm{f}\in\partial\mathrm{P}} \mathsf{K}\nabla u \cdot \mathbf{n}_{\mathrm{P},\mathrm{f}} \frac{|\mathrm{f}|}{2} \big(v_{\mathrm{v}'} + v_{\mathrm{v}''}\big).$$

Towards a local consistency condition

The high-order setting, $m > 1$, $d = 2$

Let K be constant on P and integrate by parts on cell P:

$$\int_{\mathrm{P}} \mathrm{K}\nabla u \cdot \nabla v \, dV = - \int_{\mathrm{P}} \underbrace{\mathrm{div}(\mathrm{K}\nabla u)}_{\textbf{not zero!}} v \, dV + \sum_{\mathrm{f} \in \partial \mathrm{f}} \int_{\mathrm{f}} \underbrace{\mathrm{K}\nabla u \cdot \mathbf{n}_{\mathrm{P,f}}}_{\textbf{not constant!}} v \, ds.$$

If u is a polynomial of degree m on P:

- $\mathrm{div}(\mathrm{K}\nabla u)$ is a polynomial of degree $m - 2$;
- $\mathrm{K}\nabla u \cdot \mathbf{n}_{\mathrm{P,f}}$ is a polynomial of degree $m - 1$;

Divergence term

Internal degrees of freedom, $m > 1$, $d = 2$

1. We use the **moments of v** to express the integral over P:

if

$$\mathrm{div}(\mathsf{K}\nabla u) = a_0 1 + a_1 x + a_2 y + \ldots \in \mathbb{P}_{m-2}(\mathrm{P})$$

then

$$\int_{\mathrm{P}} \mathrm{div}(\mathsf{K}\nabla u)\, v\, dV = a_0 \underbrace{\int_{\mathrm{P}} 1 v\, dV}_{\hat{v}_{\mathrm{P},0}} + a_1 \underbrace{\int_{\mathrm{P}} x v\, dV}_{\hat{v}_{\mathrm{P},1,x}} + a_2 \underbrace{\int_{\mathrm{P}} y v\, dV}_{\hat{v}_{\mathrm{P},1,y}} + \ldots$$

$$= a_0 \hat{\mathbf{v}}_{\mathrm{P},\mathbf{0}} + a_1 \hat{\mathbf{v}}_{\mathrm{P},\mathbf{1},\mathbf{x}} + a_2 \hat{\mathbf{v}}_{\mathrm{P},\mathbf{1},\mathbf{y}} + \ldots$$

This choice suggests us to define

- $m(m-1)/2$ **internal** degrees of freedom $\approx \hat{\mathbf{v}}_{\mathrm{P},\mathbf{0}}, \hat{\mathbf{v}}_{\mathrm{P},\mathbf{1},\mathbf{x}}, \hat{\mathbf{v}}_{\mathrm{P},\mathbf{1},\mathbf{y}}, \ldots$

Boundary term

Nodal degrees of freedom, $m > 1$, $d = 2$

2. We use a **Gauss-Lobatto formula** with $m+1$ nodes and weights $\{(\mathbf{x}_{\mathrm{f},q}, w_{\mathrm{f},q})\}$ on every (2D) face $\mathrm{f} \in \partial\mathrm{P}$ for:

$$\int_{\mathrm{f}} \mathrm{K}\nabla u \cdot \mathbf{n}_{\mathrm{P,f}}\, v\, dV \approx \sum_{q=0}^{m+1} w_{\mathrm{f},q} \mathrm{K}\nabla u(\mathbf{x}_{\mathrm{f},q}) \cdot \mathbf{n}_{\mathrm{P,f}}\, v(\mathbf{x}_{\mathrm{f},q}).$$

This choice suggests us to define:

- one degree of freedom per vertex,
  $$v_{\mathrm{f},0} = v_{\mathrm{v}'} \approx v(\mathbf{x}_{\mathrm{v}'}),\ v_{\mathrm{f},m+1} = v_{\mathrm{v}''} \approx v(\mathbf{x}_{\mathrm{v}''});$$
- $(m-1)$ nodal degrees of freedom per face of P,
  $$v_{\mathrm{f},q} \approx v(\mathbf{x}_{\mathrm{f},q}) \text{ for } q = 1, 2, \dots m-1.$$

The local consistency condition

The high-order setting, $m > 1$, $d = 2$

RECALL THAT:

$$\mathcal{A}_h\big(u_h, v_h\big) = \sum_{\mathrm{P}} \mathcal{A}_{h,\mathrm{P}}\big(u_{h,\mathrm{P}}, v_{h,\mathrm{P}}\big).$$

WE REQUIRE:

1. *local consistency*: for every $u \in \mathbb{P}_m(\mathrm{P})$ ($m \geq 1$) and every discrete field $v_{h,\mathrm{P}} \in \mathcal{V}_h$ there holds:

$$\mathcal{A}_{h,\mathrm{P}}\big(v_{h,\mathrm{P}}, u^{\mathrm{I}}\big) := \underbrace{-\sum_{j=0}^{m(m-1)/2} a_j \hat{v}_{\mathrm{P},j}}_{\textbf{divergence}} + \underbrace{\sum_{\mathrm{f}\in\partial\mathrm{P}} \sum_{q=0}^{m+1} w_{\mathrm{f},q} \mathrm{K}\nabla u\big(\mathbf{x}_{\mathrm{f},q}\big) \cdot \mathbf{n}_{\mathrm{P},\mathrm{f}}\, v_{\mathrm{f},q}}_{\textbf{boundary}}.$$

The spectral stability condition

For both low and high-order methods we require:

2. *spectral stability*, there exists two positive constants σ_* and σ^* such that for every $v_{h,\mathrm{P}} \in \mathcal{V}_{h,\mathrm{P}}$ there holds:

$$\sigma_* ||v_{h,\mathrm{P}}||^2_{1,h,\mathrm{P}} \leq \mathcal{A}_{h,\mathrm{P}}\big(v_{h,\mathrm{P}}, v_{h,\mathrm{P}}\big) \leq \sigma^* ||v_{h,\mathrm{P}}||^2_{1,h,\mathrm{P}};$$

   where the mesh-dependent *norm* $||\cdot||_{1,h,\mathrm{P}}$ mimics $|\cdot|_{1,\mathrm{P}}$.

Algebraic consistency condition: $\mathsf{M}\mathsf{N} = \mathsf{R}$

Let M be a **symmetric** and **semi-positive definite** matrix such that

$$\mathcal{A}_{h,\mathrm{P}}\big(u_{h,\mathrm{P}}, v_{h,\mathrm{P}}\big) = v_{h,\mathrm{P}}^{T}\mathsf{M}u_{h,\mathrm{P}}.$$

- For $u \in \{x, y, x^2, xy, y^2, \dots\}$ for any $v_{h,\mathrm{P}}$

  - we write

  $$\mathcal{A}_{h,\mathrm{P}}\big(v_{h,\mathrm{P}}, u^{\mathtt{I}}\big) = v_{h,\mathrm{P}}^{T}\mathsf{M}\,\mathbf{N}_u \quad \text{where} \quad \mathbf{N}_u = (u)^{\mathtt{I}} \quad (\text{"dofs" of } u);$$

  - we impose the *local consistency condition*:

  $$\mathcal{A}_{h,\mathrm{P}}\big(u^{\mathtt{I}}, v_h\big) = \dots = v_{h,\mathrm{P}}^{T}\,\mathbf{R}_u$$

  - we obtain by comparison:

  $$\mathsf{M}\,\mathbf{N}_u = \mathbf{R}_u$$

A family of schemes

- Using $\mathsf{N} = [\mathbf{N}_1, \mathbf{N}_2, \ldots]$, $\mathsf{R} = [\mathbf{R}_1, \mathbf{R}_2, \ldots]$, we have: $\mathsf{M}\,\mathsf{N} = \mathsf{R}$ with

$$(\mathsf{R}^T\mathsf{N})_{ij} = \int_{\mathsf{P}} \mathsf{K}\nabla u_i \cdot \nabla u_j \, dV \qquad \text{where } u_i, u_j \in \{x, y, \ldots\}.$$

- M (symmetric and semi-positive definite) is given by

$$\mathsf{M} = \underbrace{\mathsf{R}(\mathsf{R}^T\mathsf{N})^{-1}\mathsf{R}^T}_{\mathsf{MN}=\mathsf{R}!} + \underbrace{\delta\mathsf{M}}_{\text{stability!}} \qquad \text{with} \qquad \delta\mathsf{M}\,\mathsf{N} = 0.,$$

  where $\delta\mathsf{M}$ is a **symmetric matrix of parameters**.

- A one-parameter (γ) choice for $\delta\mathsf{M}$ is given by:

$$\delta\mathsf{M} = \gamma\big(\mathsf{I} - \mathsf{N}(\mathsf{N}^T\mathsf{N})^{-1}\mathsf{N}^T\big).$$

Connection with other schemes

- The low-order MFD method ($m = 1$)
  - on **triangles** (and tetrahedra in 3-D) coincides with the linear Galerkin finite element method (no free parameters);
  - on **squares** is a one-parameter family of schemes including (noted Cangiani-Russo):
    - the bi-linear finite element method (Q_1);
    - the 5-point finite difference laplacian;
    - the 9-point finite difference laplacian.
- For $m > 1$?

4. The linear functional $(f,\mathbf{v}_h)_h$

The low-order case ($m=1$, $d=2,3$)

Recall that $\big(f,v_h\big)_h \approx \int_\Omega fv\,dV$.

- We assemble $\big(f,v_h\big)_h$ from **local contribution**:

$$\big(f,v_h\big)_h := \sum_{\mathrm{P}} \big(f,v_h\big)_{h,\mathrm{P}}.$$

- We approximate the forcing term by its average on P:

$$f \approx \frac{1}{|\mathrm{P}|}\int_{\mathrm{P}} f\,dV =: \bar{f}_{\mathrm{P}};$$

- we use a *first-order* accurate integration formula based on vertex values: $\{\mathbf{x}_{\mathrm{v}}, w_{\mathrm{P,v}}\}$

$$\int_{\mathrm{P}} fv\,dV \approx \bar{f}_{\mathrm{P}} \int_{\mathrm{P}} v\,dV \approx |\mathrm{P}|\,\bar{f}_{\mathrm{P}} \sum_{\mathrm{v}\in\partial\mathrm{P}} w_{\mathrm{P,v}}v(\mathbf{x}_{\mathrm{v}}).$$

4. The linear functional $(f, \mathbf{v}_h)_h$

The low-order case ($m = 1$, $d = 2, 3$)

- Recall that $\big(f, v_h\big)_h := \sum_{\mathrm{P}} \big(f, v_h\big)_{h,\mathrm{P}}$, where

$$\big(f, v_h\big)_{h,\mathrm{P}} \approx \int_{\mathrm{P}} fv\, dV, \quad \text{and} \quad \int_{\mathrm{P}} fv\, dV \approx |\mathrm{P}|\, \overline{f}_{\mathrm{P}} \sum_{\mathrm{v}\in\partial\mathrm{P}} w_{\mathrm{P},\mathrm{v}} v_{\mathrm{v}}$$

- Thus, for every cell P we define

$$\big(f, v_h\big)_{h,\mathrm{P}} := |\mathrm{P}|\, \overline{f}_{\mathrm{P}} \sum_{\mathrm{v}\in\partial\mathrm{P}} w_{\mathrm{P},\mathrm{v}} v_{\mathrm{v}} \qquad \forall v_h \in \mathcal{V}_h$$

$$|\mathrm{P}|\, \overline{f}_{\mathrm{P}} = \int_{\mathrm{P}} f\, dV$$

$w_{\mathrm{P},\mathrm{v}}$ 1-st order integration weights.

4. The linear functional $(f, \mathbf{v}_h)_h$

The high-order case ($m > 1$, $d = 2$)

Recall that

$$\left(f, v_h\right)_h := \sum_{\mathrm{P}} \left(f, v_h\right)_{h,\mathrm{P}} \quad \text{where} \quad \left(f, v_h\right)_{h,\mathrm{P}} \approx \int_{\mathrm{P}} fv\, dV.$$

- Direct extension from $m = 1$ would require a high order accurate integration formula over P based on vertex values and moments of v.
- For $m > 1$ we consider the **orthogonal projection** of f onto the polynomials of degree $m - 2$:

$$f = b_0 1 + b_1 x + b_2 y + \ldots \in \mathbb{P}_{m-2}(\mathrm{P})$$

- and use the moments of v to express the r.h.s. integral:

$$\int_{\mathrm{P}} f\, v\, dV \approx b_0 \underbrace{\int_{\mathrm{P}} 1 v\, dV}_{\hat{v}_{\mathrm{P},0}} + b_1 \underbrace{\int_{\mathrm{P}} x v\, dV}_{\hat{v}_{\mathrm{P},1,x}} + b_2 \underbrace{\int_{\mathrm{P}} y v\, dV}_{\hat{v}_{\mathrm{P},1,y}} + \ldots$$

$$= b_0 \hat{\mathbf{v}}_{\mathrm{P},\mathbf{0}} + b_1 \hat{\mathbf{v}}_{\mathrm{P},\mathbf{1},\mathbf{x}} + b_2 \hat{\mathbf{v}}_{\mathrm{P},\mathbf{1},\mathbf{y}} + \ldots$$

4. The linear functional $(f, \mathbf{v}_h)_h$

The high-order case ($m > 1$, $d = 2$)

RECALL THAT

$$\big(f, v_h\big)_h := \sum_{\mathrm{P}} \big(f, v_h\big)_{h,\mathrm{P}} \quad \text{where} \quad \big(f, v_h\big)_{h,\mathrm{P}} \approx \int_{\mathrm{P}} fv\, dV.$$

THUS, FOR EVERY CELL P WE DEFINE

$$\big(f, v_h\big)_{h,\mathrm{P}} := \sum_{j} b_j \hat{v}_j \qquad \forall v_h \in \mathcal{V}_h.$$

(b_j) projection of f onto $(m-2)$-degree poynomials

$\hat{v}_j$ moment degrees of freedom of v_h

A mesh-dependent norm

We consider the mesh-dependent norm

$$||v_h||^2_{1,h} = \sum_{\mathrm{P}\in\Omega_h} ||v_h||^2_{1,h,\mathrm{P}}$$

that mimics the $|\cdot|_{1,\Omega}$ semi-norm;

- for the low-order method ($m = 1$, $d = 2, 3$), $\mathrm{e} = (\mathrm{v}', \mathrm{v}'')$ being an edge,

$$||v_h||^2_{1,h,\mathrm{P}} = ||\mathcal{GRAD}_h(v_h)||^2_{h,\mathrm{P}} = h_\mathrm{P} \sum_{\mathrm{e}\in\partial\mathrm{P}} |v_{\mathrm{v}''} - v_{\mathrm{v}'}|^2 ;$$

- for the high-order method ($m > 1$, $d = 2$), $\mathrm{f} = (\mathrm{v}', \mathrm{v}'')$ being an edge (2D face),

$$||v_h||^2_{1,h,\mathrm{P}} = h_\mathrm{P} \sum_{\mathrm{f}\in\partial\mathrm{P}} \left\| \frac{\partial v_{h,\mathrm{f}}}{\partial s} \right\|^2_{L^2(\mathrm{f})} + \left[\text{"moments"}\right]$$

Convergence results

- for the **low-order** method ($m = 1$, $d = 2, 3$):

$$\|u^I - u_h\|_{1,h} < Ch\big(\, |f|_{0,\Omega} + |u|_{1,\Omega} + |u|_{2,\Omega} \,\big);$$

(Brezzi, Buffa, Lipnikov, M2AN (2009)),

- for the **high-order** method ($m > 1$, $d = 2$):

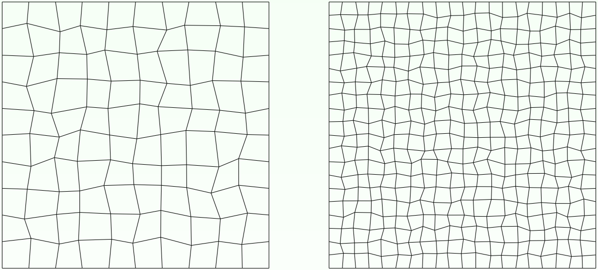
$$\|u^I - u_h\|_{1,h} < Ch^{\mathbf{m}}\, |u|_{m+1,\Omega}\,;$$

(with L. Beirao da Veiga and K. Lipnikov, in progress)

Numerical experiments

Meshes with randomized quadrilaterals

- **Meshes:**



- **Exact solution:** $u(x,y) = \left(x - e^{2(x-1)}\right)\left(y^2 - e^{3(y-1)}\right)$
- **Diffusion tensor**

$$\mathsf{K} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Results

Randomized quadrilaterals, $\|\cdot\|_{1,h}$ errors, constant K

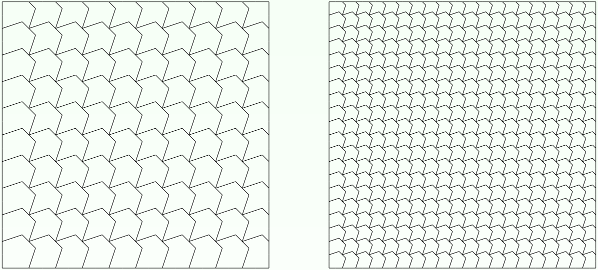
		$\mathbf{m} = 2$		$\mathbf{m} = 3$	
n	h	Error	Rate	Error	Rate
0	$1.922\,10^{-1}$	$1.416\,10^{-1}$	$--$	$7.454\,10^{-2}$	$--$
1	$9.705\,10^{-2}$	$2.441\,10^{-2}$	2.57	$8.632\,10^{-3}$	3.15
2	$4.838\,10^{-2}$	$5.366\,10^{-3}$	2.18	$1.536\,10^{-3}$	2.48
3	$2.467\,10^{-2}$	$1.399\,10^{-3}$	1.99	$1.739\,10^{-4}$	3.23
4	$1.263\,10^{-2}$	$3.524\,10^{-4}$	**2.06**	$2.227\,10^{-5}$	**3.07**

		$\mathbf{m} = 4$		$\mathbf{m} = 5$	
n	h	Error	Rate	Error	Rate
0	$1.922\,10^{-1}$	$1.031\,10^{-2}$	$--$	$4.567\,10^{-3}$	$--$
1	$9.705\,10^{-2}$	$1.690\,10^{-3}$	2.65	$2.674\,10^{-4}$	4.15
2	$4.838\,10^{-2}$	$1.273\,10^{-4}$	3.71	$1.336\,10^{-5}$	4.30
3	$2.467\,10^{-2}$	$8.279\,10^{-6}$	4.06	$4.586\,10^{-7}$	**5.01**
4	$1.263\,10^{-2}$	$5.545\,10^{-7}$	**4.04**	–	–

Numerical experiments

Meshes with non-convex polygons

- **Meshes:**



- **Exact solution:** $u(x, y) = e^{-2\pi y} \sin(2\pi x)$
- **Diffusion tensor**

$$\mathsf{K} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathsf{K}(x, y) = \begin{pmatrix} (x+1)^2 + y^2 & -xy \\ -xy & (x+1)^2 \end{pmatrix}$$

Results

Non-convex polygons, $|| \cdot ||_{1,h}$ errors, constant K

		$\mathbf{m} = \mathbf{2}$		$\mathbf{m} = \mathbf{3}$	
n	h	Error	Rate	Error	Rate
0	$1.458\,10^{-1}$	2.858	$--$	1.007	$--$
1	$7.289\,10^{-2}$	$7.867\,10^{-1}$	1.86	$2.819\,10^{-1}$	1.84
2	$3.644\,10^{-2}$	$2.049\,10^{-1}$	1.94	$5.597\,10^{-2}$	2.33
3	$1.822\,10^{-2}$	$5.289\,10^{-2}$	**1.95**	$8.897\,10^{-3}$	**2.65**

		$\mathbf{m} = \mathbf{4}$		$\mathbf{m} = \mathbf{5}$	
n	h	Error	Rate	Error	Rate
0	$1.458\,10^{-1}$	$1.943\,10^{-1}$	$--$	$2.282\,10^{-2}$	$--$
1	$7.289\,10^{-2}$	$1.276\,10^{-2}$	3.93	$1.128\,10^{-3}$	4.34
2	$3.644\,10^{-2}$	$7.075\,10^{-4}$	4.17	$4.406\,10^{-5}$	**4.68**
3	$1.822\,10^{-2}$	$3.950\,10^{-5}$	**4.16**	–	–

Results

Non-convex polygons, $\|\cdot\|_{1,h}$ errors, non-constant K

		$\mathbf{m}=\mathbf{2}$		$\mathbf{m}=\mathbf{3}$	
n	h	Error	Rate	Error	Rate
0	$1.458\,10^{-1}$	3.007	$--$	$9.873\,10^{-1}$	$--$
1	$7.289\,10^{-2}$	$8.081\,10^{-1}$	1.89	$2.760\,10^{-1}$	1.84
2	$3.644\,10^{-2}$	$2.071\,10^{-1}$	1.96	$5.621\,10^{-2}$	2.29
3	$1.822\,10^{-2}$	$5.303\,10^{-2}$	**1.97**	$9.083\,10^{-3}$	**2.63**

		$\mathbf{m}=\mathbf{4}$		$\mathbf{m}=\mathbf{5}$	
n	h	Error	Rate	Error	Rate
0	$1.458\,10^{-1}$	$2.059\,10^{-1}$	$--$	$1.988\,10^{-2}$	$--$
1	$7.289\,10^{-2}$	$1.367\,10^{-2}$	3.92	$1.016\,10^{-3}$	4.29
2	$3.644\,10^{-2}$	$7.562\,10^{-4}$	4.18	$3.924\,10^{-5}$	**4.69**
3	$1.822\,10^{-2}$	$4.210\,10^{-5}$	**4.17**	–	–

Conclusions

- We are developing a new family of **nodal** mimetic finite difference methods such that:
  (i) the low-order formulation, using vertex values to represent linear polynomials, works in 2-D and 3-D;
  (ii) an high-order formulation, using nodal values and moments to represent m-degree polynomials, works in 2-D.

- Possible developments:
  (i) extension to 3-D;
  (ii) new formulations using moments also for face terms;
  (iii) p-refinements;
  (iv) . . .

Errors versus number of degrees of freedom

Meshes with randomized quadrilaterals