

TRIMESTRE INTENSIVO INDAM
NON-STANDARD NUMERICAL METHODS FOR PDE'S

Approximation by plane waves

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Trefftz methods for Helmholtz

The **Trefftz** methods use basis functions that are solutions of the PDE under examination in each element.

Homogeneous **Helmholtz** equation: $-\Delta u - \omega^2 u = 0$

- **plane wave** basis functions (UWVF, PWDG, DEM, VTQR, ...)

$$x \mapsto e^{i\omega d \cdot x}, \quad |d| = 1;$$

- **circular / spherical / corner wave** basis functions (least-squares Trefftz, ...);
- **singular / fundamental** basis functions (MFS, ...);
- ...

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The best approximation estimate

The analysis of plane waves Trefftz methods requires **best approximation estimates**:

$$-\Delta u - \omega^2 u = 0 \quad \text{in } D, \quad \text{diam } D = h, \quad p \in \mathbb{N},$$

$$\inf_{\alpha \in \mathbb{C}^p} \left\| u - \sum_{k=1}^p \alpha_k e^{i\omega d_k \cdot x} \right\|_{H^1(D)} \leq C \epsilon(h, p) \|u\|_{H^{K+1}(D)},$$

with $\epsilon(h, p) \xrightarrow[p \rightarrow \infty]{h \rightarrow 0} 0$.

What is available

Only few results are available:

- (CESSENAT AND DESPRÉS 1998), approach with Taylor polyn.:
 h -convergence, 2D, L^2 -norm, the order is not sharp;
- (MELENK 1995), approach with Vekua theory:
 p -convergence for plane w., h and p for circular w., 2D.

Improvement and generalization needed, we follow the second approach.

Our goals:

- estimates for plane and circular/spherical waves;
- estimates both in h and p ;
- estimates in 2 and 3 dimensions;
- explicit dependence of the bounds on ω .

Outline

- Vekua theory;
- approximation by generalized harmonic polynomials;
- approximation by plane waves;
- approximation for Maxwell equations (only some ideas!).

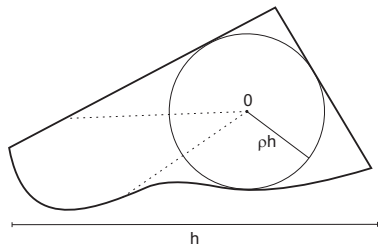
See: (HIPTMAIR, MOIOLA, PERUGIA, SAM Report 2009-27).

Part I

Vekua theory

The Vekua theory in N dimensions

$D \subset \mathbb{R}^N$, $N \geq 2$,
star-shaped.



Given $\omega > 0$, define two continuous functions:

$$M_1, M_2 : D \times [0, 1] \rightarrow \mathbb{R}$$

$$M_1(x, t) = -\frac{\omega|x|}{2} \frac{\sqrt{t}^{N-2}}{\sqrt{1-t}} J_1(\omega|x|\sqrt{1-t}),$$

$$M_2(x, t) = -\frac{i\omega|x|}{2} \frac{\sqrt{t}^{N-3}}{\sqrt{1-t}} J_1(i\omega|x|\sqrt{t(1-t)}).$$

J_1 is the ordinary Bessel function of the first kind and order 1.

The Vekua operators

$$V_1, V_2 : C(D) \rightarrow C(D),$$

$$V_j[\phi](\mathbf{x}) := \phi(\mathbf{x}) + \int_0^1 M_j(\mathbf{x}, t)\phi(t\mathbf{x}) dt, \quad \text{a.e. } \mathbf{x} \in D, j = 1, 2.$$

Short Vekua's story:

- Vekua 1942, Helmholtz equation in N dimensions, no proofs;
- Vekua 1948 (translated in 1967), elliptic equations in 2 dimensions, one page for N -dimensional Helmholtz;
- Henrici 1957, elliptic equations only in 2 dimensions.

→ N -dimensional version almost forgotten.

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The properties of Vekua operators

1 $V_2 = (V_1)^{-1}$

2 $(-\Delta - \omega^2) u = 0 \iff \Delta V_2[u] = 0$

Main idea of Vekua theory:



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Main idea of Vekua theory:

$$\text{Helmholtz solutions} \begin{array}{c} \xleftarrow{V_1} \\ \xrightarrow{V_2} \end{array} \text{Harmonic functions}$$

The continuity of Vekua operators

Weighted Sobolev norms:

$$\|u\|_{j,\omega,D} := \left(\sum_{k=0}^j \omega^{2(j-k)} |u|_{k,D}^2 \right)^{\frac{1}{2}}, \quad \omega > 0, j \in \mathbb{N}.$$

$\Delta\phi = 0, \quad (-\Delta - \omega^2)u = 0:$

$$\|V_1[\phi]\|_{j,\omega,D} \leq C_1(N,j,\rho) (1 + (\omega h)^2) \|\phi\|_{j,\omega,D} \quad j \geq 0,$$

$$\|V_2[u]\|_{j,\omega,D} \leq C_2(N,j,\rho) (1 + (\omega h)^4) e^{\frac{3}{4}\omega h} \|u\|_{j,\omega,D} \quad j \geq 1,$$

$$N = 2, 3, \quad u, \phi \in H^j(D).$$

Part II

Approximation by GHPs

The approximation by GHPs

$$-\Delta u - \omega^2 u = 0, \quad u \in H^{k+1}(D),$$

↓ V_2

$V_2[u]$ is harmonic \implies can be approximated
by harmonic polynomials
(Bramble-Hilbert, complex analysis techniques, ...),

↓ V_1

u can be approximated by

$$V_1 \left[\begin{array}{l} \text{harmonic} \\ \text{polynomials} \end{array} \right] =: \begin{array}{l} \text{generalized} \\ \text{harmonic} \\ \text{polynomials} \end{array} \quad (\text{GHPs}).$$

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V_1 [harmonic polynomials] =: **generalized harmonic polynomials** (GHPs).

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Generalized harmonic polynomials

In 2D, the GHPs are **circular waves**:

$$Q(x) = e^{il\psi} J_l(\omega r), \quad \text{in polar coordinates } x = re^{i\psi}, \quad l \in \mathbb{Z},$$

J_l = Bessel functions.

In 3D, the GHPs are **spherical waves**:

$$Q(x) = Y_{l,m}\left(\frac{x}{|x|}\right) j_l(\omega|x|), \quad 0 \leq |m| \leq l \in \mathbb{N},$$

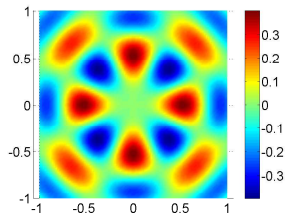
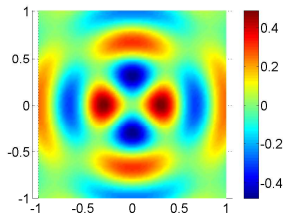
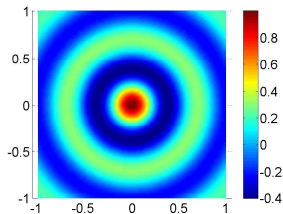
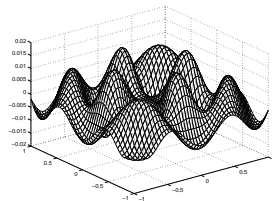
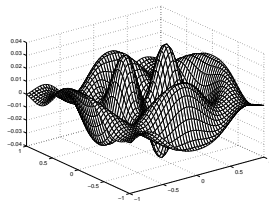
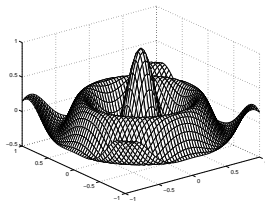
$Y_{l,m}$ = spherical harmonics,
 j_l = spherical Bessel functions.

Both belong to the family of the **Herglotz functions**.

The GHPs are also called Fourier-Bessel functions.

Generalized harmonic polynomials

Real part of GHPs in $[-1, 1]^2$: $V_1[z^l]$, $l = 0, 2, 4$, $\omega = 10$



The approximation by GHPs: h -convergence

$$\begin{aligned} P \in \left\{ \begin{array}{l} \text{inf} \\ \text{harmonic} \\ \text{polynomials} \\ \text{of degree } \leq L \end{array} \right\} & \|u - V_1[P]\|_{j,\omega,D} \leq C \inf_P \|V_2[u] - P\|_{j,\omega,D} && \text{contin. of } V_1, \\ & \leq C h^{k+1-j} \epsilon(L) \|V_2[u]\|_{k+1,\omega,D} && \text{harmonic} \\ & \leq C h^{k+1-j} \epsilon(L) \|u\|_{k+1,\omega,D} && \text{approx. results,} \\ & && \text{contin. of } V_2. \end{aligned}$$

For the h -convergence it is enough to use a harmonic version of the Bramble-Hilbert theorem.

The constant C depends on ωh , not on ω alone:

$$C = C \cdot (1 + \omega h)^{j+6} e^{\frac{3}{4}\omega h}.$$

Harmonic approximation: p -convergence, 2D

In 2 dimensions, if D satisfies the 'exterior cone condition', then:

$$\epsilon(L) = \left(\frac{\log(L+2)}{L+2} \right)^{\lambda(k+1-j)}.$$

If D is convex, $\lambda = 1$. Otherwise $\lambda = \min \frac{\text{re-entrant corner of } D}{\pi}$.

Sharp p -estimate! (MELENK)

In 2D we can use complex analysis:

- conformal mappings $B_1 \leftrightarrow D$;
- ϕ harmonic and real $\rightarrow \phi = \text{Re}(\Phi \text{ holomorphic})$;
- complex interpolation on B_1 ;
- every (complex) polynomial is harmonic;
- ...

Harmonic approximation: p -convergence, N -D

An analogous result holds in N dimensions:

$$\epsilon(L) = L^{-\lambda(k+1-j)},$$

where $\lambda > 0$ is a geometric **unknown** parameter.

Follows from:

- an exponential approximation result for compact subsets by (BAGBY, BOS AND LEVENBERG 1996) in L^∞ -norm;
- harmonic dilation and deformation technique.

If u is the restriction of a solution in a larger domain (2 or 3D), the convergence in L is **exponential**.

The approximation by GHPs: h & p -convergence

Harmonic polynomial approximation:

2-3D	h -conv.	harmonic Bramble-Hilbert: easy
2D	p -conv.	sharp estimate by Melenk: complex analysis
3D	p -conv.	new estimates, order of convergence not explicit

↓ Vekua continuity

If $-\Delta u - \omega^2 u = 0$, $N = 2, 3$, $0 \leq j \leq k \leq L$

$$\inf_{\mathcal{Q} \in \left\{ \substack{\text{GHPs} \\ \text{of degree } \leq L} \right\}} \|u - \mathcal{Q}\|_{j, \omega, D} \leq C(\omega h) h^{k+1-j} L^{-\lambda(k+1-j)} \|u\|_{k+1, \omega, D},$$

The rate $\lambda > 0$ depends only on the shape of D .
If it is convex and 2D, then $\lambda = 1 - \varepsilon$.

Best approx. estimate for **spherical waves Trefftz methods!**

Part III

Approximation by plane waves

The approximation of GHPs by plane waves

Link between plane waves and circular/spherical waves:
Jacobi-Anger expansion

$$\text{2D} \quad e^{iz \cos \theta} = \sum_{l \in \mathbb{Z}} i^l J_l(z) e^{il\theta}, \quad z \in \mathbb{C}, \theta \in \mathbb{R},$$

$$\text{3D} \quad \underbrace{e^{ir\xi \cdot \eta}}_{\text{plane wave}} = 4\pi \sum_{l \geq 0} \sum_{m=-l}^l i^l \underbrace{j_l(r) Y_{l,m}(\xi)}_{\text{GHP}} \overline{Y_{l,m}(\eta)}, \quad \xi, \eta \in \mathbb{S}^2, r \geq 0.$$

We need the other way round:

GHP = linear combination of plane waves

- truncation of J-A expansion,
- solution of a linear system,
- residual estimates.

The approximation of GHPs by plane waves: 2D

$$\begin{aligned} \text{GHP} &= \sum_{k=-q}^q \alpha_k e^{i\omega x \cdot d_k} && (x = |x|e^{i\psi}, d_k = (\cos \theta_k, \sin \theta_k)) \\ &= \underbrace{\sum_{l=-L}^L \alpha_l J_l(\omega|x|) e^{il\psi}}_{V[P], \text{ degree } L} - \underbrace{\sum_{l \in \mathbb{Z}} J_l(\omega|x|) e^{il\psi} i^l \sum_{k=-q}^q \alpha_k e^{-il\theta_k}}_{\text{Jacobi-Anger}} \\ &= - \sum_{|l| > q} i^l J_l(\omega|x|) e^{il\psi} \sum_{k=-q}^q \alpha_k e^{-il\theta_k}, \end{aligned}$$

where the vector α_k is solution of a **Vandermonde** linear system:

$$\{e^{-il\theta_k}\}_{l,k} \cdot \alpha_k = i^{-l} \alpha_l$$

- bound on the inverse matrix;
- control on the minimal angular distance between plane waves directions (non equispaced case).

The approximation of GHPs by plane waves: 2D

Given a harmonic polynomial P of degree at most $L(\geq K)$, there exists $\alpha \in \mathbb{C}^{2q+1}$ such that

$$\left\| V_1[P] - \sum_{k=1}^{2q+1} \alpha_k e^{i\omega x \cdot d_k} \right\|_{L^\infty(B_{2h})} \leq C_{(\rho, L, \omega h)} h^{K-1} \left(\frac{c_0 (\omega h)^2}{q+1} \right)^{\frac{q+1}{2}} \|P\|_{K, \omega, D}.$$

The convergence is **faster than exponential in q** .

The constants C and c_0 can be made completely explicit.

Cauchy estimates }
Vekua continuity } \rightarrow bound in Sobolev norms.

The choice of the directions in 3D

3D Jacobi-Anger gives the matrix $\{M\}_{l,m;k} = Y_{l,m}(\mathbf{d}_k)$
that depends on the choice of the directions.

Problem: an upper bound on $\|M^{-1}\|$ is needed but M is not even always invertible!

Solution:

- there exists an optimal choice s.t. $\|M^{-1}\|_1 \leq 2\sqrt{\pi} p$;
- it corresponds to the **extremal systems** of Sloan and Womersley for quadrature on S^2 ;
- some simple choices of points give good result,
heuristic: \mathbf{d}_k have to be as "equispaced" as possible.

With this choice \rightarrow analogous results as in 2D.

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The final approximation by plane waves

$$-\Delta u - \omega^2 u = 0$$



GHPs



Plane waves

Vekua theory: algebraic in h and p ,

Jacobi-Anger: algebraic in h ,
> exponential in p .

Final estimate

$$\inf_{\alpha \in \mathbb{C}^p} \left\| u - \sum_{k=1}^p \alpha_k e^{i\omega x \cdot d_k} \right\|_{j,\omega,D} \leq C h^{K+1-j} q^{-\lambda(K+1-j)} \|u\|_{K+1,\omega,D}$$

In 2D: $p = 2q + 1$, $\lambda(D)$ explicit, $\forall d_k$.

In 3D: $p = (q + 1)^2$, $\lambda(D)$ unknown, special d_k .

Approximation for Helmholtz eq.: conclusions

What we have proved:

- hp -estimates for circular/spherical and plane waves in 2D and 3D,
- all the constants are explicit in ωh ,
- all the orders in h are sharp.

Work in progress:

- explicit order of convergence in p in 3D convex domains,
- Vekua theory and approximation for Maxwell equation.

Part IV

Approximation for Maxwell equations

Maxwell equation

The vector field \mathbf{u} is solution of Maxwell equations

$$\text{curl curl } \mathbf{u} - \omega^2 \mathbf{u} = 0$$

if and only if

$$\begin{cases} -\Delta \mathbf{u}_j - \omega^2 \mathbf{u}_j = 0, & j = 1, 2, 3, \\ \text{div } \mathbf{u} = 0. \end{cases}$$

$\begin{cases} p \text{ directions} \\ 3p \text{ plane waves} \end{cases} \longrightarrow$ same approximation as for Helmholtz,
non-Trefftz functions!

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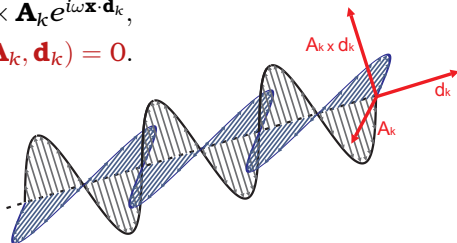
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Maxwell plane waves

Basis of Maxwell plane wave functions:

$$\mathbf{A}_k e^{i\omega \mathbf{x} \cdot \mathbf{d}_k}, \quad \mathbf{d}_k \times \mathbf{A}_k e^{i\omega \mathbf{x} \cdot \mathbf{d}_k},$$
$$|\mathbf{A}_k| = |\mathbf{d}_k| = 1, \quad (\mathbf{A}_k, \mathbf{d}_k) = 0.$$



Goal: prove convergence using $2p$ plane waves and preserving the Trefftz property.

P.W. approximation: lazy approach

1 \mathbf{u} Maxwell \Rightarrow $\text{curl } \mathbf{u}$ Maxwell \Rightarrow $\text{curl } \mathbf{u}$ Helmholtz

$$\left\| \text{curl } \mathbf{u} - \begin{array}{c} \text{Helmholtz} \\ \text{vector p.w.} \end{array} \right\|_{j,\omega,D} \leq C(hq^{-\lambda})^{K+1-j} \|\text{curl } \mathbf{u}\|_{K+1,\omega,D}$$

2 With $j \geq 1$, apply curl and reduce j (bad!):

$$\left\| \text{curl curl } \mathbf{u} - \text{curl} \begin{array}{c} \text{Helmholtz} \\ \text{vector p.w.} \end{array} \right\|_{j-1,\omega,D} \leq C(hq^{-\lambda})^{K+1-j} \|\text{curl } \mathbf{u}\|_{K+1,\omega,D}$$

\Downarrow

$$3 \left\| \omega^2 \mathbf{u} - \text{Maxwell p.w.} \right\|_{j-1,\omega,D} \leq C(hq^{-\lambda})^{K+1-j} \|\text{curl } \mathbf{u}\|_{K+1,\omega,D}$$

Mismatch between Sobolev indices and convergence order:
not sharp!

Vector spherical harmonics

Two different orthogonal basis for $(L^2(S^2))^3$:

$$\left\{ \begin{array}{l} \mathbf{Y}_{l,m}(\mathbf{x}) = Y_{l,m}(\mathbf{x}) \frac{\mathbf{x}}{|\mathbf{x}|} \\ \mathbf{\Psi}_{l,m}(\mathbf{x}) = |\mathbf{x}| \nabla Y_{l,m}(\mathbf{x}) \\ \mathbf{\Phi}_{l,m}(\mathbf{x}) = \mathbf{x} \times \nabla Y_{l,m}(\mathbf{x}) \end{array} \right\} \in L^2_T(S^2)$$
$$\left\{ \begin{array}{l} \mathbf{I}_{l,m}(\mathbf{x}) = (l+1) \mathbf{Y}_{l+1,m}(\mathbf{x}) + \mathbf{\Psi}_{l+1,m}(\mathbf{x}) \\ \mathbf{T}_{l,m}(\mathbf{x}) = -\mathbf{\Phi}_{l,m}(\mathbf{x}) \\ \mathbf{N}_{l,m}(\mathbf{x}) = l \mathbf{Y}_{l-1,m}(\mathbf{x}) - \mathbf{\Psi}_{l-1,m}(\mathbf{x}) \end{array} \right\} \in L^2_T(S^2)$$

$\mathbf{\Phi}_{l,m}, \mathbf{\Psi}_{l,m}$ tangential to S^2 , $\mathbf{Y}_{l,m}$ orthogonal,

$\mathbf{I}_{l,m}, \mathbf{T}_{l,m}, \mathbf{N}_{l,m}$ are traces of l -homogeneous harmonic vector polynomials.

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Maxwell-GHPs

Herglotz functions theory \rightarrow basis of Maxwell-GHPs:

$$\begin{cases} \mathbf{b}_{l,m}^1(\mathbf{x}) = j_l(\omega r) \Phi_{l,m} \\ \mathbf{b}_{l,m}^2(\mathbf{x}) = \frac{l+1}{2l+1} j_{l-1}(\omega r) \mathbf{I}_{l-1,m} + \frac{l}{2l+1} j_{l+1}(\omega r) \mathbf{N}_{l+1,m}. \end{cases}$$

In $\mathbf{b}_{l,m}^2$ there are two Bessel functions of different indices:

$\mathbf{V}_2[\mathbf{b}_{l,m}^2]$ is a non-homogeneous polynomial !

\rightarrow approximation theory is a bit more complicated.

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Approximation in h by Maxwell-GHPs

- (VERFÜRTH 1999) recursive definition of approximating polynomial $\mathbf{P}_{m,B}$ on D : orders in h ;
- for harmonic vector fields $\mathbf{P}_{m,B}$ is the Taylor polynomial!
- approximate $\mathbf{P}_{m,B}$ with $\{\mathbf{V}_2[\mathbf{b}_{l,m}^1]\}_{l \leq L} \cup \{\mathbf{V}_2[\mathbf{b}_{l,m}^2]\}_{l \leq L+1}$;
- Vekua continuity \Rightarrow **h -estimate** for Maxwell solutions:

$$\|\mathbf{u} - \mathbf{Q}_L\|_{j,\omega,D} \leq C(\omega h) h^{L+1-j} \|\mathbf{u}\|_{L+1,\omega,D}$$

- same order as Helmholtz, using a few more functions:

$$2(L+1)^2 - 2 + 2L + 3.$$

p -estimate impossible with this approach without any new ideas for the Helmholtz case.

Approximation in h by Maxwell plane waves

GHPs \leftrightarrow plane waves approximation relies on Jacobi-Anger exp.

We need a **vector** version of the Jacobi-Anger expansion that “preserves” **Maxwell** property:

$$e^{i\omega \mathbf{x} \cdot \mathbf{d}} \text{Id} = 4\pi \sum_{l \geq 0} \sum_{|m| \leq l} i^l \left(\frac{1}{l(l+1)} \mathbf{b}_{l,m}^1(\omega \mathbf{x}) \overline{\Phi_{l,m}(\mathbf{d})} \right. \\ \left. - \frac{i}{l(l+1)} \mathbf{b}_{l,m}^2(\omega \mathbf{x}) \overline{\Psi_{l,m}(\mathbf{d})} - i \mathbf{b}_{l,m}^\perp(\omega \mathbf{x}) \overline{\mathbf{Y}_{l,m}(\mathbf{d})} \right)$$

($\mathbf{b}_{l,m}^\perp$ are the non-Maxwell vector GHPs).

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GHPs \leftrightarrow plane waves approximation relies on Jacobi-Anger exp.

We need a **vector** version of the Jacobi-Anger expansion that "preserves" **Maxwell** property:

$$e^{i\omega \mathbf{x} \cdot \mathbf{d}} \mathbf{A} = 4\pi \sum_{l \geq 0} \sum_{|m| \leq l} i^l \left(\frac{1}{l(l+1)} \mathbf{b}_{l,m}^1(\omega \mathbf{x}) \overline{\Phi_{l,m}(\mathbf{d})} \cdot \mathbf{A} - \frac{i}{l(l+1)} \mathbf{b}_{l,m}^2(\omega \mathbf{x}) \overline{\Psi_{l,m}(\mathbf{d})} \cdot \mathbf{A} - \underbrace{i \mathbf{b}_{l,m}^\perp(\omega \mathbf{x}) \overline{\mathbf{Y}_{l,m}(\mathbf{d})} \cdot \mathbf{A}}_{=0 \text{ if } \mathbf{d} \cdot \mathbf{A} = 0} \right)$$

The diagram shows a sphere labeled S^2 . A red vector \mathbf{d} originates from the center and points to the surface. Another red vector \mathbf{A} originates from the same point and points away from the sphere. The angle between \mathbf{d} and \mathbf{A} is labeled $Y_m(\mathbf{d})$.

$(\mathbf{b}_{l,m}^\perp)$ are the non-Maxwell vector GHPs).

This gives expansion of vector fields in Maxwell plane waves.

Still some (technical ?) problems in the residual estimate to prove approximation in h .

THANK YOU!