Higher-order cochain interpolation

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Outline

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Discretization: pointwise approach

Problem? Reduce the number of degrees of freedom (dof)

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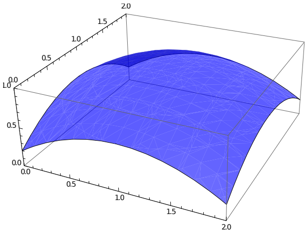
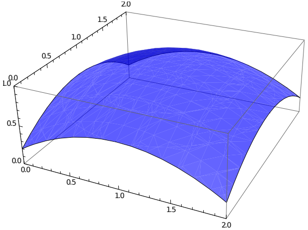


$$\phi \underset{\mathcal{I}}{\overset{\mathcal{R}}{\rightleftarrows}} \widetilde{\phi} = \mathcal{R}\phi = [\phi_i]$$

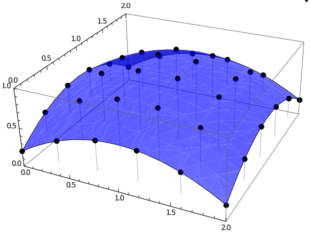
Discretization: pointwise approach

Problem? Reduce the number of degrees of freedom (dof)



$$\xrightarrow{\mathcal{R}} \quad \xleftarrow[\mathcal{I}]{} \quad \widetilde{\phi} = \mathcal{R}\phi = [\phi_i]$$

Typically pointwise



$$\xrightarrow{\mathcal{R}} \quad \xleftarrow[\mathcal{I}]{} \quad \widetilde{\phi} = \mathcal{R}\phi = [\phi_i] = [\phi(x_i)]$$

Differential operators become approximations

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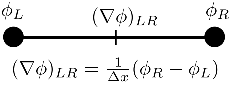
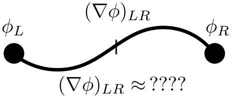
Pointwise discrete gradient (G):





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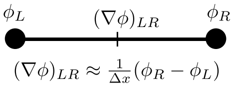

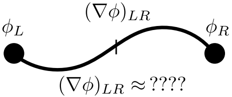

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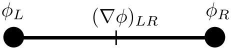
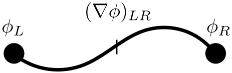
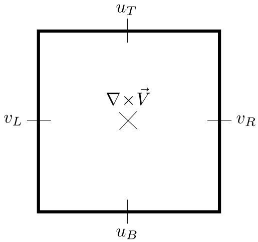


$(\nabla\phi)_{LR} \approx \frac{1}{\Delta x}(\phi_R - \phi_L)$

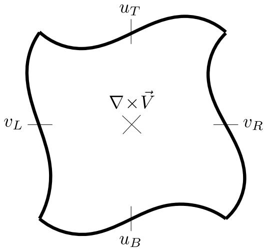


$(\nabla\phi)_{LR} \approx ????$

Pointwise discrete curl (C):



$\nabla \times \vec{V} \approx \frac{1}{\Delta x}(v_R - v_L) + \frac{1}{\Delta y}(u_B - u_T)$



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Differential operators become approximations

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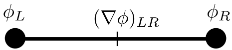

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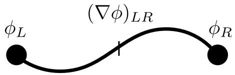

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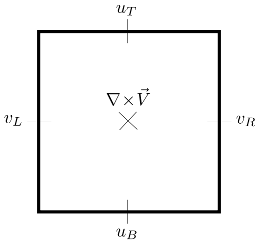
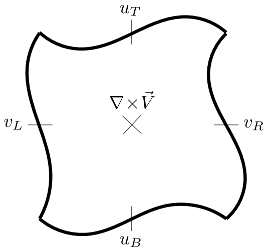


$$\nabla\times\vec{V} \approx \frac{1}{\Delta x\Delta y}[(v_R - v_L)\Delta y + (u_B - u_T)\Delta x]$$



$$\nabla\times\vec{V} \approx ???$$

Discretization: geometric approach

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Fundamental theorem of calculus

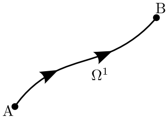
Kelvin-Stokes theorem

Gauss theorem

Discretization: geometric approach

Fundamental theorem of calculus

$$\int_{\Omega^1} \nabla\varphi \cdot \mathrm{d}\vec{s} = \varphi(\Omega^1_B) - \varphi(\Omega^1_A)$$



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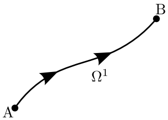
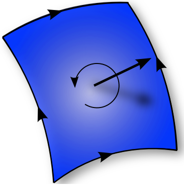


Kelvin-Stokes theorem

$$\int_{\Omega^2} (\nabla \times \vec{V}) \cdot \vec{n}\,\mathrm{d}S = \int_{\partial\Omega^2} \vec{V} \cdot \mathrm{d}\vec{s}$$



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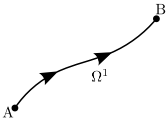

Kelvin-Stokes theorem

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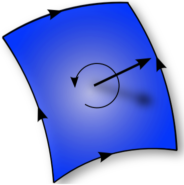
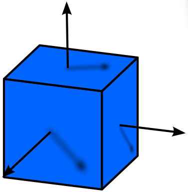


Gauss theorem

$$\int_{\Omega^3} (\nabla\cdot\vec{V})\mathrm{d}V = \int_{\partial\Omega^3} \vec{V}\cdot\vec{n}\,\mathrm{d}S$$



Differential geometry: Generalized Stokes Theorem

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$$\int_{\Omega^{k+1}} \mathrm{d}\omega^k = \int_{\partial\Omega^{k+1}} \omega^k$$

With $\omega^k \in \Lambda^k$, space of continuous differential k-forms.

Differential geometry: Generalized Stokes Theorem

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$$\langle \mathrm{d}\omega^k, \Omega^{k+1} \rangle = \langle \omega^k, \partial\Omega^{k+1} \rangle$$

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With $\omega^k \in \Lambda^k$, space of continuous differential k-forms.

$$k = 0: \quad \int_L \mathrm{d}\phi^0 = \phi^0(B) - \phi^0(A) \quad \mathrm{d}: \Lambda^0 \mapsto \Lambda^1 \quad \text{(maps points to lines)}$$

$$k = 1: \quad \int_S \mathrm{d}\alpha^1 = \int_{\partial S} \alpha^1 \quad \mathrm{d}: \Lambda^1 \mapsto \Lambda^2 \quad \text{(maps lines to surfaces)}$$

$$k = 2: \quad \int_V \mathrm{d}\beta^2 = \int_{\partial V} \beta^2 \quad \mathrm{d}: \Lambda^2 \mapsto \Lambda^3 \quad \text{(maps surfaces to volumes)}$$

Differential geometry: Generalized Stokes Theorem

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$$k = 2: \quad \int_V \mathrm{d}\beta^2 = \int_{\partial V} \beta^2 \quad \mathrm{d} : \Lambda^2 \mapsto \Lambda^3 \quad \text{(maps surfaces to volumes)}$$

Constitutes an exact sequence (de Rham complex), since $\mathrm{dd} = 0$

$$\mathbb{R} \hookrightarrow \Lambda^0 \xrightarrow[\nabla]{\mathrm{d}} \Lambda^1 \xrightarrow[\nabla\times]{\mathrm{d}} \Lambda^2 \xrightarrow[\nabla\cdot]{\mathrm{d}} \Lambda^3 \xrightarrow{\mathrm{d}} 0$$

Chains and cochains and the de Rham map

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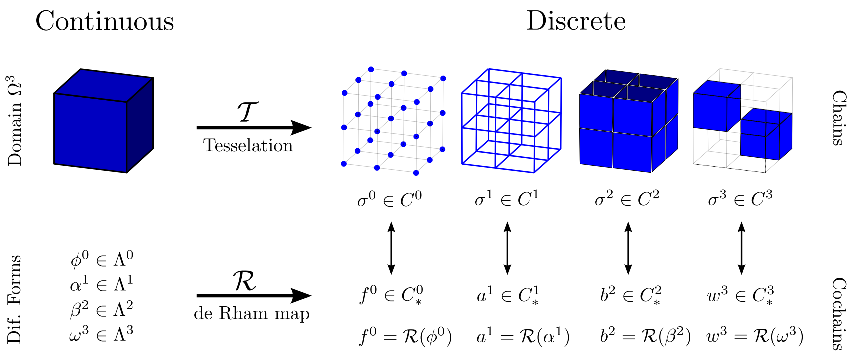
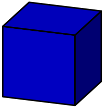


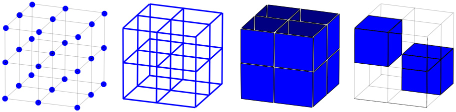
Chains and cochains and the de Rham map

Continuous | Discrete

Domain Ω^3



$\xrightarrow{\mathcal{T}}$ Tesselation



Chains

$\sigma^0 \in C^0$ $\quad \sigma^1 \in C^1$ $\quad \sigma^2 \in C^2$ $\quad \sigma^3 \in C^3$

$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$

Dif. Forms

$\phi^0 \in \Lambda^0$
$\alpha^1 \in \Lambda^1$
$\beta^2 \in \Lambda^2$
$\omega^3 \in \Lambda^3$

$\xrightarrow{\mathcal{R}}$ de Rham map

$f^0 \in C^0_*$ $\quad a^1 \in C^1_*$ $\quad b^2 \in C^2_*$ $\quad w^3 \in C^3_*$

$f^0 = \mathcal{R}(\phi^0)$ $\quad a^1 = \mathcal{R}(\alpha^1)$ $\quad b^2 = \mathcal{R}(\beta^2)$ $\quad w^3 = \mathcal{R}(\omega^3)$

Cochains

$$\mathcal{R} : \Lambda^k \mapsto C^k_*$$

$$\mathcal{R}(\alpha^k) = \left[\int_{\sigma^k_i \in \sigma^k} \alpha^k\right] = \left[a^k_i\right] = a^k$$

Exact differential operators

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$$\omega^{k+1} = \mathrm{d}\alpha^{k}$$

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$$\mathcal{R}(\omega^{k+1}) = \mathcal{R}(\mathrm{d}\alpha^{k})$$

Exact differential operators

$$\begin{aligned}\omega^{k+1} &= \mathrm{d}\alpha^{k} \\ \mathcal{R}(\omega^{k+1}) &= \mathcal{R}(\mathrm{d}\alpha^{k}) \\ &= \left[\int_{\sigma_i^{k+1}} \mathrm{d}\alpha^{k}\right]\end{aligned}$$

Exact differential operators

$$
\begin{aligned}
\omega^{k+1} &= \mathrm{d}\alpha^{k} \\
\mathcal{R}(\omega^{k+1}) &= \mathcal{R}(\mathrm{d}\alpha^{k}) \\
&= \left[\int_{\sigma_i^{k+1}} \mathrm{d}\alpha^{k}\right] \\
&= \left[\int_{\partial\sigma_i^{k+1}} \alpha^{k}\right]
\end{aligned}
$$

Exact differential operators

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\end{aligned}
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&= \delta\left[\int_{\sigma_i^{k}} \alpha^k\right] \\
\mathcal{R}(\omega^{k+1}) &= \delta\mathcal{R}(\alpha^k)
\end{aligned}
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Exact differential operators

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\mathcal{R}(\omega^{k+1}) &= \delta\mathcal{R}(\alpha^k) \\
\Rightarrow \mathcal{R}\mathrm{d} &= \delta\mathcal{R}
\end{aligned}
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Exact differential operators

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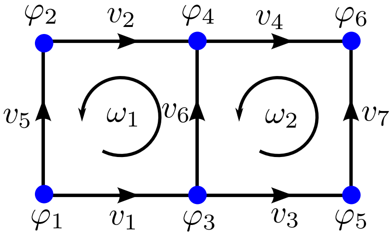
\mathcal{R}(\omega^{k+1}) &= \delta\mathcal{R}(\alpha^k) \\
\Rightarrow \mathcal{R}\mathrm{d} &= \delta\mathcal{R} \\
\Rightarrow \quad \delta\delta &= 0
\end{aligned}
$$

Exact differential operators



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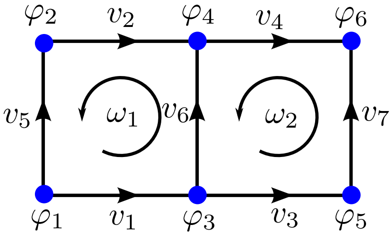
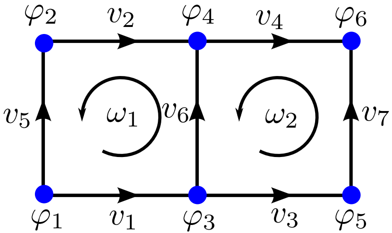


Gradient $\nu^1 = \mathrm{d}\phi^0 \rightarrow v^1 = \delta f^0$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix}$$

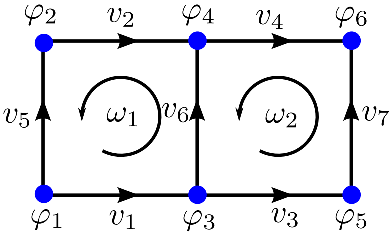
Exact differential operators



Curl $\omega^2 = \mathrm{d}\nu^1 \rightarrow w^2 = \delta v^1$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}$$

Exact differential operators



$$\mathrm{dd} = 0 \rightarrow \delta\delta = 0$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{bmatrix} = 0$$

Exact differential operators

Discrete exact complex

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & \Lambda^0 & \xrightarrow{\mathrm{d}} & \Lambda^1 & \xrightarrow{\mathrm{d}} & \Lambda^2 & \xrightarrow{\mathrm{d}} & \Lambda^3 & \longrightarrow & 0 \\
 & & \downarrow \mathcal{R} & & \downarrow \mathcal{R} & & \downarrow \mathcal{R} & & \downarrow \mathcal{R} & & \\
\mathbb{R} & \longrightarrow & C_*^0 & \xrightarrow{\delta} & C_*^1 & \xrightarrow{\delta} & C_*^2 & \xrightarrow{\delta} & C_*^3 & \longrightarrow & 0
\end{array}
$$

Cochain reconstruction: the Whitney map, $\mathcal{I}$

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Problem? Given a cochain, get a piecewise smooth reconstruction

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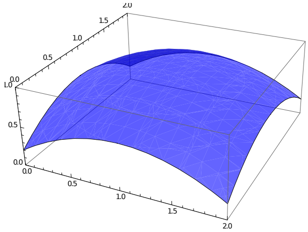
$$\widetilde{\phi} = \mathcal{R}\phi = [\phi_i] \xrightarrow{\mathcal{I}}$$



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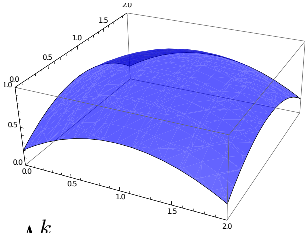

$$\mathcal{I} : C^k_* \mapsto \Lambda^k$$

$$\mathcal{I}(a^k) = \sum_i a_i\, \epsilon_i^k(\boldsymbol{x}), \quad \epsilon_i^k(\boldsymbol{x}) \in \Lambda_p^k \subset \Lambda^k$$

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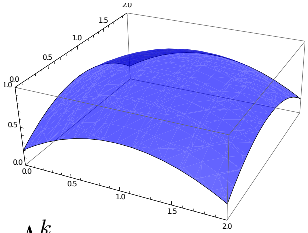

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Such that:

$$\mathcal{R} \circ \mathcal{I} = \mathbb{1}, \quad \mathcal{I} \circ \mathcal{R} = \mathbb{1} + \mathcal{O}(h^p) \quad \text{and} \quad \mathrm{d}\mathcal{I} = \mathcal{I}\delta$$

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$$\mathcal{R} \circ \mathcal{I} = \mathbb{1}$$

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$\mathcal{R} \circ \mathcal{I} = \mathbb{1}$

$$\mathcal{R} \circ \mathcal{I}(a^k) = \mathcal{R}[\mathcal{I}(a^k)]$$

Cochain reconstruction: the Whitney map, $\mathcal{I}$

$\mathcal{R} \circ \mathcal{I} = \mathbb{1}$

$$\begin{aligned}\mathcal{R} \circ \mathcal{I}(a^k) &= \mathcal{R}[\mathcal{I}(a^k)] \\ &= \mathcal{R}[\sum_i a_i \, \epsilon_i^k(\boldsymbol{x})]\end{aligned}$$

Cochain reconstruction: the Whitney map, $\mathcal{I}$

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\end{aligned}
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\mathcal{R} \circ \mathcal{I}(a^k) &= \mathcal{R}[\mathcal{I}(a^k)] \\
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&= \sum_i a_i \, \mathcal{R}[\epsilon_i^k(\boldsymbol{x})]
\end{aligned}$$

$$\Rightarrow \mathcal{R}[\epsilon_i^k] = \left[\int_{\sigma_j^k} \epsilon_i^k \right] = \delta_{ij} \quad \text{(Kronecker delta)}$$

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Cochain reconstruction: the Whitney map, $\mathcal{I}$

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$$\begin{aligned}
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\end{aligned}$$

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$$\Rightarrow \mathrm{d}\epsilon_j^k = \sum_i [\delta]_{ij} \epsilon_i^{k+1}$$

Exact de Rham complex

Exact de Rham complex

Polynomial exact complex

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & \Lambda_p^0 & \xrightarrow{\mathrm{d}} & \Lambda_p^1 & \xrightarrow{\mathrm{d}} & \Lambda_p^2 & \xrightarrow{\mathrm{d}} & \Lambda_p^3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathcal{R}} & & \downarrow{\scriptstyle \mathcal{R}} & & \downarrow{\scriptstyle \mathcal{R}} & & \downarrow{\scriptstyle \mathcal{R}} & & \\
\mathbb{R} & \longrightarrow & C_*^0 & \xrightarrow{\delta} & C_*^1 & \xrightarrow{\delta} & C_*^2 & \xrightarrow{\delta} & C_*^3 & \longrightarrow & 0
\end{array}
$$

Edge basis functions

Edge basis functions

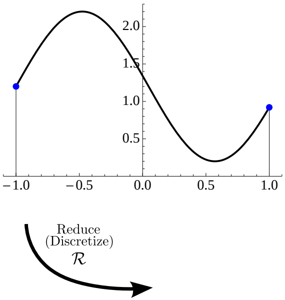
Nodal quantities: Lagrange interpolants

$$h_i^p(x_j) = \delta_{ij}$$

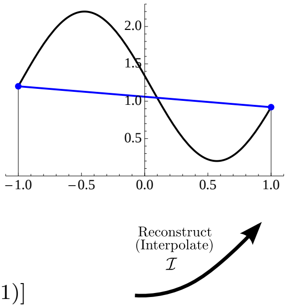
$i, j = 1, \ldots, p+1$

Edge basis functions

Nodal quantities: p=1

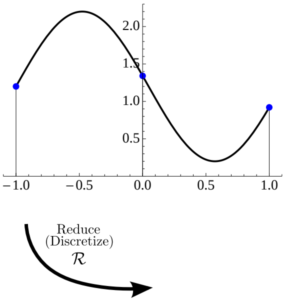


$$[\widetilde{\varphi}_i] = [\varphi(-1), \varphi(1)]$$

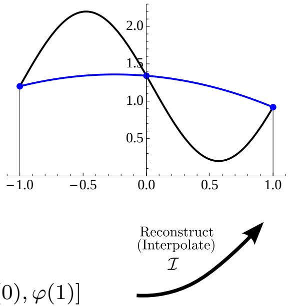


Edge basis functions

Nodal quantities: p=2

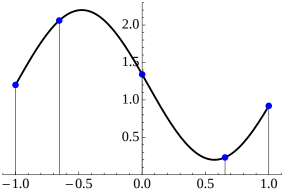


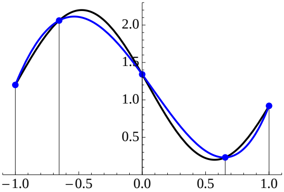
$$[\widetilde{\varphi}_i] = [\varphi(-1), \varphi(0), \varphi(1)]$$

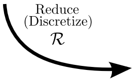


Edge basis functions

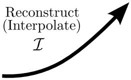
Nodal quantities: p=4





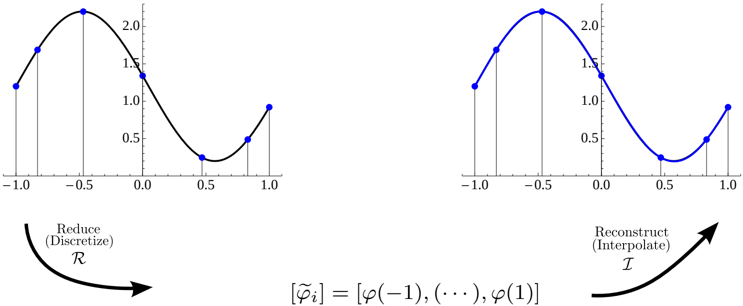


$$[\widetilde{\varphi}_i] = [\varphi(-1), (\cdots), \varphi(1)]$$



Edge basis functions

Nodal quantities: p=6



$$[\widetilde{\varphi}_i] = [\varphi(-1), (\cdots), \varphi(1)]$$

Edge basis functions

Volume quantities: edge functions

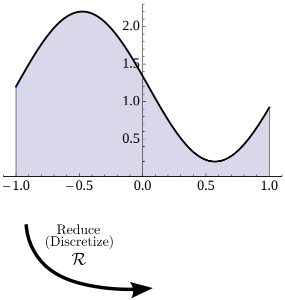
$$\int_{x_j}^{x_{j+1}} e_i^p(x) = \delta_{ij}$$

$$e_i^p(x) = -\sum_{n=1}^{i} \frac{d}{dx} h_n^{p+1}(x)$$

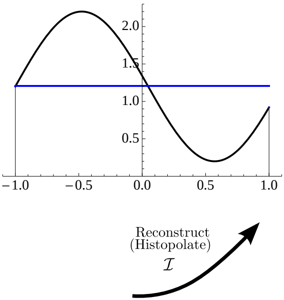
$i, j = 1, \dots, p$

Edge basis functions

Volume quantities: p=0

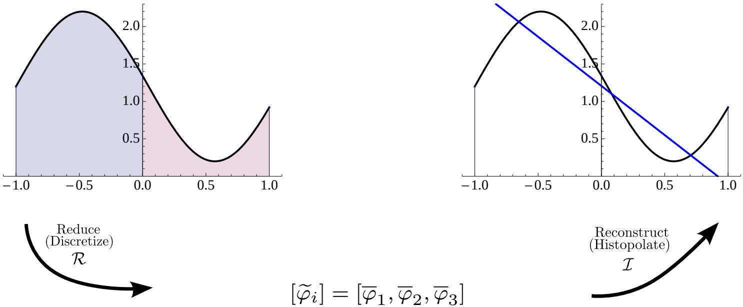


$$[\widetilde{\varphi}_i] = [\overline{\varphi}_1, \overline{\varphi}_2]$$



Edge basis functions

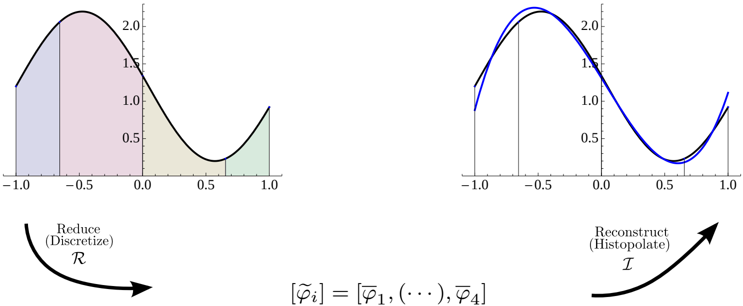
Volume quantities: p=1



$$[\widetilde{\varphi}_i] = [\overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3]$$

Edge basis functions

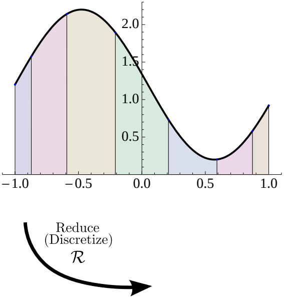
Volume quantities: p=3



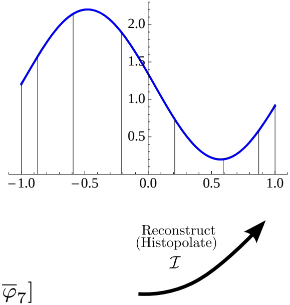
$$[\widetilde{\varphi}_i] = [\overline{\varphi}_1, (\cdots), \overline{\varphi}_4]$$

Edge basis functions

Volume quantities: p=6



$$[\widetilde{\varphi}_i] = [\overline{\varphi}_1, (\cdots), \overline{\varphi}_7]$$



Higher dimensions: (2D case)

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Nodal quantities:

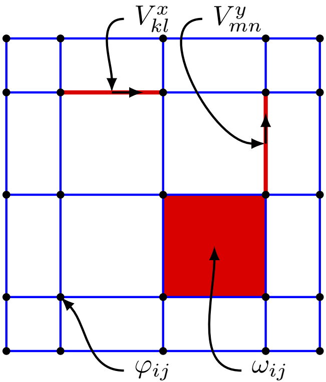
$$\varphi(x,y) \approx \widetilde{\varphi}^p(x,y) = \sum_{ij} \varphi_{ij} h_i^p(x) h_j^p(y)$$

1-form quantities:

$$\vec{V}(x,y) \approx \widetilde{\vec{V}}^p(x,y) = \sum_{ij} \overline{V}_{ij}^x e_i^{p-1}(x) h_j^p(y)\mathrm{d}x + \sum_{ij} \overline{V}_{ij}^y h_i^p(x) e_j^{p-1}(y)\mathrm{d}y$$

Volume quantities:

$$\omega(x,y) \approx \widetilde{\omega}^p(x,y) = \sum_{ij} \overline{\omega}_{ij} e_i^{p-1}(x) e_j^{p-1}(y)\mathrm{d}x\mathrm{d}y$$



Finite volumes and Galerkin

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$$\mathrm{d}\phi^0 = v^1$$

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$$\langle \mathrm{d}\phi^0, \epsilon^1_j \rangle = \langle v^1, \epsilon^1_j \rangle$$

Finite volumes and Galerkin

$$\mathrm{d}\phi^0 = v^1$$

$$\langle \mathrm{d}\phi^0, \epsilon^1_j \rangle = \langle v^1, \epsilon^1_j \rangle$$

$$\langle \mathrm{d}\phi^0 - v^1, \epsilon^1_j \rangle = 0$$

Finite volumes and Galerkin

$$
\begin{aligned}
\mathrm{d}\phi^0 &= v^1 \\
\langle \mathrm{d}\phi^0, \epsilon^1_j \rangle &= \langle v^1, \epsilon^1_j \rangle \\
\langle \mathrm{d}\phi^0 - v^1, \epsilon^1_j \rangle &= 0
\end{aligned}
$$

$$
\phi^0 = \sum_i \phi_i \epsilon^0_i, \quad v^1 = \sum_i v_i \epsilon^1_i
$$

Finite volumes and Galerkin

$$\langle \mathrm{d}\phi^0 - v^1, \epsilon_j^1 \rangle = 0$$

Finite volumes and Galerkin

$$\langle \mathrm{d}\phi^0 - v^1, \epsilon^1_j \rangle = 0$$
$$\langle \mathrm{d} \sum_i \phi_i \epsilon^0_i - \sum_i v_i \epsilon^1_i, \epsilon^1_j \rangle = 0$$

Finite volumes and Galerkin

$$\langle \mathrm{d}\phi^0 - v^1, \epsilon_j^1 \rangle = 0$$

$$\langle \mathrm{d} \sum_i \phi_i \epsilon_i^0 - \sum_i v_i \epsilon_i^1, \epsilon_j^1 \rangle = 0$$

$$\langle \sum_{ik} [\delta]_{ik} \phi_k \epsilon_i^1 - \sum_i v_i \epsilon_i^1, \epsilon_j^1 \rangle = 0$$

Finite volumes and Galerkin

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$$\left(\sum_k [\delta]_{ik} \phi_k - v_i \right) \langle \epsilon_i^1, \epsilon_j^1 \rangle = 0$$

Finite volumes and Galerkin

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$$\langle \sum_{ik} [\delta]_{ik} \phi_k \epsilon_i^1 - \sum_i v_i \epsilon_i^1, \epsilon_j^1 \rangle = 0$$

$$\sum_{ik} [\delta]_{ik} \phi_k \langle \epsilon_i^1, \epsilon_j^1 \rangle - \sum_i v_i \langle \epsilon_i^1, \epsilon_j^1 \rangle = 0$$

$$\left(\sum_k [\delta]_{ik} \phi_k - v_i \right) \langle \epsilon_i^1, \epsilon_j^1 \rangle = 0$$

$$\Rightarrow \sum_k [\delta]_{ik} \phi_k - v_i = 0, \quad \forall i$$

Conclusions

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- Algebraic topology as a natural discrete analogue of differential geometry.
- Geometrical reduction operator, $\mathcal{R}$, produces exact differential operators
- Reduction operator, $\mathcal{R}$, Whitney map, $\mathcal{W}$, and exterior derivative, d, have nice commuting properties.
- Edge basis functions for finite elements yield finite volume like discretizations .

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J. Kreeft presentation on *Higher-order discretization of Laplace operator* for application of this ideas.

Thank you

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