Quadrature for high order finite elements on pyramids

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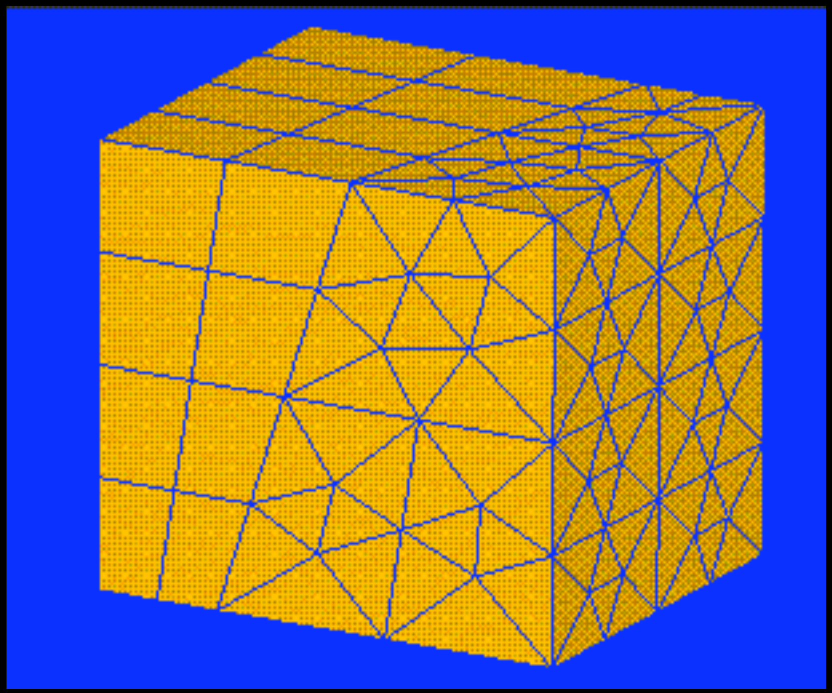
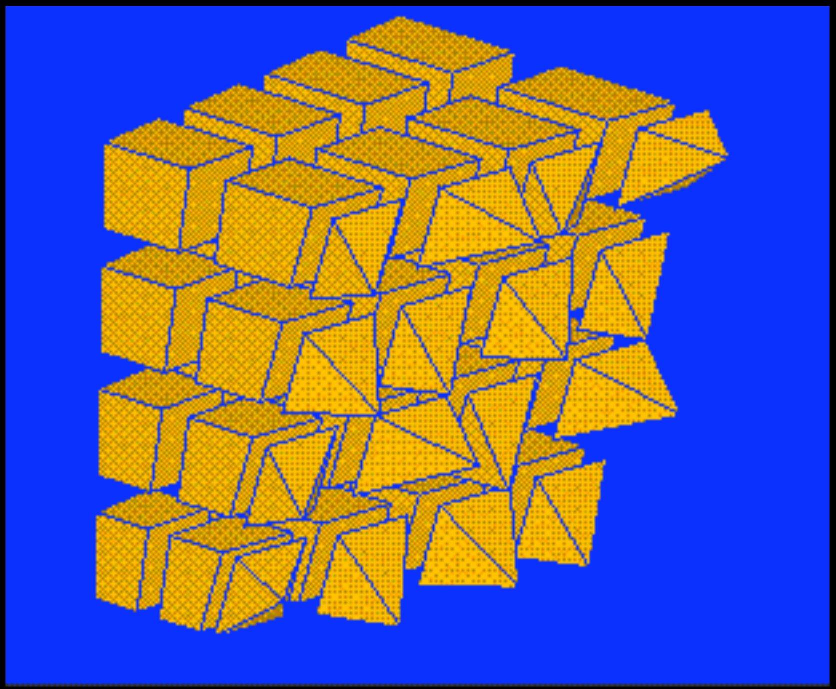
Non-standard numerical methods for PDEs. Pavia, June 30

Why construct finite elements on pyramidal domains?





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http://www.andrew.cmu.edu/user/sowen/hextet/hextotet2.htm

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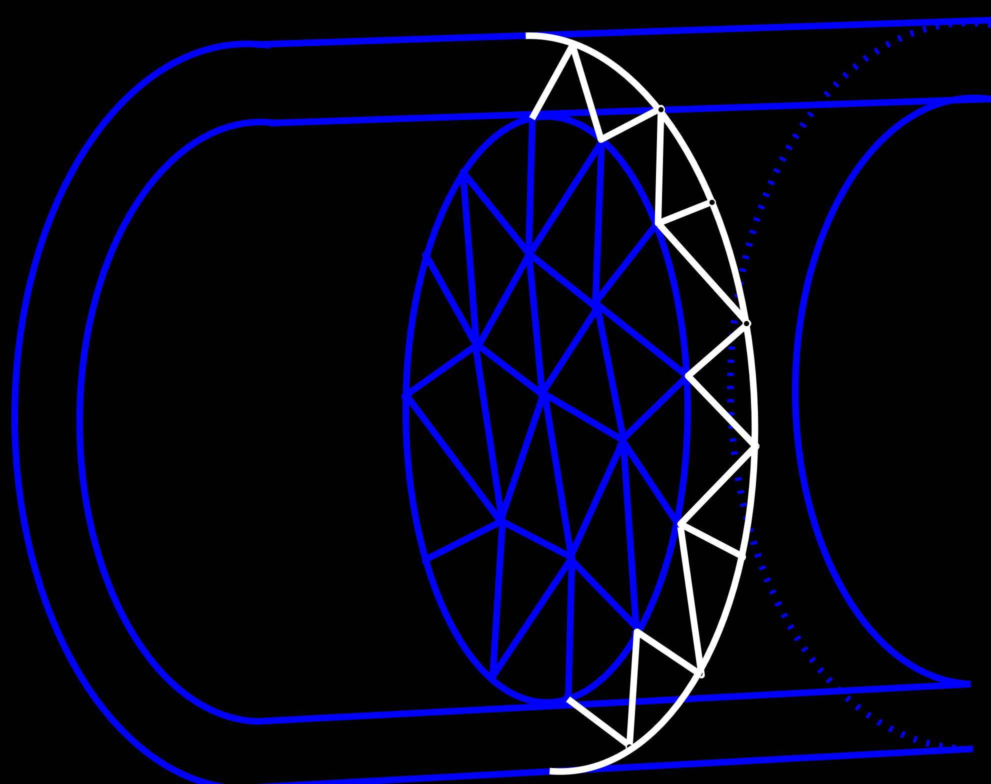
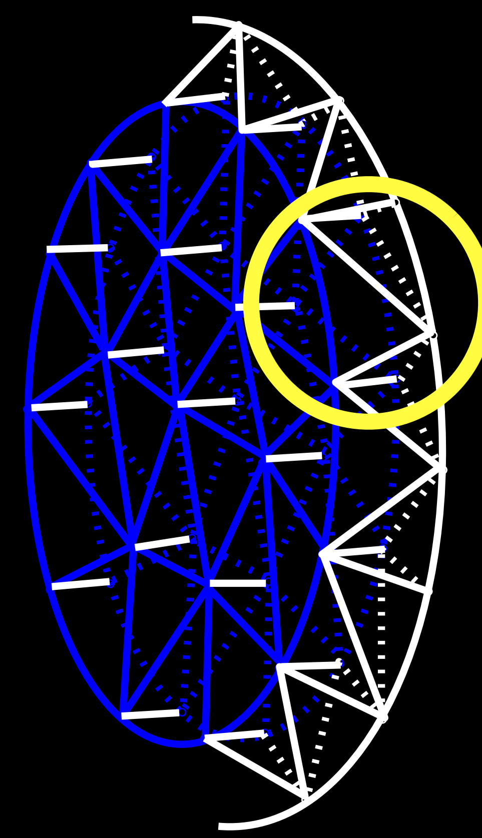




Gatto, Demkowicz, 2010

Continuous elements on pyramids

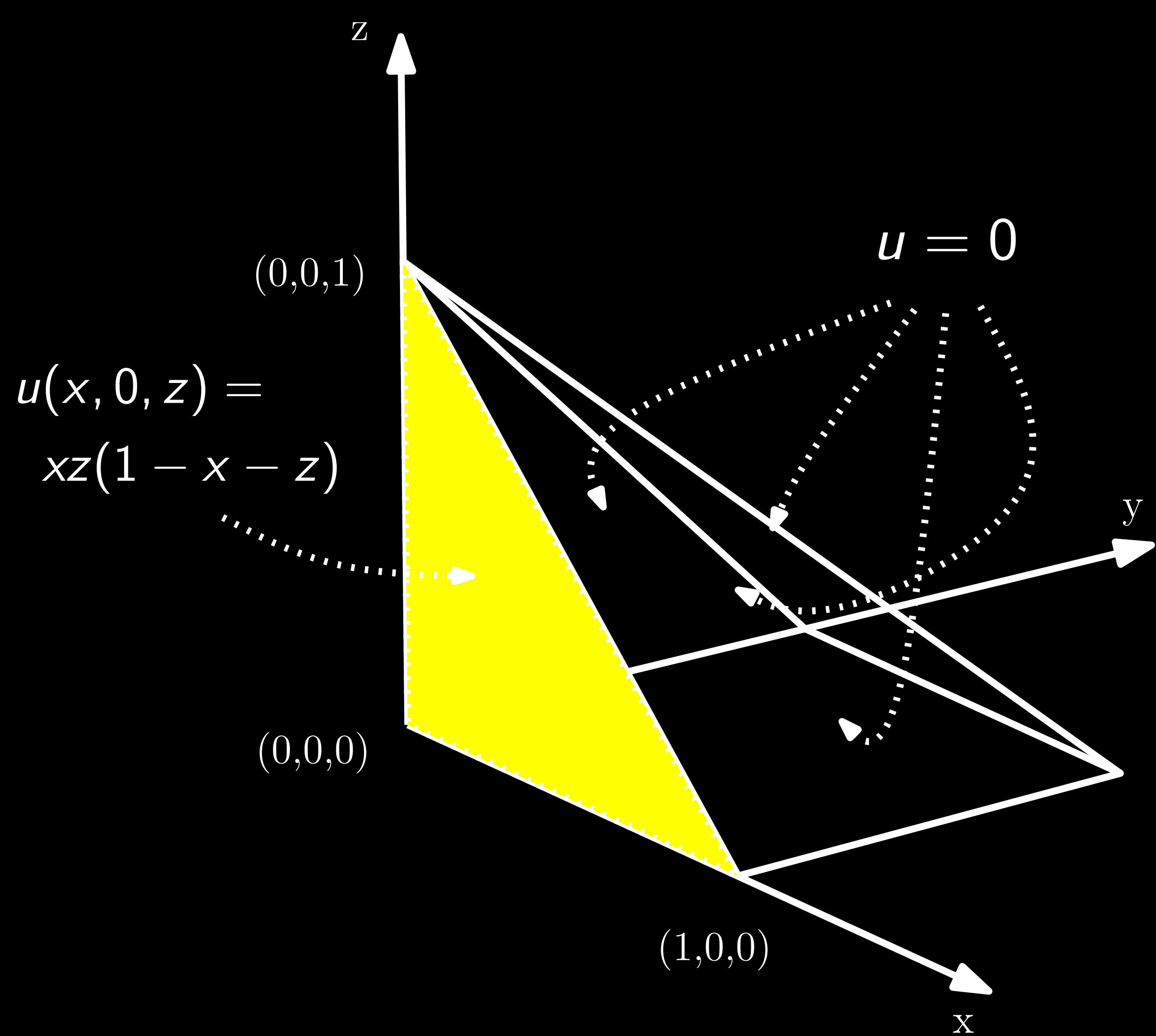
- Wachspress 1975. General polyhedral elements with 3-vertices. Generalises to pyramids.
- Bedrosian 1992. First and second order pyramidal elements.
- Macro-element based approaches. Wieners 1997 ...
- Sherwin 1997; Chatzi 2000. Attempts at high order.
- Bergot, Cohen, Durufle, 2010. Includes survey. “Optimal” high order elements.

Conforming pyramidal elements for the de Rham complex

$$
\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\nabla} & \mathbf{H}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \mathbf{H}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2(\Omega) \\
\Pi^{(0)}\big\downarrow & & \Pi^{(1)}\big\downarrow & & \Pi^{(2)}\big\downarrow & & \Pi^{(3)}\big\downarrow \\
\mathcal{V}^{(0)}(\Omega) & \xrightarrow{\nabla} & \mathcal{V}^{(1)}(\Omega) & \xrightarrow{\nabla\times} & \mathcal{V}^{(2)}(\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{V}^{(3)}(\Omega)
\end{array}
$$

- Graglia, 1999. First and second order edge and face elements.
- Hiptmair, 1999. First order elements. Proof of commutativity.
- Zaglmayr. High order: local exact sequences.
- Nigam and Phillips, 2007, 2010. High order: infinite pyramid.
- Bossavit, 2008. Canonical construction of first order.

Theorem: *There is no (high order) conforming pyramidal continuous finite element whose approximation space consists purely of polynomials.* Bedrossian 92, Wieners 97, Warren 02.



$$u(x, y, z) = \frac{xz(1-x-z)(1-y-z)}{1-z}$$

$$p(x, y, z) = xz(1-x-z)(1-y-z)\,(r(x, z) + ys(x, y, z))$$

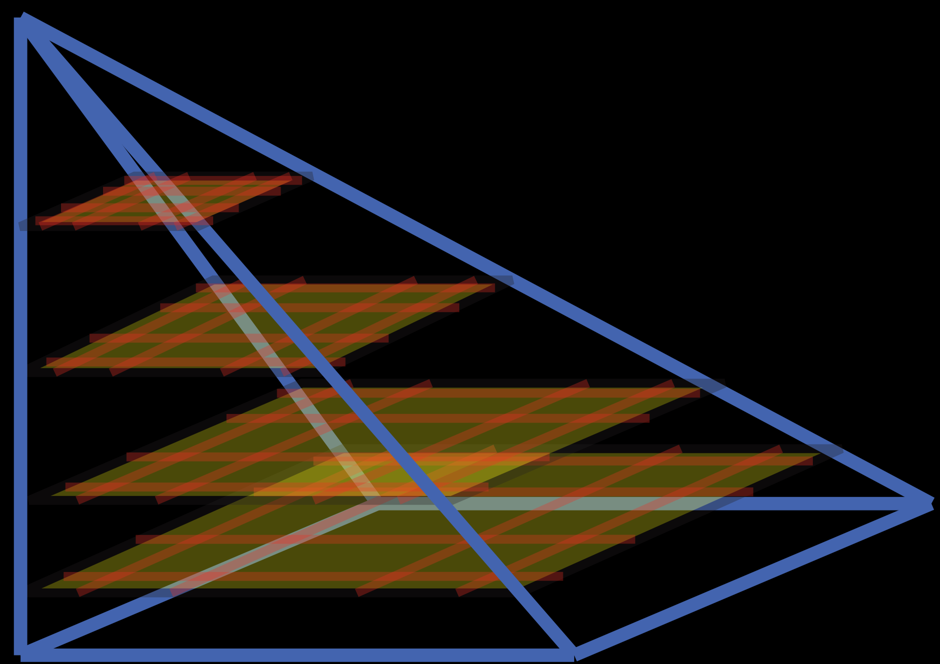
$$p(x, 0, z) = xz(1-x-z)(1-z)r(x, z) = u(x, 0, z) = xz(1-x-z)$$

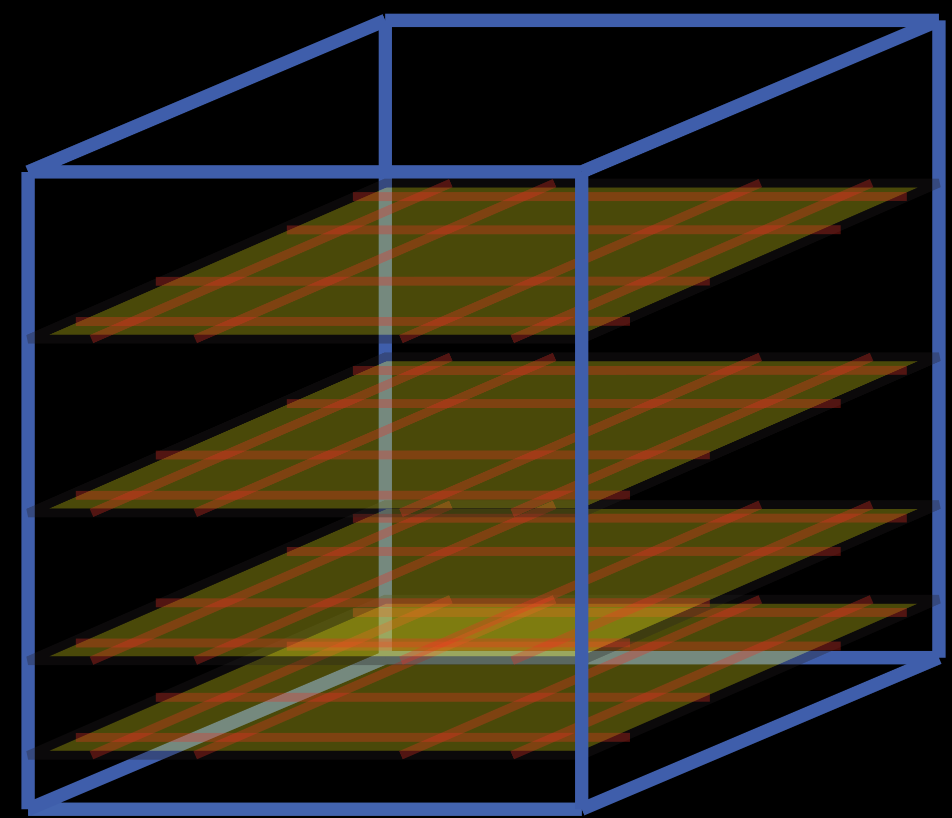
$$\Rightarrow r(x, z) = \frac{1}{1-z}$$

Quadrature intuition

- "Our basic objective is to give sufficient conditions on the quadrature scheme which ensure that the effect of the numerical integration does not decrease [the] order of convergence", Ciarlet 1978.
- Rule of thumb: Choose a quadrature method that integrates products of any pair of shape functions exactly.

Quadrature on pyramids





$P_k^{(0,0)} \times P_k^{(0,0)} \times P_k^{(2,0)}$

- Conical product formulae (Stroud, 72).
- Duffy transform + Gauss Legendre / Jacobi.
- k^3 evaluations.

Conical product formulae

- *k*th degree formula is exact for polynomials of degree 2*k* on the pyramid.
- Exact for products of any pair of *k*th order (Nigam-Phillips) pyramidal shape functions from each family of elements, including the rational functions.
- Numerical evidence that they perform well for continuous elements (Bergot et al, 2010).

As much of the relevant bits of the classical framework for analysing the effects of quadrature on finite elements as will fit onto one slide.

Example problem:

Find $u \in H_0^1(\Omega)$ such that $a(u, v) := \int \sum_{ij} A^{ij} \partial_i u \partial_j v = f(v) \; \forall v \in H_0^1(\Omega)$.

$\mathcal{T}_h$ is a tesselation for a domain, Ω. For a given order, k, $V_{h,k}$ is assembled using k^{th} order elements on each $K \in \mathcal{T}_h$. Write $S_{h,k}(\cdot) = \sum_{K \in \mathcal{T}_h} S_{k;K}(\cdot)$ where $S_{k;K}(\cdot)$ are per-element quadrature rules. Discrete problem:

Find $u_h \in V_{h,k}$ such that $a_h(u, v) := S_h\left(\sum_{ij} A^{ij} \partial_i u \partial_j v\right) = f(v_h) \; \forall v_h \in V_{h,k}$.

First Strang Lemma:

$$\|u - u_h\|_1 \leq C \inf_{v_h \in V_{h,k}} \left(\|u - v_h\|_1 + \sup_{w_h \in V_{h,k}} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_1} \right)$$

Write $E_{k;K}(\cdot) = S_{k;K}(\cdot) - \int_K \cdot$. For all $p \in P^{2k}$, $E_{k,K}(p) = 0$. Apply Bramble Hilbert Lemma and scaling argument to obtain local estimate:

$$|E_{k;K}[A^{ij} \partial_i w \partial_j v]| \leq C h^k \|A\|_{k,\infty,K} \|v\|_{k,K} |w|_{1,K} \quad \forall v, w \in V_{h,k}$$

Sum over all elements. Assume an $O(h^k)$ best approximation estimate and bounded interpolation operator, $\Pi : H^1(\Omega) \to V_{h,k}$.

$$\|u - u_h\|_1 \leq C h^k \left(|u|_{k+1} + \|A\|_{k,\infty,K} \|u\|_{k+1} \right)$$

What goes wrong on pyramids...

- Consistency error is controlled using local error estimates:

$$|E_{k,K}[A^{ij}\partial_i w \partial_j v]| \leq Ch^k \|A\|_{k,\infty,K} \|v\|_{k,K} |w|_{1,K}$$

- Zeros in the denominator of the rational functions at the tip of the pyramid mean that we can't take high derivatives.
- Continuous element shape function associated with vertex (1,1,0) is $v(x,y,z) = \frac{xy}{1-z}$.

$$\int_{\hat{K}} \left(\frac{\partial^3 v}{\partial z^3} \right)^2 d\mathbf{x} = \int_0^1 \int_0^{1-z} \int_0^{1-z} \left(\frac{-6xy}{(1-z)^4} \right)^2 dxdydz$$

$$= \int_0^1 \frac{9}{(1-z)^2} dz.$$

Rational functions can't be ignored

- Temptation: Apply local error estimate to polynomials on each element.
- But ... rational functions are essential to interpolation
- ... and interpolation is essential to controlling the consistency error.

$$
\begin{aligned}
\|u - u_h\|_1 &\leq C \inf_{v_h \in V_{h,k}} \left(\|u - v_h\|_1 + \sup_{w_h \in V_{h,k}} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_1} \right) \\
&\leq C \inf_{v_h \in V_{h,k}} \left(\|u - v_h\|_1 + h^k \sum_{K \in \mathcal{T}_h} \|A\|_{k,\infty,K} \|v_h\|_{k,K} \right) \\
&\leq C \left(\|u - \Pi u\|_1 + h^k \sum_{K \in \mathcal{T}_h} \|A\|_{k,\infty,K} \|\Pi u\|_{k,K} \right) \\
&\leq C h^k \left(|u|_{k+1} + \|A\|_{k,\infty} \|u\|_{k+1} \right)
\end{aligned}
$$

A new family of elements

- Smaller than the original Nigam-P. elements.
- Same approximation properties.
- Approximation space for continuous element is the same as Bergot et al optimal element (and Zaglmayr)
- Much more concise definition

New approximation spaces

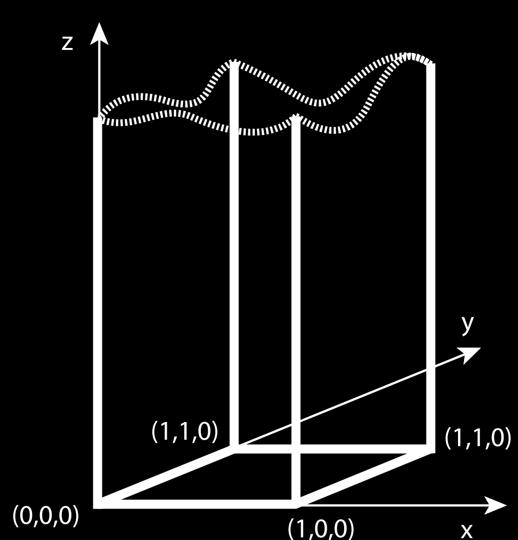
$$\mathcal{R}_k^{(0)}(K_\infty) = Q_k^{[k,k]}$$

$$\mathcal{R}_k^{(1)}(K_\infty) = \left(Q_{k+1}^{[k-1,k]} \times Q_{k+1}^{[k,k-1]} \times \{0\}\right) \oplus \{\nabla u : u \in Q_k^{[k,k]}\}$$

$$\mathcal{R}_k^{(2)}(K_\infty) = \left(\{0\} \times \{0\} \times Q_{k+2}^{[k-1,k-1]}\right) \oplus \left\{\nabla \times u : u \in \left(Q_{k+1}^{[k-1,k]} \times Q_{k+1}^{[k,k-1]} \times \{0\}\right)\right\}$$

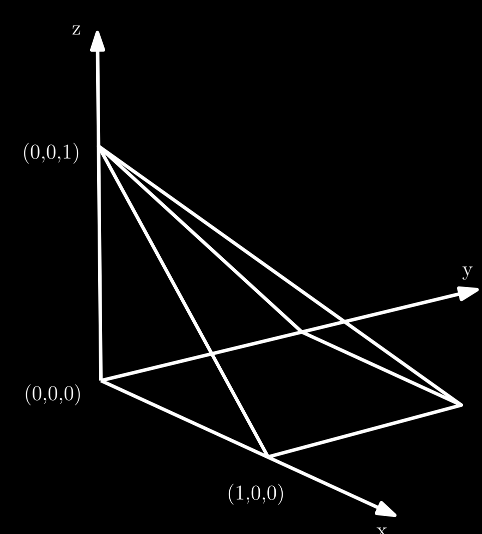
$$\mathcal{R}_k^{(3)}(K_\infty) = Q_{k+3}^{[k-1,k-1]}$$

$$Q_k^{[l,m]} = \text{span}\left\{\frac{x^a y^b (1+z)^c}{(1+z)^k} \quad a+c \le l \text{ and } b+c \le m\right\}$$



$$\phi : K_\infty \to K$$

$$\phi^* : \mathcal{R}_k^{(s)}(K) \to \mathcal{R}_k^{(s)}(K_\infty)$$



- Approximation spaces defined on infinite pyramid, K_∞, in terms of k-weighted polynomials
- Approximation spaces on finite pyramid, K, defined as pre-image of K_∞ under pullback mappings

Resolution

- Observation: On the reference pyramid, the components of each k^{th} order approximation live in spaces spanned by "rational monomials":

$$e_{abr}(x, y, z) := x^a y^b (1 - z)^{r-a-b} \quad r \le k \text{ and } a, b \le r + 1$$

- Regularity increases with r.

$$e_{abr} \in H^{r+1}(\hat{K}) \quad \forall a, b$$

$$|E_{k,K}[A^{ij} \partial_i e_{abr} \partial_j v]| \le C h^r \|A\|_{r,\infty,K} \|e\|_{r,K} |w|_{1,K}$$

- Can let the components of the tensor A^{ij} be element-wise polynomial. Quadrature is still exact:

$$E_{k,K}(A^{ij} \partial_i e_{abr} \partial_j w) = 0 \quad \forall A^{ij} \in P^{k-r}, \quad w \in V_k(K)$$

- Can reapply Bramble Hilbert Lemma and scaling argument to get "missing" h^{k-r}. So consistency error is $O(h^k)$.

Summary

- There exist high order conforming pyramidal finite elements for the de Rham complex that satisfy a commuting diagram property.
- Using Stroud's *k*th degree conical product rules to assemble the stiffness matrix for methods using *k*th degree pyramidal elements will not decrease the order of convergence.
- On face value, not a surprising result: consistent with the rule of thumb.
- Justification of the result is maybe a little surprising.
- Nigam and Phillips. *Numerical integration for high order pyramidal finite elements. arXiv:1003.0495v1*