Some estimates for hpk-refinement in Isogeometric Analysis

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Non-Standard Numerical Methods for PDEs
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Outline of the talk

B-splines on the reference domain $[0, 1]$

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- $\mathcal{S}^p_{\boldsymbol{k}} = \mathrm{span}\{B_1, \ldots, B_n\}$ piecewise polynomials of degree p with continuous derivatives up to the order $k_i - 1$ at knot ζ_i.

$$\mathcal{S}^p_k = \mathcal{S}^p_{\boldsymbol{k}} \text{ with } k_1 = \cdots = k_m = k.$$

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- $\mathcal{S}^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2} \equiv \mathcal{S}^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}(\mathcal{Q}_h) := \mathcal{S}^{p_1}_{\boldsymbol{k}_1} \otimes \mathcal{S}^{p_2}_{\boldsymbol{k}_2} = \mathrm{span}\{B_{ij}\}_{i=1,j=1}^{n_1,\ n_2}$.

  $\mathcal{S}^{p_1,p_2}_{k_1,k_2} = \mathcal{S}^{p_1,p_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}$ with $k_{1,d} = \cdots = k_{m_d,d} = k_d$, $d = 1, 2$.

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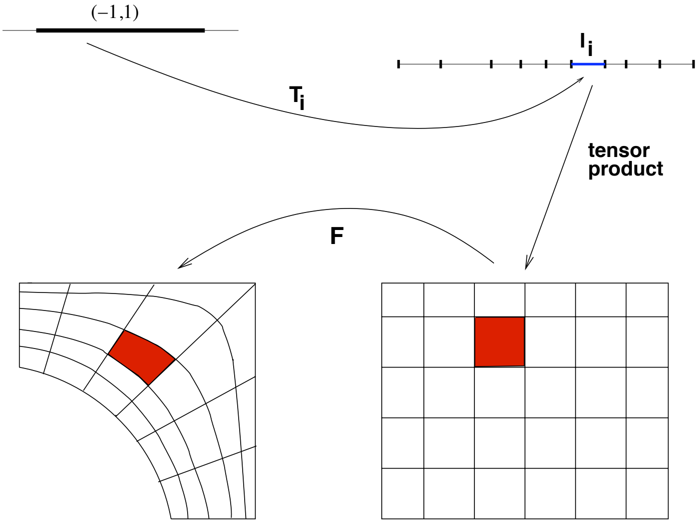

Legendre polynomials

Definition (Legendre polynomial of degree i)

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- Orthogonal projection of order $N \in \mathbf{N}$:

$$\pi_N\left(\sum_{i=0}^{\infty} \hat{\varphi}_i L_i(x)\right) = \sum_{i=0}^{N} \hat{\varphi}_i L_i(x).$$

Primitives of Legendre polynomials

Definition

Given a nonnegative integer n, we define the n-th primitive of L_i, $\Psi_{i,n}$, in a recursive way,

$$\Psi_{i,0}(x) = L_i(x), \quad \Psi_{i,n}(x) = \int_{-1}^{x} \Psi_{i,n-1}(\xi)\, d\xi, \quad n = 1, 2, \ldots$$

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The new projection operator

Definition

- p, k *nonnegative integers;*
- $\mathcal{S}^p(\Lambda)$ *set of polynomials of degree* $\leqslant p$ *in* $\Lambda = (-1, 1)$*;*

$\widehat{\pi}_{p,k} : H^k(\Lambda) \to \mathcal{S}^p(\Lambda)$ *is defined as:*

$$(\widehat{\pi}_{p,k}u)^{(k)}(x) = \pi_{p-k}u^{(k)}(x), \; x \in \Lambda,$$
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- If $u^{(k)}(x) = \sum_{i=0}^{\infty} \alpha_i L_i(x)$, then

$$\widehat{\pi}_{p,k}u(x) = \sum_{i=0}^{p-k} \alpha_i \Psi_{i,k}(x) + \sum_{m=0}^{k-1} u^{(m)}(-1) \frac{(x+1)^m}{m!}$$

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- If $p \geqslant 2k-1$, $(\widehat{\pi}_{p,k}u)^{(\ell)}(1) = u^{(\ell)}(1)$, $\ell = 0, 1, \dots, k-1$.

Spline approximation on the reference domain $[0, 1]$

Definition

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- $\{0 = \zeta_1 < \zeta_2 < \cdots < \zeta_m = 1\}$, $I_i = (\zeta_i, \zeta_{i+1})$, $1 \leqslant i \leqslant m - 1$;
- $T_i \colon \Lambda \to I_i$ *linear mapping.*

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Remark

$(\pi_{p,k} u)^{(\ell)}(\zeta_i) = u^{(\ell)}(\zeta_i)$, $1 \leqslant i \leqslant m$, $0 \leqslant \ell \leqslant k - 1$.

Error estimate for $\pi_{p,k}$

Theorem

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- $\{0 = \zeta_1 < \cdots < \zeta_m = 1\}$, $I_i = (\zeta_i, \zeta_{i+1})$, $h_i = \zeta_{i+1} - \zeta_i$;
- $u^{(k)} \in H^s(I_i)$ *for some* $0 \leqslant s \leqslant \kappa = p - k + 1$.

Then, for $\ell = 0, \ldots, k$,

$$\|u^{(\ell)} - (\pi_{p,k} u)^{(\ell)}\|^2_{L^2(I_i)} \leqslant \left(\frac{h_i}{2}\right)^{2(s+k-\ell)} \frac{(\kappa - s)!}{(\kappa + s)!} \frac{(\kappa - (k - \ell))!}{(\kappa + (k - \ell))!} |u^{(k)}|^2_{H^s(I_i)}.$$

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Consequently, for $u \in H^\sigma(I_i)$, $k \leqslant \sigma \leqslant p+1$, *and* $\ell = 0,\dots,k$, *there exists a constant* C *independent of* u, ℓ, σ, h_i, p *and* k, *s.t.*

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Error estimate for $\widehat{\pi}_{p,k}$
in certain weighted norms

Error estimate for $\pi_{p,k}$: Some comments on the proof

- $k = 1$: C. Schwab, *p- and hp- Finite Element Methods. Theory and applications in Solid and Fluid Mechanics*, Oxford University Press, Oxford, 1998.
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Error estimate for $\widehat{\pi}_{p,k}$
$$\underbrace{\text{in (unweighted) Sobolev seminorms} \quad + \quad \text{scaling}}$$

$\Downarrow$

local error estimate for $\pi_{p,k}$

$\pi_{p,1}$ versus $\pi_{\tilde{p},\frac{\tilde{p}+1}{2}}$

$$|u - \pi_{p,k}u|_{H^\ell(0,1)} \leqslant Ch^{\sigma-\ell}(p-k+1)^{-(\sigma-\ell)}|u|_{H^\sigma(0,1)}.$$

$\pi_{p,1}$ versus $\pi_{\tilde{p},\frac{\tilde{p}+1}{2}}$

$$|u - \pi_{p,k} u|_{H^\ell(0,1)} \leqslant C h^{\sigma-\ell} (p - k + 1)^{-(\sigma-\ell)} |u|_{H^\sigma(0,1)}.$$

$$N = \dim(\mathcal{S}_k^p) = \frac{p-k+1}{h} + k \quad \text{(uniform mesh)}$$

$\pi_{p,1}$ versus $\pi_{\tilde{p},\frac{\tilde{p}+1}{2}}$

$$|u - \pi_{p,k}u|_{L^2(0,1)} \leqslant C(N-k)^{-(p+1)}|u|_{H^{p+1}(0,1)}$$

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- **Same degree, different mesh:** $N = \dim(\mathcal{S}^p_1) = \dim(\mathcal{S}^p_{\frac{p+1}{2}})$

$$|u - \pi_{p,1}u|_{L^2(0,1)} \leqslant CN^{-(p+1)}|u|_{H^{p+1}(0,1)}$$
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- **Different degree, same mesh:** $N = \dim(\mathcal{S}_1^p) \approx \dim(\mathcal{S}_p^{2p-1})$

$$|u - \pi_{p,1}u|_{L^2(0,1)} \leqslant CN^{-(p+1)}|u|_{H^{p+1}(0,1)}$$
$$|u - \pi_{2p-1,p}u|_{L^2(0,1)} \leqslant CN^{-2p}|u|_{H^{2p}(0,1)}$$

$\pi_{p,1}$ versus $\pi_{2p-1,p}$: Analytic functions

- u analytic in $[0,1] \implies u$ analytic in $\mathcal{E}_R$
  $\mathcal{E}_R$ ellipse with foci 0 and 1 and semiaxis sum $R > 1/2$
- $\rho = \rho(R) > 0$ s.t. $B(x,\rho) \subset \mathcal{E}_R, \forall x \in [0,1]$
- $|u|_{H^s(0,1)} \leqslant C\sqrt{s}\left(\frac{s}{\rho e}\right)^s \max_{z\in\mathcal{E}_R} |u(z)|$

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$$\frac{N^{-(2p)}\sqrt{2p}\left(\frac{2p}{\rho e}\right)^{2p}}{N^{-(p+1)}\sqrt{p+1}\left(\frac{p+1}{\rho e}\right)^{p+1}} \leqslant C\left(\frac{4h}{\rho e}\right)^{p-1}$$

Spline approximation on the reference domain $\widehat{\Omega} = [0,1]^2$

Definition

- $p = (p_1, p_2)$, $k = (k_1, k_2)$, $2k_d - 1 \leqslant p_d$, $d = 1, 2$;
- $Q_{ij} = I_i \times J_j = (\zeta_{i,1}, \zeta_{i+1,1}) \times (\zeta_{j,2}, \zeta_{j+1,2}) \in \mathcal{Q}_h$,
- $H^{k_1,k_2}(Q_{ij}) = H^{k_1}(I_i, H^{k_2}(J_j))$
- $\mathcal{S}^{p_1,p_2}(Q_{ij}) = \{u \colon Q_{ij} \to \mathbb{R} : \; u(\cdot, y) \in \mathcal{S}^{p_1}(I_i), \; u(x, \cdot) \in \mathcal{S}^{p_2}(J_j)\}$.

The (local) projection operator $\Pi^{ij}_{p,k} \colon H^{k_1,k_2}(Q_{ij}) \to \mathcal{S}^{p_1,p_2}(Q_{ij})$ *is:*

$$\Pi^{ij}_{p,k} = \pi^{i}_{p_1,k_1} \otimes \pi^{j}_{p_2,k_2}.$$

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The (global) projection operator $\Pi_{p,k}\colon H^{k_1,k_2}(\widehat{\Omega}) \to \mathcal{S}^{p_1,p_2}_{k_1,k_2}(\mathcal{Q}_h)$ *is:*

$$(\Pi_{p,k} v)|_{Q_{ij}} = (\Pi^{ij}_{p,k} v), \forall\, Q_{ij} \in \mathcal{Q}_h.$$

Error estimate for $\Pi_{p,k}$

Theorem

- $p_1 = p_2 = p$;
- k_1, k_2 *be nonnegative integers,* $2k_d - 1 \leqslant p$ *for* $d = 1, 2$
  $k_* = \min\{k_1, k_2\}$, $k^* = \max\{k_1, k_2\}$;
- $Q_{ij} = (\zeta_{i,1}, \zeta_{i+1,1}) \times (\zeta_{j,2}, \zeta_{j+1,2})$,
  $h_{ij} = \max\{\zeta_{i+1,1} - \zeta_{i,1}, \zeta_{j+1,2} - \zeta_{j,2}\}$;
- $v \in H^\sigma(Q_{ij})$ *with* $k_1 + k_2 \leqslant \sigma \leqslant p + 1$.

Then, for all integers $0 \leqslant \ell \leqslant k_*$*, there exists a positive constant* C*, independent of* v, σ, ℓ, h_{ij}, p, k_1 *and* k_2*, such that,*

$$|v - \Pi_{p,k} v|_{H^\ell(Q_{ij})} \leqslant C {h_{ij}}^{\sigma-\ell} (p - k^* + 1)^{-(\sigma-\ell)} |v|_{H^\sigma(Q_{ij})}.$$

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$$|v - \Pi_{p,k} v|_{H^\ell(\widehat{\Omega})} \leqslant C h^{\sigma-\ell} (p - k^* + 1)^{-(\sigma-\ell)} |v|_{H^\sigma(\widehat{\Omega})}.$$

NURBS approximation in the physical domain $\Omega \subset \mathbb{R}^2$

Definition

- $p = (p_1, p_2)$, $k = (k_1, k_2)$, $2k_d - 1 \leqslant p_d$, $d = 1, 2$;
- $K_{ij} = \mathbf{F}\left(Q_{ij}\right)$, $Q_{ij} \in \mathcal{Q}_h$,

The (local) projection operator for functions defined on K_{ij} *is:*

$\Pi_{\mathcal{V}}^{ij} : H^{\sigma}(K_{ij}) \to \mathcal{V}_{k_1,k_2}^{p_1,p_2}$, $\sigma \geqslant k_1 + k_2$

$$\Pi_{\mathcal{V}}^{ij}(v) \circ \mathbf{F} = \frac{\Pi_{p,k}^{ij}\left(w\left(v \circ \mathbf{F}\right)\right)}{w}.$$

NURBS approximation in the physical domain $\Omega \subset \mathbb{R}^2$

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The (local) projection operator for functions defined on K_{ij} is:

$\Pi^{ij}_{\mathcal{V}} : H^{\sigma}(K_{ij}) \to \mathcal{V}^{p_1,p_2}_{k_1,k_2}$, $\sigma \geqslant k_1 + k_2$

$$\Pi^{ij}_{\mathcal{V}}(v) \circ \mathbf{F} = \frac{\Pi^{ij}_{p,k}(w\,(v \circ \mathbf{F}))}{w}.$$

The (global) projection operator for functions defined on Ω is

$\Pi_{\mathcal{V}} : H^{\sigma}(\Omega) \to \mathcal{V}^{p_1,p_2}_{k_1,k_2}$, $\sigma \geqslant k_1 + k_2$,

$$\Pi_{\mathcal{V}}(v)|_{K_{ij}} = \Pi^{ij}_{\mathcal{V}}(v|_{K_{ij}}) \qquad \forall K_{ij} \in \mathcal{K}_h.$$

Error estimate for $\Pi_{\mathcal{V}}$

Theorem

- $p_1 = p_2 = p$;
- k_1, k_2 *nonnegative integers,* $2k_d - 1 \leqslant p$ *for* $d = 1, 2$ $k_* = \min\{k_1, k_2\}$, $k^* = \max\{k_1, k_2\}$;
- w *and* $\mathbf{F}$ *fixed at the coarsest level of discretization;*
- $K \in \mathcal{K}_h$, $h_K =$ *diam* K;
- $v \in H^\sigma(K)$ *with* $k_1 + k_2 \leqslant \sigma \leqslant p + 1$;

Then for $\ell = 0, \ldots, k_*$, *there exists a constant* $C = C(w, \mathbf{F})$, *independent of* v, σ, ℓ, h_K, p, k_1 *and* k_2 *such that*

$$|v - \Pi_{\mathcal{V}}(v)|_{H^\ell(K)} \leqslant C h_K{}^{\sigma-\ell}(p - k^* + 1)^{-(\sigma-\ell)} \|v\|_{H^\sigma(K)}.$$

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$$|v - \Pi_{\mathcal{V}}(v)|_{H^\ell(\Omega)} \leqslant Ch^{\sigma-\ell}(p - k^* + 1)^{-(\sigma-\ell)} \|v\|_{H^\sigma(\Omega)}.$$

Concluding remarks

- We have constructed new projection operators onto B-splines and NURBS spaces and given error estimates in Sobolev norms which are explicit in the three discretization parameters: degree p, regularity k and mesh size h.

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- We have constructed new projection operators onto B-splines and NURBS spaces and given error estimates in Sobolev norms which are explicit in the three discretization parameters: degree p, regularity k and mesh size h.
- We have showed superior approximation properties when regularity is increased.
- A restriction on the regularity must be imposed, namely $2k-1 \leqslant p$.

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- We have showed superior approximation properties when regularity is increased.
- A restriction on the regularity must be imposed, namely $2k-1 \leqslant p$.
- The case of higher regularity, up to $k = p$ remains open.

Thank you
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your attention