

Analysis of the Discrete Maximum Principle in the Mimetic Finite Difference Method for Elliptic Problems

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Outline

- Motivation and Problem Formulation
- Mimetic finite difference method
- Algebraic monotonicity conditions
- Simplicial, Quadrilateral and AMR meshes
- Numerical experiments

Problem Formulation

Diffusion equation

$$-\operatorname{div} \mathbb{K} \operatorname{grad} c = f \quad \text{in} \quad \Omega \subset \mathbb{R}^2$$

- $\mathbb{K}(x)$ - diffusion tensor, such that

$$\alpha_1(\xi, \xi) \leq (\mathbb{K}\xi, \xi) \leq \alpha_2(\xi, \xi) \quad \forall \xi, x \in \mathbb{R}^2, \alpha_1, \alpha_2 > 0$$

- $c(x)$ - unknown scalar function;
- $f(x)$ - source function;



Maximum principle

$$\mathcal{L}c = -\operatorname{div} \mathbb{K} \operatorname{grad} c = f \quad \text{in} \quad \Omega$$



First Hopf lemma (1927)

*If $f(x) \geq 0$ in Ω then there is **no local minima** in Ω ,*

Maximum principle

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*If $f(x) \geq 0$ in Ω then there is **no local minima** in Ω ,*

Special case (monotonicity):

if $f(x) \geq 0$ in Ω and $c(x) \geq 0$ on $\partial\Omega$ then $c(x) \geq 0$ in Ω .



Maximum principle

$$\mathcal{L} c = -\operatorname{div} \mathbb{K} \operatorname{grad} c = f \quad \text{in} \quad \Omega$$

$$\text{Discrete case: } \mathcal{A}^h C^h = F^h$$



First Hopf lemma (1927)

If $f(x) \geq 0$ in Ω then there is **no local minima** in Ω ,

$$\text{Discrete case: } \text{If } F^h \leq 0 \Rightarrow \max_{\bar{\Omega}} C^h \leq \max_{\partial\Omega} C^h$$

$$\text{If } F^h \geq 0 \Rightarrow \min_{\bar{\Omega}} C^h \geq \min_{\partial\Omega} C^h$$

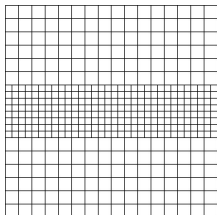
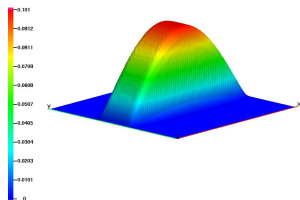
Special case (monotonicity):

if $f(x) \geq 0$ in Ω and $c(x) \geq 0$ on $\partial\Omega$ then $c(x) \geq 0$ in Ω .

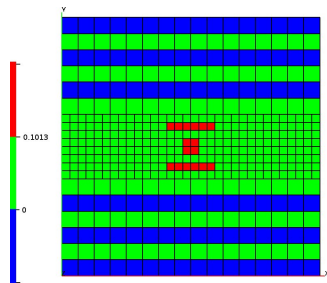
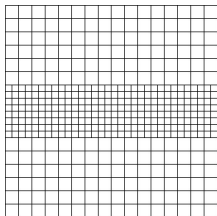
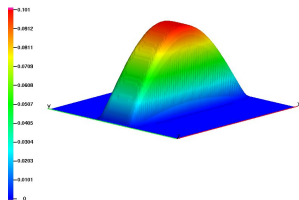
$$\text{Discrete case: } c \geq 0 \Rightarrow C^h \geq 0$$



Violation of the discrete maximum principle



Violation of the discrete maximum principle



Minimum of numerical solution

$-6.203e-03$

Mixed discretization of diffusion equation

$$\begin{aligned} \mathbf{u} &= -\mathbb{K} \operatorname{grad} c && \text{Darcy's law} \\ \operatorname{div} \mathbf{u} &= f && \text{Mass conservation law} \end{aligned}$$

Weak formulation

$$\begin{aligned} \int_{\Omega} \mathbb{K}^{-1} \mathbf{u} \mathbf{v} - \int_{\Omega} c \operatorname{div} \mathbf{v} + \int_{\partial \Omega} c \mathbf{v} \cdot \mathbf{n} &= 0 && \mathbf{v} \in H(\operatorname{div}, \Omega) \\ \int_{\Omega} \operatorname{div} \mathbf{u} q &= \int_{\Omega} f q && q \in L_2(\Omega) \end{aligned}$$

Discretization

$$\begin{aligned} \int_P \mathbb{K}^{-1} \mathbf{u}^h \mathbf{v}^h &= \mathbf{U}^T \mathbb{M}_{P,u} \mathbf{V} && \int_P c^h q^h = \mathbf{C}^T \mathbb{M}_{P,c} \mathbf{Q}^T \\ \int_{\partial P} c^h \mathbf{v}^h \cdot \mathbf{n} &= \mathbf{C}^T \mathbb{M}_{\partial P,c} \mathbf{V} \end{aligned}$$

Static Condensation

Weak form

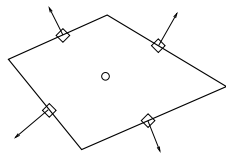
$$\int_P \mathbb{K}^{-1} \mathbf{u} \mathbf{v} - \int_P c \operatorname{div} \mathbf{v} + \int_{\partial P} c \mathbf{v} \cdot \mathbf{n} = 0$$

$$\int_P \operatorname{div} \mathbf{u} q = \int_P f q + B.C.$$

Algebraic form

$$\mathbb{M}_{P,U} \mathbf{U}_P - \mathbb{B}_P \mathbf{C}_P + \mathbb{C}_P \mathbf{C}_{\lambda,P} = 0$$

$$\mathbb{B}_P^T \mathbf{U}_P = F_P + B.C. \quad \text{and} \quad \text{flux continuity condition}$$



Static Condensation

Weak form

$$\int_P \mathbb{K}^{-1} \mathbf{u} \mathbf{v} - \int_P c \operatorname{div} \mathbf{v} + \int_{\partial P} c \mathbf{v} \cdot \mathbf{n} = 0$$

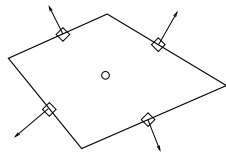
$$\int_P \operatorname{div} \mathbf{u} \mathbf{q} = \int_P f \mathbf{q} + B.C.$$

Algebraic form

$$\mathbb{M}_{P,U} \mathbf{U}_P - \mathbb{B}_P \mathbf{C}_P + \mathbb{C}_P \mathbf{C}_{\lambda,P} = 0$$

$$\mathbb{B}_P^T \mathbf{U}_P = F_P + B.C. \quad \text{and} \quad \text{flux continuity condition}$$

where $\mathbb{B}_P = \begin{pmatrix} |e_1| \\ |e_2| \\ \vdots \\ |e_n| \end{pmatrix}$ and $\mathbb{C}_P = \begin{pmatrix} |e_1| \\ |e_2| \\ \vdots \\ |e_n| \end{pmatrix}$



Static Condensation

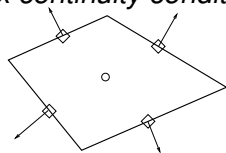
Algebraic form

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$$\Downarrow$$

$$\mathbb{S}_P \mathbf{C}_{\lambda,P} + \mathbb{C}_P \mathbf{U}_P = \mathbf{G}_P$$



where $\mathbb{W} = \mathbb{M}^{-1}$

$$\mathbb{S}_P = \mathbb{C} \left(\mathbb{W} - (\mathbb{B}^T \mathbb{W} \mathbb{B})^{-1} \mathbb{W} \mathbb{B} \mathbb{B}^T \mathbb{W} \right) \mathbb{C}^T$$

$$\mathbf{G}_P = (\mathbb{B}^T \mathbb{W} \mathbb{B})^{-1} \mathbb{C} \mathbb{W} \mathbb{B} \mathbf{F}$$

Static Condensation

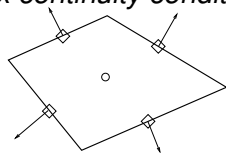
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↓

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where $\mathbb{W} = \mathbb{M}^{-1}$

$$\mathbb{S}_P = \mathbb{C} \left(\mathbb{W} - (\mathbb{B}^T \mathbb{W} \mathbb{B})^{-1} \mathbb{W} \mathbb{B} \mathbb{B}^T \mathbb{W} \right) \mathbb{C}^T \quad \mathbf{G}_P = (\mathbb{B}^T \mathbb{W} \mathbb{B})^{-1} \mathbb{C} \mathbb{W} \mathbb{B} \mathbf{F}$$

$$\mathbb{S} = \sum_{P \in \Omega_h} \mathcal{N}_P \mathbb{S}_P \mathcal{N}_P^T$$

$$\mathbf{G} = \sum_{P \in \Omega_h} \mathcal{N}_P \mathbf{G}_P \mathcal{N}_P^T$$

Static Condensation

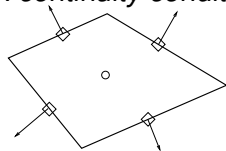
Algebraic form

$$\mathbb{M}_{P,u} \mathbf{U}_P - \mathbb{B}_P \mathbf{C}_P + \mathbb{C}_P \mathbf{C}_{\lambda,P} = 0$$

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$$\mathbb{S}_P \mathbf{C}_{\lambda,P} + \mathbb{C}_P \mathbf{U}_P = \mathbf{G}_P$$



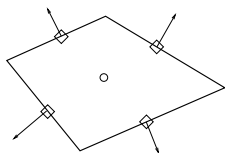
where $\mathbb{W} = \mathbb{M}^{-1}$

$$\mathbb{S}_P = \mathbb{C} \left(\mathbb{W} - (\mathbb{B}^T \mathbb{W} \mathbb{B})^{-1} \mathbb{W} \mathbb{B} \mathbb{B}^T \mathbb{W} \right) \mathbb{C}^T$$

$$\mathbf{G}_P = (\mathbb{B}^T \mathbb{W} \mathbb{B})^{-1} \mathbb{C} \mathbb{W} \mathbb{B} \mathbf{F}$$

$$\mathbb{S} \mathbf{C}_{\lambda} = \tilde{\mathbf{G}}$$

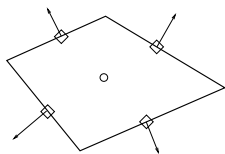
Mass Matrix and its inverse



$$\mathbb{R}_{ij} = \int_{e_j} x_i \, ds \quad \mathbb{N}_{ij} = (\mathbb{K} \text{grad } x_i) \cdot \mathbf{n}_{e_j}$$

$$\mathbb{G}^T \mathbb{N} = 0 \quad \mathbb{D}^T \mathbb{R} = 0$$

Mass Matrix and its inverse

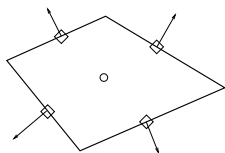


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$$\mathbb{G}^T \mathbb{N} = 0 \quad \mathbb{D}^T \mathbb{R} = 0$$

$$\mathbb{M}_{P,u} = \frac{1}{|P|} \mathbb{R} \mathbb{K}^{-1} \mathbb{R}^T + \mathbb{G} \tilde{\mathbb{U}} \mathbb{G}^T$$

Mass Matrix and its inverse



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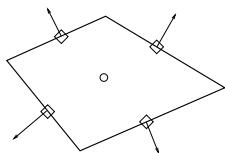
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$$\mathbb{M}_{P,u} = \frac{1}{|P|} \mathbb{R} \mathbb{K}^{-1} \mathbb{R}^T + \mathbb{G} \tilde{\mathbb{U}} \mathbb{G}^T$$

$$\mathbb{M}_{P,u}^{-1} = \mathbb{W}_{P,u} = \frac{1}{|P|} \mathbb{N} \mathbb{K}^{-1} \mathbb{N}^T + \mathbb{D} \mathbb{U} \mathbb{D}^T$$



Mass Matrix and its inverse



$$\mathbf{R}_{ij} = \int_{e_j} x_i \, ds \quad \mathbf{N}_{ij} = (\mathbf{K} \text{grad } x_i) \cdot \mathbf{n}_{e_j}$$

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Can we choose a subfamily which provides a monotone discretization?

Assumptions

A1. Matrix \mathbb{W}_P satisfies the geometric constraint:

$$w_{jj}|e_j| + \sum_{i \neq j} w_{ij}|e_i| \geq 0 \quad \forall j = 1, \dots, n;$$

A2. Matrix \mathbb{W}_P is a Z-matrix:

$$w_{ij} \leq 0 \text{ for } i \neq j \text{ with } i, j = 1, \dots, n.$$

$$\mathbb{W}_{P,u} = \mathbb{W}_0 + \mathbb{W}_1 = \frac{1}{|P|} \mathbb{N} \mathbb{K}^{-1} \mathbb{N}^T + \mathbb{D} \mathbb{U} \mathbb{D}^T$$

If matrix \mathbb{W}_P satisfies assumptions **A1** and **A2** then

\mathbb{S}_P is a singular Z-matrix



\mathbb{S} is non-singular Z-matrix



\mathbb{S} is non-singular M-matrix

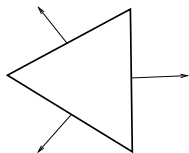


Discrete Maximum Principle

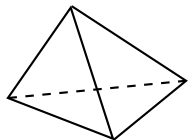
Let C_P and C_λ be the mimetic-hybrid scalar solution that approximates numerically c under assumptions **A1** and **A2**.

Theorem. If $f(x)$ is a *nonnegative*(nonpositive) function in Ω and g_D are *nonnegative*(nonpositive) functions on Γ_D then $C_P \geq 0$ (≤ 0) and $C_\lambda \geq 0$ (≤ 0) for any $P \in \Omega_h$.

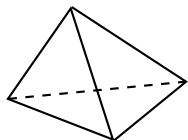
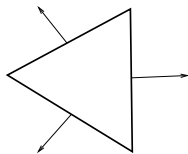
Simplex cell



$$W = \frac{1}{|P|} NK^{-1}N^T + DUD^T \quad U \in \mathbb{R}^{(n-d) \times (n-d)}$$

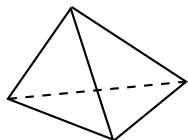
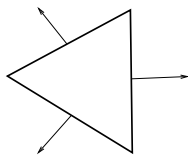


Simplex cell



$$W = \frac{1}{|P|} N K^{-1} N^T + D u D^T \quad u \in \mathbb{R}$$

Simplex cell

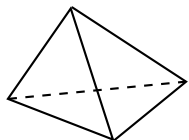
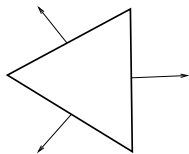


$$W = \frac{1}{|P|} N K^{-1} N^T + D u D^T \quad u \in \mathbb{R}$$

$$S = C \left(W - (B^T W B)^{-1} W B B^T W \right) C^T$$

$$= \frac{1}{|P|} C N K^{-1} N C \quad \text{since} \quad N^T B = 0$$

Simplex cell



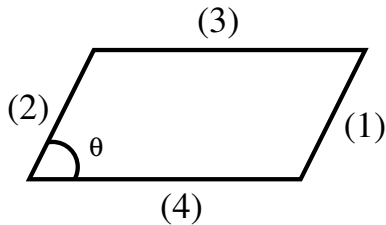
$$W = \frac{1}{|P|} N K^{-1} N^T + D u D^T \quad u \in \mathbb{R}$$

$$\begin{aligned} S &= C \left(W - (B^T W B)^{-1} W B B^T W \right) C^T \\ &= \frac{1}{|P|} C N K^{-1} N C \quad \text{since} \quad N^T B = 0 \end{aligned}$$

A1. reduces to $u \geq 0$.

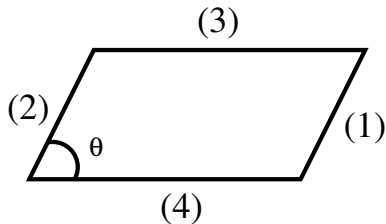
A2. reduces to $\mathbf{n}_{e_i}^T \mathbb{K} \mathbf{n}_{e_j} \leq 0$.

Quadrilateral cells



$$W = \frac{1}{|P|} \mathbf{N} \mathbf{K}^{-1} \mathbf{N}^T + \mathbf{D} \mathbf{U} \mathbf{D}^T \quad \mathbf{U} \in \mathbb{R}^{2 \times 2}$$

Quadrilateral cells

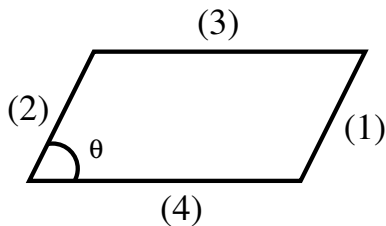


$$\mathbb{W} = \frac{1}{|P|} \mathbb{N} \mathbb{K}^{-1} \mathbb{N}^T + \mathbb{D} \mathbb{U} \mathbb{D}^T \quad \mathbb{U} \in \mathbb{R}^{2 \times 2}$$

$$\mathbb{D} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \mathbb{R}^T \mathbb{D} = 0$$

$$\mathbb{K}_{ij}^\theta = \mathbf{n}_{e_i}^T \mathbb{K} \mathbf{n}_{e_j}$$

Quadrilateral cells



$$\mathbb{W} = \frac{1}{|\mathcal{P}|} \mathbb{N} \mathbb{K}^{-1} \mathbb{N}^T + \mathbb{D} \mathbb{U} \mathbb{D}^T \quad \mathbb{U} \in \mathbb{R}^{2 \times 2}$$

$$\mathbb{D} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \mathbb{R}^T \mathbb{D} = 0$$

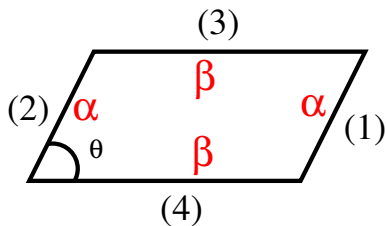
$$\mathbb{K}_{ij}^\theta = \mathbf{n}_{e_i}^T \mathbb{K} \mathbf{n}_{e_j}$$

$$\mathbb{W} = \begin{pmatrix} \mathbb{W}^{11} & \mathbb{W}^{12} \\ \mathbb{W}^{21} & \mathbb{W}^{22} \end{pmatrix}$$

$$\mathbb{W}^{ij} = \frac{\mathbb{K}_{ij}^\theta}{|\mathcal{P}|} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + u_{ij} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$



Quadrilateral cells



$$\mathbb{W} = \frac{1}{|\mathcal{P}|} \mathbb{N} \mathbb{K}^{-1} \mathbb{N}^T + \mathbb{D} \mathbb{U} \mathbb{D}^T \quad \mathbb{U} \in \mathbb{R}^{2 \times 2}$$

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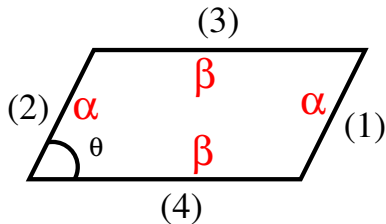
$$\mathbb{K}_{ij}^\theta = \mathbf{n}_{e_i}^T \mathbb{K} \mathbf{n}_{e_j}$$

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Quadrilateral cells



$$W = \frac{1}{|P|} NK^{-1}N^T + DUD^T \quad U \in \mathbb{R}^{2 \times 2}$$

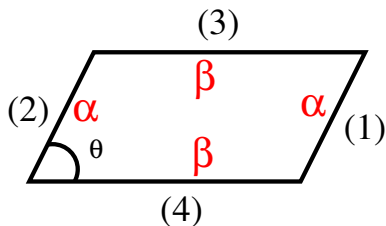
A1 and **A2** reduce to

$$0 \leq |P|U \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \leq \tilde{K}^\theta \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\tilde{K}^\theta = \begin{pmatrix} K_{11}^\theta & -|K_{12}^\theta| \\ -|K_{21}^\theta| & K_{22}^\theta \end{pmatrix}$$



Quadrilateral cells



$$W = \frac{1}{|P|} NK^{-1}N^T + DUD^T \quad U \in \mathbb{R}^{2 \times 2}$$

A1 and **A2** reduce to

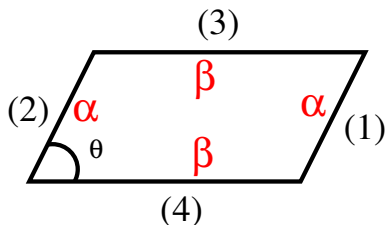
$$0 \leq |P|U \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \leq \tilde{K}^\theta \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\tilde{K}^\theta = \begin{pmatrix} K_{11}^\theta & -|K_{12}^\theta| \\ -|K_{21}^\theta| & K_{22}^\theta \end{pmatrix}$$

Necessary monotonicity condition:

$$\tilde{K}^\theta \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \geq 0$$

Quadrilateral cells



I. Aavatsmark et al (2008) for MPFA method

$$\mathbf{a}_1 = \alpha \mathbf{n}_{e_1} \quad \mathbf{a}_2 = \beta \mathbf{n}_{e_3}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \frac{1}{|P|} (\mathbf{a}_1 \mathbf{a}_2)^T \mathbb{K} (\mathbf{a}_1 \mathbf{a}_2)$$

$$|c| \leq \min(a, b)$$

$$\mathbf{W} = \frac{1}{|P|} \mathbf{N} \mathbb{K}^{-1} \mathbf{N}^T + \mathbf{D} \mathbf{U} \mathbf{D}^T \quad \mathbf{U} \in \mathbb{R}^{2 \times 2}$$

A1 and **A2** reduce to

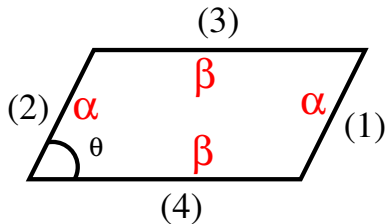
$$0 \leq |P| \mathbf{U} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \leq \tilde{\mathbb{K}}^\theta \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\tilde{\mathbb{K}}^\theta = \begin{pmatrix} \mathbb{K}_{11}^\theta & -|\mathbb{K}_{12}^\theta| \\ -|\mathbb{K}_{21}^\theta| & \mathbb{K}_{22}^\theta \end{pmatrix}$$

Necessary monotonicity condition:

$$\tilde{\mathbb{K}}^\theta \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \geq 0$$

Quadrilateral cells

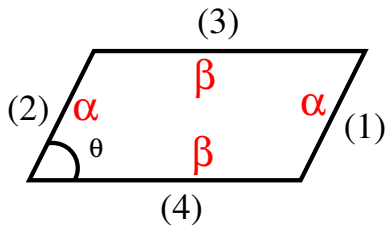


$$W = \frac{1}{|P|} N K^{-1} N^T + D U D^T \quad U \in \mathbb{R}^{2 \times 2}$$

Monotone MFD method:

$$U = \frac{1}{|P|} \tilde{K}^\theta$$

Quadrilateral cells



$$W = \frac{1}{|P|} NK^{-1}N^T + DUD^T \quad U \in \mathbb{R}^{2 \times 2}$$

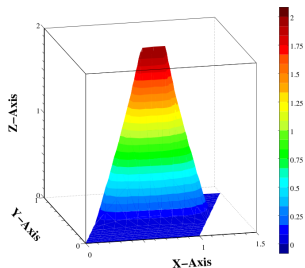
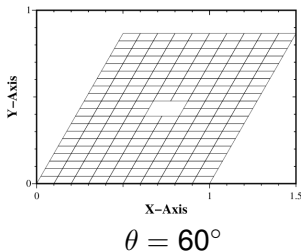
Monotone MFD method:

$$U = \frac{1}{|P|} \tilde{K}^\theta$$

Standard MFD method:

$$U = \frac{\text{tr}(\mathbb{K})}{|P|} \mathbb{I}$$

Numerical test



$$-\operatorname{div} \mathbb{K} \operatorname{grad} c = 0$$

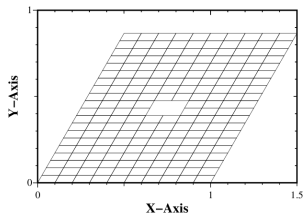
$$\mathbb{K} = \begin{pmatrix} 100 & 0 \\ 0 & 1 \end{pmatrix}$$

$c = 0$ on outer boundary

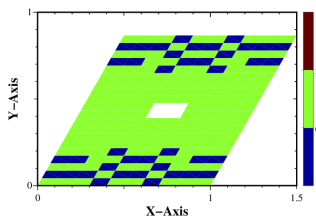
$c = 2$ on internal boundary



Numerical test



$\theta = 60^\circ$



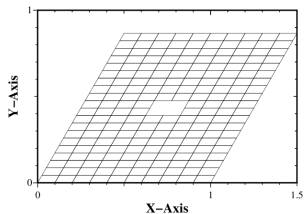
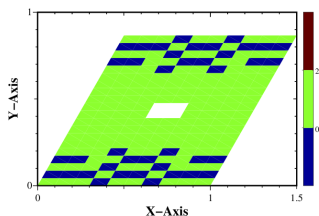
$$-\operatorname{div} \mathbb{K} \operatorname{grad} c = 0$$

$$\mathbb{K} = \begin{pmatrix} 100 & 0 \\ 0 & 1 \end{pmatrix}$$

$c = 0$ on outer boundary

$c = 2$ on internal boundary

Numerical test


 $\theta = 60^\circ$


$$-\operatorname{div} \mathbb{K} \operatorname{grad} c = 0$$

$$\mathbb{K} = \begin{pmatrix} 100 & 0 \\ 0 & 1 \end{pmatrix}$$

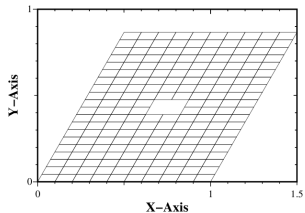
 $c = 0$ on outer boundary

 $c = 2$ on internal boundary

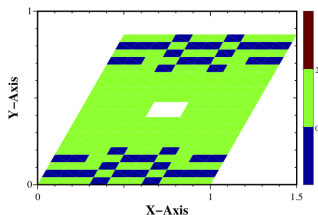
θ	Original MFD		Monotone MFD	
	$\min(C_P)$	$\max(C_P)$	$\min(C_P)$	$\max(C_P)$
90	$-5.986 \cdot 10^{-3}$	1.711	$6.989 \cdot 10^{-7}$	1.715
80	$-5.571 \cdot 10^{-3}$	1.712	$7.718 \cdot 10^{-7}$	1.716
70	$-3.666 \cdot 10^{-3}$	1.711	$1.165 \cdot 10^{-6}$	1.716
60	$-1.134 \cdot 10^{-3}$	1.710	$2.396 \cdot 10^{-6}$	1.715



Numerical test



$\theta = 60^\circ$



$$-\operatorname{div} \mathbb{K} \operatorname{grad} c = 0$$

$$\mathbb{K} = \begin{pmatrix} 100 & 0 \\ 0 & 1 \end{pmatrix}$$

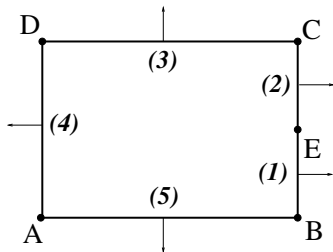
$c = 0$ on outer boundary

$c = 2$ on internal boundary

θ	Original MFD		Monotone MFD	
	#Neg	%	$\min(C_P)$	$\max(C_P)$
90	53036	29.2	$6.989 \cdot 10^{-7}$	1.715
80	52338	28.8	$7.718 \cdot 10^{-7}$	1.716
70	50386	27.8	$1.165 \cdot 10^{-6}$	1.716
60	46726	25.7	$2.396 \cdot 10^{-6}$	1.715



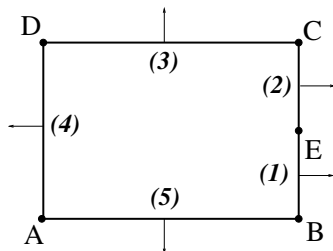
AMR cell



$$W = \frac{1}{|P|} \mathbf{N} \mathbf{K}^{-1} \mathbf{N}^T + \mathbf{D} \mathbf{U} \mathbf{D}^T \quad \mathbf{U} \in \mathbb{R}^{3 \times 3}$$

$$\mathbf{K} = \begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} \quad r = \frac{AB}{AD}$$

AMR cell



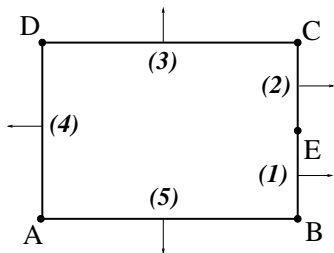
$$W = \frac{1}{|P|} \mathbf{N} \mathbf{K}^{-1} \mathbf{N}^T + \mathbf{D} \mathbf{U} \mathbf{D}^T \quad \mathbf{U} \in \mathbb{R}^{3 \times 3}$$

$$\mathbf{K} = \begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} \quad r = \frac{AB}{AD}$$

$$\mathbf{U} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{23} \\ u_{13} & u_{23} & u_{33} \end{pmatrix}$$



AMR cell



$$W = \frac{1}{|P|} NK^{-1}N^T + DUD^T \quad U \in \mathbb{R}^{3 \times 3}$$

$$K = \begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} \quad r = \frac{AB}{AD}$$

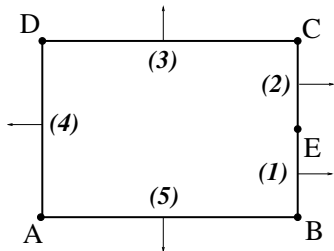
$$W_{11} = W_{22} \quad W_{33} = W_{55}$$

$$W_{15} = W_{23} \quad W_{14} = W_{24}$$

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{12} \\ u_{13} & u_{12} & u_{11} \end{pmatrix}$$



AMR cell



$$W_{11} = W_{22} \quad W_{33} = W_{55}$$

$$W_{15} = W_{23} \quad W_{14} = W_{24}$$

$$W = \frac{1}{|P|} NK^{-1}N^T + DUD^T \quad U \in \mathbb{R}^{3 \times 3}$$

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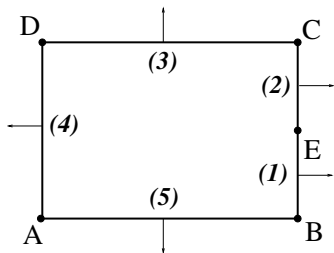
$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{12} \\ u_{13} & u_{12} & u_{11} \end{pmatrix}$$

Necessary monotonicity condition:

$$r^2 < 4 \frac{k_y}{k_x}$$



AMR cell



$$W_{11} = W_{22} \quad W_{33} = W_{55}$$

$$W_{15} = W_{23} \quad W_{14} = W_{24}$$

$$W = \frac{1}{|P|} \mathbf{N} \mathbf{K}^{-1} \mathbf{N}^T + \mathbf{D} \mathbf{U} \mathbf{D}^T \quad \mathbf{U} \in \mathbb{R}^{3 \times 3}$$

$$\mathbf{K} = \begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} \quad r = \frac{AB}{AD}$$

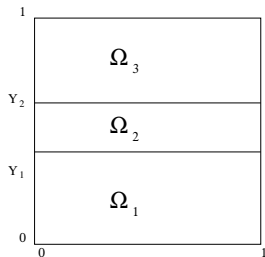
$$\mathbf{U} = \frac{1}{|P|} \begin{pmatrix} \frac{k_y}{r^2} + \frac{k_x}{4} & -\frac{k_x}{2} & \frac{k_y}{r^2} \\ -\frac{k_x}{2} & k_x & -\frac{k_x}{2} \\ \frac{k_y}{r^2} & -\frac{k_x}{2} & \frac{k_y}{r^2} + \frac{k_x}{4} \end{pmatrix}$$

Necessary monotonicity condition:

$$r^2 < 4 \frac{k_y}{k_x}$$



Numerical Test



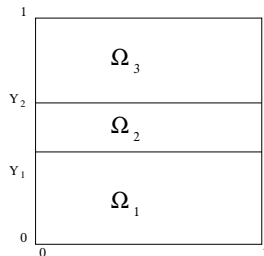
$$-\operatorname{div} \mathbb{K} \operatorname{grad} c = f$$

$$\mathbb{K} = \begin{pmatrix} 1000 & 0 \\ 0 & 1 \end{pmatrix},$$

$$f(x, y) = \begin{cases} 0, & (x, y) \in \Omega_1 \cup \Omega_3, \\ 1000 \sin(\pi x) & (x, y) \in \Omega_2. \end{cases}$$



Numerical Test



$$-\operatorname{div} \mathbb{K} \operatorname{grad} c = f$$

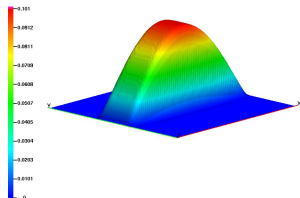
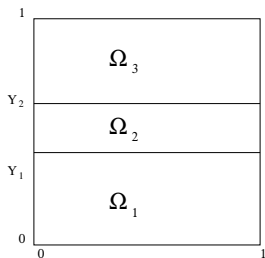
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$$c(x, y) = \phi(y) \sin(\pi x)$$



Numerical Test



$$-\operatorname{div} \mathbb{K} \operatorname{grad} c = f$$

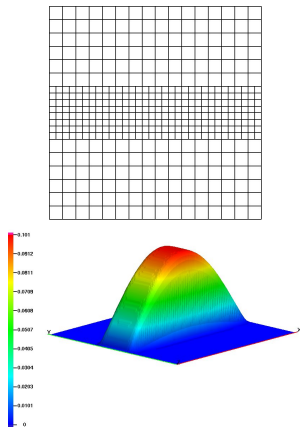
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Numerical Test



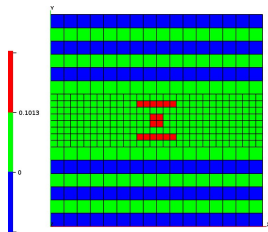
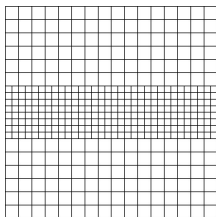
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Numerical Test



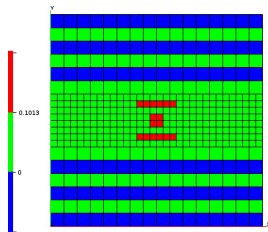
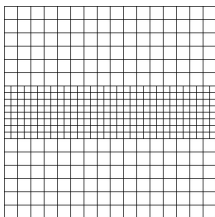
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Numerical Test



$$-\operatorname{div} \mathbb{K} \operatorname{grad} c = f$$

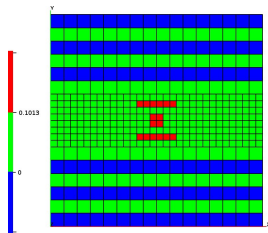
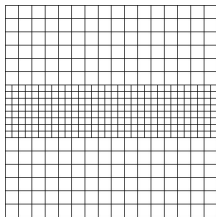
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h	Original MFD		Monotone MFD	
	$\min(C_P)$	$\max(C_P)$	$\min(C_P)$	$\max(C_P)$
1/16	-6.203e-03	1.055e-01	2.879e-12	1.012e-01
1/32	-4.219e-03	1.013e-01	8.927e-16	1.013e-01
1/64	-1.418e-13	1.013e-01	2.172e-18	1.013e-01
1/128	-7.573e-14	1.013e-01	7.343e-20	1.013e-01
1/256	9.855e-21	1.013e-01	9.928e-21	1.013e-01



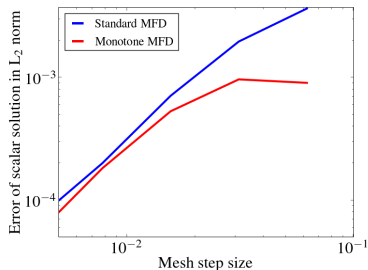
Numerical Test



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$$\mathbb{K} = \begin{pmatrix} 1000 & 0 \\ 0 & 1 \end{pmatrix},$$

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Conclusions

- We define a subfamily of MFD discretization which provides a monotone discretization under the necessary conditions on
 - parallelogram meshes (2D)
 - rectangular AMR meshes (2D)
 - tilted parallelepiped meshes (3D)
- Future work
 - Extension to more general meshes
 - Numerical optimization
 - Relax the assumptions

