Cochain projections in finite element exterior calculus

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Bounded Hilbert complexes

$$0 \rightarrow V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} V^n \rightarrow 0$$
Bounded Hilbert complexes

\[ 0 \to V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} V^n \to 0 \]

Examples:
The de Rham complex \( (\Omega \subset \mathbb{R}^3) \):

\[ 0 \to H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0 \]

Spaces with vanishing trace:

\[ 0 \to \dot{H}^1(\Omega) \xrightarrow{\text{grad}} \dot{H}(\text{curl}, \Omega) \xrightarrow{\text{curl}} \dot{H}(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0 \]
Closed Hilbert complexes

\[ 0 \to V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} V^n \to 0 \]

Assumptions: \( V^k \subset W^k \), where \((W, d)\) is a \textit{closed} Hilbert complex, with associated domain complex \((V, d)\).
Closed Hilbert complexes

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Hodge decomposition:

\[ V^k = \mathcal{Z}^k \oplus \mathcal{Z}^k \perp = d(V^{k-1}) \oplus \mathcal{H}^k \oplus \mathcal{Z}^k \perp \]

The Poincaré inequality

\[ \| \nu \| \leq c_P \| d^k \nu \|, \quad \nu \in \mathcal{Z}^k \perp, \]

follows from Banach’s bounded inverse theorem. Here, \( \| \cdot \| = \| \cdot \|_W \).
The abstract Hodge Laplacian (corresponds to a mixed formulation of \((dd^* + d^*d)u = f\)):
Abstract Hodge Laplacian

The abstract Hodge Laplacian (corresponds to a mixed formulation of \((dd^* + d^*d)u = f\)): Find \((\sigma, u, p) \in V^{k-1} \times V^k \times \mathcal{H}^k\) satisfying

\[
\langle \sigma, \tau \rangle - \langle d\tau, u \rangle = 0, \quad \tau \in V^{k-1},
\]

\[
\langle d\sigma, \nu \rangle + \langle du, d\nu \rangle + \langle \nu, p \rangle = \langle f, \nu \rangle, \quad \nu \in V^k,
\]

\[
\langle u, q \rangle = 0, \quad q \in \mathcal{H}^k.
\]

Here \(\langle \cdot, \cdot \rangle\) denotes the \(W\) inner products. This problem is \textit{well-posed}, thanks to the Poincaré inequality.
Approximation of Hilbert complexes

Consider a commuting diagram of the form

\[ \begin{array}{ccccccc}
0 & \rightarrow & V^0 & \xrightarrow{d} & V^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & V^n & \rightarrow & 0 \\
\downarrow \pi_h & & \downarrow \pi_h & & \downarrow \pi_h & & \downarrow \pi_h & & \\
0 & \rightarrow & V_h^0 & \xrightarrow{d} & V_h^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & V_h^n & \rightarrow & 0
\end{array} \]

where the “discrete space” \( V_h^k \subset V^k \), and the operators \( \pi_h^k \) are projections.
Approximation of Hilbert complexes

Consider a commuting diagram of the form

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0 \rightarrow V^0 \overset{d}{\rightarrow} V^1 \overset{d}{\rightarrow} \cdots \overset{d}{\rightarrow} V^n \rightarrow 0
\]

\[
\downarrow \pi_h \quad \downarrow \pi_h \quad \downarrow \pi_h
\]

\[
0 \rightarrow V^0_h \overset{d}{\rightarrow} V^1_h \overset{d}{\rightarrow} \cdots \overset{d}{\rightarrow} V^n_h \rightarrow 0
\]

where the “discrete space” \(V^k_h \subset V^k\), and the operators \(\pi^k_h\) are projections.

Assumption: The operators \(\pi_h\) are uniformly bounded (wrt. \(h\)) in \(\mathcal{L}(V)\).

As a result the discrete Poincaré inequality holds, and the discrete Hodge Laplace problem is stable.
Discrete Poincaré inequalities

The discrete Poincaré inequality:

\[ \| \nu \| \leq c \| d\nu \| \quad \nu \in \mathcal{Z}_h^+ \subset V_h. \] (1)

(1) \iff \exists \text{ cochain projections } \{ \pi_h \} \text{ uniformly bounded in } \mathcal{L}(V)

Here \( P_h \) is the orthogonal projection onto \( \mathcal{Z}_h^+ \). The importance of (2) is that it is equivalent to operator convergence (and therefore eigenvalue convergence).
Discrete Poincaré inequalities

The discrete Poincaré inequality:

\[ \| \nu \| \leq c \| d\nu \| \quad \nu \in \mathcal{Z}_h^\perp \subset V_h. \]  

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(1) \Leftrightarrow \exists \text{ cochain projections } \{ \pi_h \} \text{ uniformly bounded in } \mathcal{L}(V)

**Strengthened** discrete Poincaré inequality:

\[ \| \nu \| \leq c \| \hat{\nu} \| = c \| d\nu \|_{(V^*)'}, \]

(2)

for all \((\nu, \hat{\nu}) \in \mathcal{Z}_h^\perp \times \mathcal{Z}^\perp\) with \(d\hat{\nu} = d\nu\).
Discrete Poincaré inequalities

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for all \((\nu, \hat{\nu}) \in Z_h^\perp \times Z^\perp \text{ with } d\hat{\nu} = d\nu.

(2) \iff \exists \text{ coch. proj.} \{ \pi_h \} \text{ with} \ P_h \pi_h \text{ uniformly bounded in} \ L(W)

Here \(P_h\) is the orthogonal projection onto \(Z_h^\perp W\). The importance of (2) is that it is equivalent to operator convergence (and therefore eigenvalue convergence).
Finite element exterior calculus

\[ V^k = H \Lambda^k(\Omega) \text{ and } W^k = L^2 \Lambda^k(\Omega), \text{ where } \Omega \subset \mathbb{R}^n, \text{ and the operator } d \text{ is the exterior derivative.} \]

The discrete spaces \( V^k_h \) are either of the form \( V^k_h = P_r \Lambda^k(T_h) \) or \( V^k_h = P_r^{-} \Lambda^k(T_h) \).
Finite element exterior calculus

\[ V^k = H \Lambda^k(\Omega) \] and \[ W^k = L^2 \Lambda^k(\Omega) , \] where \( \Omega \subset \mathbb{R}^n \), and the operator \( d \) is the exterior derivative.

The discrete spaces \( V^k_h \) are either of the form \( V^k_h = P_r \Lambda^k(\mathcal{T}_h) \) or \( V^k_h = P_r^{-} \Lambda^k(\mathcal{T}_h) \).

They form a complex if they are composed as follows:

\[
\begin{align*}
\left\{ P_{r+1} \Lambda^{k-1}(\mathcal{T}) \right\} & \quad \quad \text{or} \quad \quad \left\{ P_{r+1} \Lambda^k(\mathcal{T}) \right\} \\
\left\{ P_{r+1}^{-} \Lambda^{k-1}(\mathcal{T}) \right\} & \quad \quad \rightarrow \quad \quad \left\{ P_{r} \Lambda^k(\mathcal{T}) \right\}
\end{align*}
\]
Degrees of freedom

An element \( u \in P_r \Lambda^k(\mathcal{T}_h) \) is uniquely determined by

\[
\int_f \text{tr}_f u \wedge \nu, \quad \nu \in P_{r+k-\dim f}^{-} \Lambda^{\dim f-k}(f), \quad f \in \Delta(\mathcal{T}_h).
\]

An element \( u \in P_{r}^{-} \Lambda^k(\mathcal{T}_h) \) is uniquely determined by

\[
\int_f \text{tr}_f u \wedge \nu, \quad \nu \in P_{r+k-\dim f-1} \Lambda^{\dim f-k}(f), \quad f \in \Delta(\mathcal{T}_h).
\]
Degrees of freedom

An element \( u \in P_r \Lambda^k(T_h) \) is uniquely determined by

\[
\int_f \text{tr}_f u \wedge \nu, \quad \nu \in P_{r+k-\dim f}^\Lambda \dim f^{-k}(f), \ f \in \Delta(T_h).
\]

An element \( u \in P_r^- \Lambda^k(T_h) \) is uniquely determined by

\[
\int_f \text{tr}_f u \wedge \nu, \quad \nu \in P_{r+k-\dim f-1} \dim f^{-k}(f), \ f \in \Delta(T_h).
\]

To simplify the notation we let \( P \Lambda^k(T_h) \) be a family of spaces either of the form \( P_r \Lambda^k(T_h) \) or \( P_r^- \Lambda^k(T_h) \).

Furthermore, the test space associated the face \( f \in \Delta(T_h) \) is simply denoted \( P'(f,k) \). Hence, an element \( u \in P \Lambda^k(T_h) \) is uniquely determined by

\[
\int_f \text{tr}_f u \wedge \nu, \quad \nu \in P'(f,k), \ f \in \Delta(T_h).
\]
The canonicalprojection

The canonical projections, $I_h^k$ onto $P\Lambda^k(T_h)$ are determined by these degrees of freedom. More precisely,

$$\int_f \text{tr}_f (u - I_h^k u) \wedge \nu = 0, \quad \nu \in P'(f, k), \quad f \in \Delta(T_h).$$

These projections are well defined, and they commute with $d$ in the situations where the spaces $P\Lambda^k(T_h)$ make up a complex.
The canonical projection

The canonical projections, $I^k_h$ onto $P\Lambda^k(T_h)$ are determined by these degrees of freedom. More precisely,

$$\int_f \text{tr}_f (u - I^k_h u) \wedge \nu = 0, \quad \nu \in P'(f, k), \ f \in \Delta(T_h).$$

These projections are well defined, and they commute with $d$ in the situations where the spaces $P\Lambda^k(T_h)$ make up a complex.

In fact, the latter property follows from two key properties on the test spaces $P'(f, k)$:

i) $\nu \in P'(f, k + 1) \implies d\nu \in P'(f, k)$

ii) $\nu \in P'(f, k + 1)$ and $g \in \Delta_{\text{dim} f - 1}(f) \implies \text{tr}_g \nu \in P'(g, k)$
The canonical projection

The projections \( \{ I_h^k \} \) commutes as a result of Stokes theorem. For \( \nu \in P'(f, k + 1) \)

\[
\int_f I_h^{k+1} du \wedge \nu = \int_f du \wedge \nu
\]

\[
= (-1)^{k+1} \int_f u \wedge d\nu + \int_{\partial f} u \wedge \nu
\]

\[
= (-1)^{k+1} \int_f I_h^k u \wedge d\nu + \int_{\partial f} I_h^k u \wedge \nu
\]

\[
= \int_f dI_h^k u \wedge \nu
\]
The canonical projection

The projections \( \{ I_h^k \} \) commutes as a result of Stokes theorem. For \( \nu \in \mathcal{P}'(f, k + 1) \)

\[
\int_f I_{h}^{k+1} du \wedge \nu = \int_f du \wedge \nu
\]

\[
= (-1)^{k+1} \int_f u \wedge d\nu + \int_{\partial f} u \wedge \nu
\]

\[
= (-1)^{k+1} \int_f I_h^k u \wedge d\nu + \int_{\partial f} I_h^k u \wedge \nu
\]

\[
= \int_f dI_h^k u \wedge \nu
\]

So the canonical projections commute with the exterior derivative, but they are \textit{not bounded} on \( H\Lambda^k \), since they require traces on all subcomplexes of \( T_h \).
Alternative projections

- projection based interpolation (Demkowicz, D-Buffa, ...)
  (*require* traces, but behaves well with respect polynomial degree)
Alternative projections

- projection based interpolation (Demkowicz, D-Buffa, ...) (require traces, but behaves well with respect polynomial degree)

- smoothed projections (Schöberl, Christiansen, A-F-W,..) (do not require traces, but seem to require bounded polynomial degree)
The Clément operator

For each $f \in \Delta(T_h)$ let

$$
\Omega_f = \bigcup \{ T \mid T \in T_h, f \in \Delta(T) \}.
$$
The Clément operator

For each $f \in \Delta(T_h)$ let

$$\Omega_f = \bigcup \{ T \mid T \in T_h, f \in \Delta(T) \}.$$

$n = 2$: 

![Diagram showing $\Omega_f$ for $n = 2$](image)
The Clément operator

The classical Clément operator (Clément 1975) is of the form

\[ \pi_h^k : L^2 \Lambda^k(\Omega) \xrightarrow{\Phi_h^k} \bigotimes_{f \in \Delta(T_h)} L^2 \Lambda^k(\Omega_f) \xrightarrow{P_h^k} \bigotimes_{f \in \Delta(T_h)} \mathcal{P} \Lambda^k(\Omega_f) \xrightarrow{I_h^k} \mathcal{P} \Lambda^k(T_h) \]

Here the \( f \)-component of \( \Phi_h^k u \) is just \( u \big|_{\Omega_f} \), and \( P_h^k \) consist of local \( L^2 \) projections onto polynomial forms.
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\]

Here the \(f\)-component of \(\Phi_h^k u\) is just \(u|_{\Omega_f}\), and \(P_h^k\) consist of local \(L^2\) projections onto polynomial forms. Finally, for \(u = (u_f)_{f \in \Delta(T_h)} \in \bigotimes_{f \in \Delta(T_h)} \mathcal{P}\Lambda^k(\Omega_f)\) we let

\[
\int_f I_h^k u \wedge \nu = \int_f u_f \wedge \nu, \quad \nu \in \mathcal{P}(f, k), \ f \in \Delta(T_h).
\]
The Clément operator

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$$\pi_h^k : L^2 \Lambda^k(\Omega) \xrightarrow{\Phi_h^k} \bigotimes_{f \in \Delta(T_h)} L^2 \Lambda^k(\Omega_f) \xrightarrow{P_h^k} \bigotimes_{f \in \Delta(T_h)} P \Lambda^k(\Omega_f) \xrightarrow{I_h^k} P \Lambda^k(T_h)$$

Here the $f$–component of $\Phi_h^k u$ is just $u|_{\Omega_f}$, and $P_h^k$ consist of local $L^2$ projections onto polynomial forms. Finally, for $u = (u_f)_{f \in \Delta(T_h)} \in \bigotimes_{f \in \Delta(T_h)} P \Lambda^k(\Omega_f)$ we let

$$\int_f I_h^k u \wedge \nu = \int_f u_f \wedge \nu, \quad \nu \in P(f,k), \; f \in \Delta(T_h).$$

This operator is $L^2$ bounded. However, it is not a projection, and it does not commute with the exterior derivative.
A commuting projection of Clément type

The purpose of the rest of this talk is to construct a commuting projection of Clément type, which will be bounded in $H\Lambda^k$.

These operators will be of the form

$$
\pi^k_h : H\Lambda^k(\Omega) \xrightarrow{\Phi^k_h} \bigotimes_{f \in \Delta(T_h)} H\Lambda^k(\Omega_f) \xrightarrow{P^k_h} \bigotimes_{f \in \Delta(T_h)} P\Lambda^k(T_f,h) \xrightarrow{I^k_h} P\Lambda^k(T_h)
$$

Here $T_{f,h}$ is the restriction of $T_h$ to $\Omega_f$. 
A commuting projection of Clément type

The purpose of the rest of this talk is to construct a commuting projection of Clément type, which will be bounded in $H^1$. These operators will be of the form

$$\pi^k_h : H^1(\Omega) \xrightarrow{\Phi^k_h} H^1(\Omega_f) \xrightarrow{P^k_h} P^1(\mathcal{T}_f,h) \xrightarrow{I^k_h} P^1(\mathcal{T}_h)$$

Here $\mathcal{T}_{f,h}$ is the restriction of $\mathcal{T}_h$ to $\Omega_f$.

The operator $P^k_h$ will be defined locally and independently on each domain $\Omega_f$. 

A commuting projection of Clément type

The purpose of the rest of this talk is to construct a commuting projection of Clément type, which will be bounded in $H\Lambda^k$.

These operators will be of the form

$$\pi_h^k: H\Lambda^k(\Omega) \xrightarrow{\Phi_h^k} \bigotimes_{f \in \Delta(T_h)} H\Lambda^k(\Omega_f) \xrightarrow{P_h^k} \bigotimes_{f \in \Delta(T_h)} P\Lambda^k(T_f,h) \xrightarrow{I_h^k} P\Lambda^k(T_h)$$

Here $T_{f,h}$ is the restriction of $T_h$ to $\Omega_f$.

The operator $P_h^k$ will be defined locally and independently on each domain $\Omega_f$. So for $u_f \in H\Lambda^k(\Omega_f)$ we define $u_{f,h} \in P\Lambda^k(T_{f,h})$ by

$$\|d(u_{f,h} - u_f)\| = \inf_{\nu \in P\Lambda^k(T_{f,h})} \|d(\nu - u_f)\|$$

subject to $\langle \nu, d\tau \rangle = \langle u_f, d\tau \rangle$, $\tau \in P\Lambda^{k-1}(T_{f,h})$. 
The mapping $u_f \mapsto u_{f,h} = (P_h^k u)_f$ can alternatively be characterized by the system:

$$\langle u_{f,h}, d\tau \rangle = \langle u_f, d\tau \rangle, \quad \tau \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}_{f,h})$$

$$\langle du_{f,h}, dv \rangle = \langle du_f, dv \rangle, \quad v \in \mathcal{P}\Lambda^k(\mathcal{T}_{f,h})$$

This can be seen as a Hodge Laplace problem, but where one component of the solution, $\sigma_{f,h} \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}_{f,h})$, is zero.
The mapping \( u_f \mapsto u_{f,h} = (P_h^k u)_f \) can alternatively be characterized by the system:

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\langle u_{f,h}, d\tau \rangle = \langle u_f, d\tau \rangle, \quad \tau \in \mathcal{P} \Lambda^{k-1}(\mathcal{T}_{f,h})
\]

\[
\langle du_{f,h}, d\nu \rangle = \langle du_f, d\nu \rangle, \quad \nu \in \mathcal{P} \Lambda^{k}(\mathcal{T}_{f,h})
\]

This can be seen as a Hodge Laplace problem, but where one component of the solution, \( \sigma_{f,h} \in \mathcal{P} \Lambda^{k-1}(\mathcal{T}_{f,h}) \), is zero. As a consequence, a proper Poincaré inequality will imply \( H\Lambda \) stability, and this can be derived by scaling (for bounded polynomial degree).
It remains to construct the operator $I^k_h : \bigotimes_{f \in \Delta(T_h)} \mathcal{P} \Lambda^k(T_f,h) \to \mathcal{P} \Lambda^k(T_h)$ so that it is $H \Lambda$ bounded and it commutes with $d$. 
It remains to construct the operator
$I^k_h : \otimes_{f \in \Delta(T_h)} \mathcal{P}\Lambda^k(T_f, h) \rightarrow \mathcal{P}\Lambda^k(T_h)$ so that it is $H\Lambda$ bounded and it commutes with $d$. Furthermore, $I^k_h$ should have the property that if

$$u_f = U|_{\Omega_f} \quad \forall f \in \Delta(T_h), \quad \text{where } U \in \mathcal{P}\Lambda^k(\Omega),$$

then $I^k_h u = U$. This will ensure that the operator

$$\pi^k_h = I^k_h \circ P^k_h \circ \Phi^k_h$$

is a projection.
It remains to construct the operator

\[ I^k_h : \bigotimes_{f \in \Delta(\mathcal{T}_h)} \mathcal{P}\Lambda^k(\mathcal{T}_f,h) \to \mathcal{P}\Lambda^k(\mathcal{T}_h) \]

so that it is \( H \Lambda \) bounded and it commutes with \( d \). Furthermore, \( I^k_h \) should have the property that if

\[ u_f = U|_{\Omega_f} \forall f \in \Delta(\mathcal{T}_h), \quad \text{where } U \in \mathcal{P}\Lambda^k(\Omega), \]

then \( I^k_h u = U \). This will ensure that the operator

\[ \pi^k_h = I^k_h \circ P^k_h \circ \Phi^k_h \]

is a projection.

The operator \( I^k_h \) will be of the form \( I^k_h = \sum f I^k_{f,h} \), where

\[ I^k_{f,h} u \in \mathcal{P}\Lambda^k(\mathcal{T}_f,h), \quad S^k_{f,h} u := \text{tr}_f I^k_{f,h} u \in \mathcal{P}\Lambda^k(f) \]

Each operator \( I_{f,h} \) will be of the form \( I_{f,h} = E^k_{f,h} \circ S^k_{f,h} \), where

\[ E^k_{f,h} : \mathcal{P}\Lambda^k(f) \to \mathcal{P}\Lambda^k(\mathcal{T}_f,h) \]

is the harmonic extension.
Harmonic extension

For \( \phi \in \tilde{\mathcal{P}}\Lambda^k(f) \) define \( E_{f,h}^k \phi = u_{f,h} \in \tilde{\mathcal{P}}\Lambda^k(T_{f,h}) \), where \( u_{f,h} \) solves

\[
\langle u_{f,h}, d\tau \rangle = 0, \quad \tau \in \tilde{\mathcal{P}}\Lambda^{k-1}(T_{f,h}),
\]
\[
\langle du_{f,h}, d\nu \rangle = 0, \quad \nu \in \tilde{\mathcal{P}}\Lambda^k(T_{f,h})
\]
\[
\text{tr}_f u_{f,h} = \phi.
\]

Here,

\[
\tilde{\mathcal{P}}\Lambda^k(T_{f,h}) = \{ \nu \in \tilde{\mathcal{P}}\Lambda^k(T_{f,h}) \mid \text{tr}_f \nu = 0 \},
\]

and all the \( L^2 \)-inner products are with respect to the domain \( \Omega_f \). This extension operator commutes with \( d \).
Harmonic extension

For $\phi \in \mathcal{P}\Lambda^k(f)$ define $E_{f,h}^k \phi = u_{f,h} \in \mathcal{P}\Lambda^k(T_{f,h})$, where $u_{f,h}$ solves

$$\langle u_{f,h}, d\tau \rangle = 0, \quad \tau \in \mathcal{P}\Lambda^{k-1}(T_{f,h}),$$
$$\langle du_{f,h}, d\nu \rangle = 0, \quad \nu \in \mathcal{P}\Lambda^k(T_{f,h})$$
$$\text{tr}_f u_{f,h} = \phi.$$ 

Here,

$$\mathcal{P}\Lambda^k(T_{f,h}) = \{ \nu \in \mathcal{P}\Lambda^k(T_{f,h}) \mid \text{tr}_f \nu = 0 \},$$

and all the $L^2$–inner products are with respect to the domain $\Omega_f$. This extension operator commutes with $d$.

Hence, it remains to define the operator

$$S_{f,h}^k = \text{tr}_f I_{f,h}^k : \bigotimes_{f \in \Delta(T_h)} \mathcal{P}\Lambda^k(T_{f,h}) \to \mathcal{P}\Lambda^k(f)$$
The operators $I_{f,h}^0$ and $S_{f,h}^0 = \text{tr}_f I_{f,h}^0$

We inductively require

$$\int_f S_{f,h}^0 u \wedge \nu = \int_f \text{tr}_f (u_f - \sum_{g \supseteq f} I_{g,h}^0 u) \wedge \nu, \quad \nu \in \mathcal{P}'(f,0).$$
The operators $I^0_{f,h}$ and $S^0_{f,h} = \text{tr}_f I^0_{f,h}$

We inductively require

$$\int_f S^0_{f,h} u \wedge \nu = \int_f \text{tr}_f (u_f - \sum_{g \subseteq f} I^0_{g,h} u) \wedge \nu, \quad \nu \in \mathcal{P}'(f,0).$$

which implies that

$$\int_f \text{tr}_f (I^0_{h} u - u_f) \wedge \nu = 0, \quad \nu \in \mathcal{P}'(f,0).$$
The operators $I_{f,h}^0$ and $S_{f,h}^0 = \text{tr} f I_{f,h}^0$

We inductively require

$$\int_f S_{f,h}^0 u \wedge \nu = \int_f \text{tr} f (u_f - \sum_{g \preceq f} I_{g,h}^0 u) \wedge \nu, \quad \nu \in \mathcal{P}'(f,0).$$

which implies that

$$\int_f \text{tr} f (I_{h}^0 u - u_f) \wedge \nu = 0, \quad \nu \in \mathcal{P}'(f,0).$$

This ensures that $\pi_{h}^0$ is a projection.
The operators $S^k_{f,h} = \text{tr}_f I^k_{f,h}$

The operators $S^k_{f,h}$ are defined inductively by

$$
\int_f S^k_{f,h} u \wedge \eta = \int_f \text{tr}_f (dI^{k-1}_h R^k_f u + R^{k+1}_f du_f - \sum_{g \subsetneq f} I^k_{g,h} u) \wedge \nu,
$$

for $\nu \in \mathcal{P}'(f,k)$. 

The operators $S_{f,h}^k = \text{tr}_f I_{f,h}^k$

The operators $S_{f,h}^k$ are defined inductively by

$$
\int_f S_{f,h}^k u \wedge \eta = \int_f \text{tr}_f (dI_{h}^{k-1} R_{f}^{k} u + R_{f}^{k+1} u - \sum_{g \subsetneq f} I_{g,h}^{k} u) \wedge \nu,
$$

for $\nu \in \mathcal{P}'(f, k)$. Here $R_{f}^{k}$ is a Poincaré operator of the form

$$
(R_{f}^{k} \nu)_x = \int_0^1 t^{k-1} \nu_{a+t(x-a) -} (x - a) \, dt
$$

where $a = a_f \in f$. 

The operators $S_{f,h}^k = \text{tr}_f I_{f,h}^k$

The operators $S_{f,h}^k$ are defined inductively by

$$\int_f S_{f,h}^k u \wedge \eta = \int_f \text{tr}_f (dI_{h}^{k-1} R_{f}^k u + R_{f}^{k+1} du_f - \sum_{g \subseteq f} I_{g,h}^k u) \wedge \nu,$$

for $\nu \in P'(f,k)$. Here $R_{f}^k$ is a Poincaré operator of the form

$$(R_{f}^k \nu)_{x} = \int_0^1 t^{k-1} \nu_{a+t(x-a)}(x-a) \, dt$$

where $a = a_f \in f$.

We have the identity

$$\nu = dR_{f}^k \nu + R_{f}^{k+1} d\nu,$$

for all $\nu \in H\Lambda^k(\Omega_g)$, $k \geq 1$ with $g \subset f$. 
\( \pi_h^k \) is a projection

From the definition above we obtain that

\[
\int_f \text{tr}_f I_h^k u \wedge \nu = \int_f \text{tr}_f (dI_h^{k-1} R^k_f u + R_f^{k+1} du_f) \wedge \nu, \quad \nu \in \mathcal{P}'(f, k).
\]
\[ \int_f \text{tr}_f I_h^k u \wedge \nu = \int_f \text{tr}_f (dI_h^{k-1} R^k u + R^{k+1} du_f) \wedge \nu, \quad \nu \in P'(f, k). \]

By an inductive argument this will imply that \( \pi^k_h \) is a projection.
$\pi^k_h$ is a cochain projection

To show that $I^k_h$ commutes with $d$ we will use the fact that

$$\int_f dI^{k-1}_h du \wedge \nu = 0, \; \nu \in \mathcal{P}'(f,k). \quad (3)$$

Here $u \in \otimes_g \mathcal{P} \Lambda^{k-2}(T_{g,h})$. 
\( \pi^k_h \) is a cochain projection

To show that \( I^k_h \) commutes with \( d \) we will use the fact that

\[
\int_f dI^k_{h}^{-1} du \wedge \nu = 0, \quad \nu \in \mathcal{P}'(f, k).
\]  

(3)

Here \( u \in \otimes_g \mathcal{P} \wedge^{k-2}(T_{g,h}) \). This will imply that

\[
\int_f I^k_h du \wedge \nu = \int_f dI^k_{h}^{-1} R^k_f du \wedge \nu
\]

\[
= \int_f dI^k_{h}^{-1} (u - dR^k_{f} u) \wedge \nu = \int_f dI^k_{h}^{-1} u, \quad \nu \in \mathcal{P}'(f, k)
\]
\[ \pi_h^k \text{ is a cochain projection} \]

To show that \( I_h^k \) commutes with \( d \) we will use the fact that

\[
\int_f dI_h^{k-1} du \wedge \nu = 0, \quad \nu \in \mathcal{P}'(f, k). \tag{3}
\]

Here \( u \in \otimes g \mathcal{P} \wedge^{k-2}(T_{g,h}) \). This will imply that

\[
\int_f I_h^k du \wedge \nu = \int_f dI_h^{k-1} R_f^k du \wedge \nu = \int_f dI_h^{k-1} (u - dR_f^{k-1} u) \wedge \nu = \int_f dI_h^{k-1} u, \quad \nu \in \mathcal{P}'(f, k)
\]

Finally, the identity (3) can be established by induction on \( k \).