Cochain projections in finite element exterior calculus

Ragnar Winther

CMA, University of Oslo
Norway

based on joint work with: Douglas N. Arnold, Minnesota, Richard S. Falk, Rutgers, and Snorre H. Christiansen, Oslo

Bounded Hilbert complexes

$$0 \to V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} V^n \to 0$$

Bounded Hilbert complexes

$$0 \to V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} V^n \to 0$$

Examples:
The de Rham complex ($\Omega \subset \mathbb{R}^3$):

$$0 \to H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

Spaces with vanishing trace:

$$0 \to \mathring{H}^1(\Omega) \xrightarrow{\text{grad}} \mathring{H}(\text{curl}, \Omega) \xrightarrow{\text{curl}} \mathring{H}(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

Closed Hilbert complexes

$$0 \to V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} V^n \to 0$$

Assumptions: $V^k \subset W^k$, where (W, d) is a *closed* Hilbert complex, with associated domain complex (V, d).

Closed Hilbert complexes

$$0 \to V^0 \xrightarrow{d^0} V^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} V^n \to 0$$

Assumptions: $V^k \subset W^k$, where (W, d) is a *closed* Hilbert complex, with associated domain complex (V, d).
Hodge decomposition:

$$V^k = \mathfrak{Z}^k \oplus \mathfrak{Z}^{k\perp} = d(V^{k-1}) \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp}$$

The Poincaré inequality

$$\|v\| \le c_P \|d^k v\|, \quad v \in \mathfrak{Z}^{k\perp},$$

follows from Banach's bounded inverse theorem. Here, $\|\cdot\| = \|\cdot\|_W$.

Abstract Hodge Laplacian

The abstract Hodge Laplacian (corresponds to a mixed formulation of $(dd^* + d^*d)u = f$):

Abstract Hodge Laplacian

The abstract Hodge Laplacian (corresponds to a mixed formulation of $(dd^* + d^*d)u = f$): Find $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ satisfying

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0, & \tau &\in V^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle v, p \rangle &= \langle f, v \rangle, & v &\in V^k, \\ \langle u, q \rangle &= 0, & q &\in \mathfrak{H}^k. \end{aligned}$$

Here $\langle \cdot, \cdot \rangle$ denotes the W inner products. This problem is *well-posed*, thanks to the Poincaré inequality.

UNIVERSITY OF OSLO

Centre of Mathematics for Applications

Approximation of Hilbert complexes

Consider a commuting diagram of the form

$$\begin{array}{ccccccccccc} 0 \to V^0 & \xrightarrow{d} & V^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & V^n \to 0 \\ \downarrow \pi_h & & \downarrow \pi_h & & & & \downarrow \pi_h \\ 0 \to V_h^0 & \xrightarrow{d} & V_h^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & V_h^n \to 0 \end{array}$$

where the “discrete space” $V_h^k \subset V^k$, and the operators π_h^k are projections.

Approximation of Hilbert complexes

Consider a commuting diagram of the form

$$\begin{array}{ccccccccc} 0 \to V^0 & \xrightarrow{d} & V^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & V^n \to 0 \\ \downarrow \pi_h & & \downarrow \pi_h & & & & \downarrow \pi_h \\ 0 \to V_h^0 & \xrightarrow{d} & V_h^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & V_h^n \to 0 \end{array}$$

where the "discrete space" $V_h^k \subset V^k$, and the operators π_h^k are projections.

Assumption: The operators π_h are *uniformly bounded* (wrt. h) in $\mathcal{L}(V)$.

As a result *the discrete Poincaré inequality* holds, and the discrete Hodge Laplace problem is *stable*.

Discrete Poincaré inequalities

The discrete Poincaré inequality:

$$\|v\| \le c\|dv\| \quad v \in \mathfrak{z}_h^{\perp} \subset V_h. \tag{1}$$

(1) $\Leftrightarrow \exists$ cochain projections $\{\pi_h\}$ uniformly bounded in $\mathcal{L}(V)$

Discrete Poincaré inequalities

The discrete Poincaré inequality:

$$\|v\| \le c\|dv\| \quad v \in \mathfrak{Z}_h^\perp \subset V_h. \tag{1}$$

(1) $\Leftrightarrow$ $\exists$ cochain projections $\{\pi_h\}$ uniformly bounded in $\mathcal{L}(V)$

Strengthened discrete Poincaré inequality:

$$\|v\| \le c\|\hat{v}\| = c\|dv\|_{(V^*)'} \tag{2}$$

for all $(v, \hat{v}) \in Z_h^\perp \times Z^\perp$ with $d\hat{v} = dv$.

Discrete Poincaré inequalities

The discrete Poincaré inequality:

$$\|v\| \le c\|dv\| \quad v \in \mathfrak{z}_h^{\perp} \subset V_h. \tag{1}$$

(1) $\Leftrightarrow \exists$ cochain projections $\{\pi_h\}$ uniformly bounded in $\mathcal{L}(V)$

Strengthened discrete Poincaré inequality:

$$\|v\| \le c\|\hat{v}\| = c\|dv\|_{(V^*)'} \tag{2}$$

for all $(v, \hat{v}) \in Z_h^{\perp} \times Z^{\perp}$ with $d\hat{v} = dv$.

(2) $\Leftrightarrow \exists$ coch. proj. $\{\pi_h\}$ with $P_h\pi_h$ uniformly bounded in $\mathcal{L}(W)$

Here P_h is the orthogonal projecion onto $\mathfrak{z}_h^{\perp W}$. The importance of (2) is that it is equivalent to operator convergence (and therefore eigenvalue convergence).

Finite element exterior calculus

$V^k = H\Lambda^k(\Omega)$ and $W^k = L^2\Lambda^k(\Omega)$, where $\Omega \subset \mathbb{R}^n$, and the operator d is the exterior derivative.
The discrete spaces V_h^k are ether of the form $V_h^k = \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ or $V_h^k = \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$.

Finite element exterior calculus

$V^k = H\Lambda^k(\Omega)$ and $W^k = L^2\Lambda^k(\Omega)$, where $\Omega \subset \mathbb{R}^n$, and the operator d is the exterior derivative.
The discrete spaces V_h^k are ether of the form $V_h^k = \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ or $V_h^k = \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$.

They form a complex if they are composed as follows:

$$\begin{Bmatrix} \mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_{r+1}^-\Lambda^{k-1}(\mathcal{T}) \end{Bmatrix} \xrightarrow{d} \begin{Bmatrix} \mathcal{P}_{r+1}^-\Lambda^{k}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_{r}\Lambda^{k}(\mathcal{T}) \end{Bmatrix}$$

Degrees of freedom

An element $u \in \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ is uniquely determined by

$$\int_f \operatorname{tr}_f u \wedge v, \quad v \in \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f - k}(f),\ f \in \Delta(\mathcal{T}_h).$$

An element $u \in \mathcal{P}^-_r\Lambda^k(\mathcal{T}_h)$ is uniquely determined by

$$\int_f \operatorname{tr}_f u \wedge v, \quad v \in \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f - k}(f),\ f \in \Delta(\mathcal{T}_h).$$

Degrees of freedom

An element $u \in \mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ is uniquely determined by

$$\int_f \operatorname{tr}_f u \wedge v, \quad v \in \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f - k}(f),\, f \in \Delta(\mathcal{T}_h).$$

An element $u \in \mathcal{P}^-_r\Lambda^k(\mathcal{T}_h)$ is uniquely determined by

$$\int_f \operatorname{tr}_f u \wedge v, \quad v \in \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f - k}(f),\, f \in \Delta(\mathcal{T}_h).$$

To simplify the notation we let $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ be a family of spaces either of the form $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ or $\mathcal{P}^-_r\Lambda^k(\mathcal{T}_h)$. Furthermore, the test space associated the face $f \in \Delta(\mathcal{T}_h)$ is simply denoted $\mathcal{P}'(f,k)$. Hence, an element $u \in \mathcal{P}\Lambda^k(\mathcal{T}_h)$ is uniquely determined by

$$\int_f \operatorname{tr}_f u \wedge v, \quad v \in \mathcal{P}'(f,k),\, f \in \Delta(\mathcal{T}_h).$$

The canonical projection

The canonical projections, I_h^k onto $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ are determined by these degrees of freedom. More precisely,

$$\int_f \operatorname{tr}_f(u - I_h^k u) \wedge v = 0, \quad v \in \mathcal{P}'(f,k),\ f \in \Delta(\mathcal{T}_h).$$

These projections are well defined, and they commute with d in the situations where the spaces $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ make up a complex.

The canonical projection

The canonical projections, I_h^k onto $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ are determined by these degrees of freedom. More precisely,

$$\int_f \mathrm{tr}_f(u - I_h^k u) \wedge v = 0, \quad v \in \mathcal{P}'(f,k),\, f \in \Delta(\mathcal{T}_h).$$

These projections are well defined, and they commute with d in the situations where the spaces $\mathcal{P}\Lambda^k(\mathcal{T}_h)$ make up a complex.

In fact, the latter property follows from two key properties on the test spaces $\mathcal{P}'(f,k)$:

i) $v \in \mathcal{P}'(f,k+1) \Longrightarrow dv \in \mathcal{P}'(f,k)$

ii) $v \in \mathcal{P}'(f,k+1)$ and $g \in \Delta_{\dim f-1}(f) \Longrightarrow \mathrm{tr}_g v \in \mathcal{P}'(g,k)$

The canonical projection

The projections $\{I_h^k\}$ commutes as a result of Stokes theorem. For $v \in \mathcal{P}'(f, k+1)$

$$
\begin{aligned}
\int_f I_h^{k+1} du \wedge v &= \int_f du \wedge v \\
&= (-1)^{k+1} \int_f u \wedge dv + \int_{\partial f} u \wedge v \\
&= (-1)^{k+1} \int_f I_h^k u \wedge dv + \int_{\partial f} I_h^k u \wedge v \\
&= \int_f d I_h^k u \wedge v
\end{aligned}
$$

The canonical projection

The projections $\{I_h^k\}$ commutes as a result of Stokes theorem. For $v \in \mathcal{P}'(f, k+1)$

$$\begin{aligned}\int_f I_h^{k+1} du \wedge v &= \int_f du \wedge v \\ &= (-1)^{k+1} \int_f u \wedge dv + \int_{\partial f} u \wedge v \\ &= (-1)^{k+1} \int_f I_h^k u \wedge dv + \int_{\partial f} I_h^k u \wedge v \\ &= \int_f dI_h^k u \wedge v\end{aligned}$$

So the canonical projections commute with the exterior derivative, but they are *not bounded* on $H\Lambda^k$, since they require traces on all subcomplexes of $\mathcal{T}_h$.

Alternative projections

- projection based interpolation (Demkowicz, D-Buffa, ...) (*require* traces, but behaves well with respect polynomial degree)

Alternative projections

- projection based interpolation (Demkowicz, D-Buffa, ...) (*require* traces, but behaves well with respect polynomial degree)
- smoothed projections (Schöberl, Christiansen, A-F-W,..) (do not require traces, but seem to *require* bounded polynomial degree)

The Clément operator

For each $f \in \Delta(\mathcal{T}_h)$ let

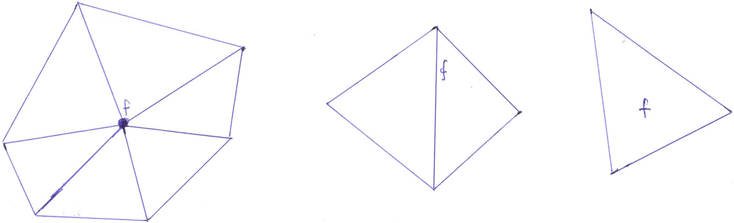
$$\Omega_f = \bigcup \{ T \mid T \in \mathcal{T}_h,\, f \in \Delta(T) \}.$$

The Clément operator

For each $f \in \Delta(\mathcal{T}_h)$ let

$$\Omega_f = \bigcup \{ T \mid T \in \mathcal{T}_h,\, f \in \Delta(T) \}.$$

$n = 2$:



The Clément operator

The classical Clément operator (Clément 1975) is of the form

$$\pi_h^k : L^2\Lambda^k(\Omega) \xrightarrow{\Phi_h^k} \bigotimes_{f\in\Delta(\mathcal{T}_h)} L^2\Lambda^k(\Omega_f) \xrightarrow{P_h^k} \bigotimes_{f\in\Delta(\mathcal{T}_h)} \mathcal{P}\Lambda^k(\Omega_f) \xrightarrow{I_h^k} \mathcal{P}\Lambda^k(\mathcal{T}_h)$$

Here the f-component of $\Phi_h^k u$ is just $u|_{\Omega_f}$, and P_h^k consist of local L^2 projections onto polynomial forms.

The Clément operator

The classical Clément operator (Clément 1975) is of the form

$$\pi_h^k : L^2\Lambda^k(\Omega) \xrightarrow{\Phi_h^k} \bigotimes_{f\in\Delta(\mathcal{T}_h)} L^2\Lambda^k(\Omega_f) \xrightarrow{P_h^k} \bigotimes_{f\in\Delta(\mathcal{T}_h)} \mathcal{P}\Lambda^k(\Omega_f) \xrightarrow{I_h^k} \mathcal{P}\Lambda^k(\mathcal{T}_h)$$

Here the f-component of $\Phi_h^k u$ is just $u|_{\Omega_f}$, and P_h^k consist of local L^2 projections onto polynomial forms. Finally, for $u = (u_f)_{f\in\Delta_(\mathcal{T}_h)} \in \bigotimes_{f\in\Delta(\mathcal{T}_h)} \mathcal{P}\Lambda^k(\Omega_f)$ we let

$$\int_f I_h^k u \wedge v = \int_f u_f \wedge v, \quad v \in \mathcal{P}(f,k),\ f \in \Delta(\mathcal{T}_h).$$

The Clément operator

The classical Clément operator (Clément 1975) is of the form

$$\pi_h^k : L^2\Lambda^k(\Omega) \xrightarrow{\Phi_h^k} \bigotimes_{f\in\Delta(\mathcal{T}_h)} L^2\Lambda^k(\Omega_f) \xrightarrow{P_h^k} \bigotimes_{f\in\Delta(\mathcal{T}_h)} \mathcal{P}\Lambda^k(\Omega_f) \xrightarrow{I_h^k} \mathcal{P}\Lambda^k(\mathcal{T}_h)$$

Here the f-component of $\Phi_h^k u$ is just $u|_{\Omega_f}$, and P_h^k consist of local L^2 projections onto polynomial forms. Finally, for $u = (u_f)_{f\in\Delta_(\mathcal{T}_h)} \in \bigotimes_{f\in\Delta(\mathcal{T}_h)} \mathcal{P}\Lambda^k(\Omega_f)$ we let

$$\int_f I_h^k u \wedge v = \int_f u_f \wedge v, \quad v \in \mathcal{P}(f,k),\, f \in \Delta(\mathcal{T}_h).$$

This operator is L^2 bounded. However, it is *not* a projection, and it does *not* commute with the exterior derivative.

A commuting projection of Clément type

The purpose of the rest of this talk is to construct a commutiong projection of Clément type, which will be bounded in $H\Lambda^k$.

These operators will be of the form

$$\pi_h^k : H\Lambda^k(\Omega) \xrightarrow{\Phi_h^k} \bigotimes_{f \in \Delta(\mathcal{T}_h)} H\Lambda^k(\Omega_f) \xrightarrow{P_h^k} \bigotimes_{f \in \Delta(\mathcal{T}_h)} \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}) \xrightarrow{I_h^k} \mathcal{P}\Lambda^k(\mathcal{T}_h)$$

Here $\mathcal{T}_{f,h}$ is the restriction of $\mathcal{T}_h$ to Ω_f.

A commuting projection of Clément type

The purpose of the rest of this talk is to construct a commutiong projection of Clément type, which will be bounded in $H\Lambda^k$.

These operators will be of the form

$$\pi_h^k : H\Lambda^k(\Omega) \xrightarrow{\Phi_h^k} \bigotimes_{f \in \Delta(\mathcal{T}_h)} H\Lambda^k(\Omega_f) \xrightarrow{P_h^k} \bigotimes_{f \in \Delta(\mathcal{T}_h)} \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}) \xrightarrow{I_h^k} \mathcal{P}\Lambda^k(\mathcal{T}_h)$$

Here $\mathcal{T}_{f,h}$ is the restriction of $\mathcal{T}_h$ to Ω_f.

The operator P_h^k will be defined locally and independently on each domain Ω_f.

A commuting projection of Clément type

The purpose of the rest of this talk is to construct a commutiong projection of Clément type, which will be bounded in $H\Lambda^k$.

These operators will be of the form

$$\pi_h^k : H\Lambda^k(\Omega) \xrightarrow{\Phi_h^k} \bigotimes_{f \in \Delta(\mathcal{T}_h)} H\Lambda^k(\Omega_f) \xrightarrow{P_h^k} \bigotimes_{f \in \Delta(\mathcal{T}_h)} \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}) \xrightarrow{I_h^k} \mathcal{P}\Lambda^k(\mathcal{T}_h)$$

Here $\mathcal{T}_{f,h}$ is the restriction of $\mathcal{T}_h$ to Ω_f.

The operator P_h^k will be defined locally and independently on each domain Ω_f. So for $u_f \in H\Lambda^k(\Omega_f)$ we define $u_{f,h} \in \mathcal{P}\Lambda^k(\mathcal{T}_{f,h})$ by

$$\|d(u_{f,h} - u_f)\| = \inf_{v \in \mathcal{P}\Lambda^k(\mathcal{T}_{f,h})} \|d(v - u_f)\|$$

$$\text{subject to } \langle v, d\tau \rangle = \langle u_f, d\tau \rangle, \quad \tau \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}_{f,h}).$$

The mapping $u_f \mapsto u_{f,h} = (P_h^k u)_f$ can alternatively be characterized by the system:

$$\langle u_{f,h}, d\tau \rangle = \langle u_f, d\tau \rangle, \quad \tau \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}_{f,h})$$
$$\langle du_{f,h}, dv \rangle = \langle du_f, dv \rangle, \quad v \in \mathcal{P}\Lambda^{k}(\mathcal{T}_{f,h})$$

This can be seen as a Hodge Laplace problem, but where one component of the solution, $\sigma_{f,h} \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}_{f,h})$, is zero.

The mapping $u_f \mapsto u_{f,h} = (P_h^k u)_f$ can alternatively be characterized by the system:

$$\langle u_{f,h}, d\tau\rangle = \langle u_f, d\tau\rangle, \quad \tau \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}_{f,h})$$
$$\langle du_{f,h}, d\nu\rangle = \langle du_f, d\nu\rangle, \quad \nu \in \mathcal{P}\Lambda^{k}(\mathcal{T}_{f,h})$$

This can be seen as a Hodge Laplace problem, but where one component of the solution, $\sigma_{f,h} \in \mathcal{P}\Lambda^{k-1}(\mathcal{T}_{f,h})$, is zero.

As a consequence, a proper Poincaré inequality will imply $H\Lambda$ stability, and this can be derived by scaling (for bounded polynomial degree).

It remains to construct the operator
$I_h^k : \bigotimes_{f \in \Delta(\mathcal{T}_h)} \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}) \to \mathcal{P}\Lambda^k(\mathcal{T}_h)$ so that it is $H\Lambda$
bounded and it commutes with d.

It remains to construct the operator $I_h^k : \bigotimes_{f \in \Delta(\mathcal{T}_h)} \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}) \to \mathcal{P}\Lambda^k(\mathcal{T}_h)$ so that it is $H\Lambda$ bounded and it commutes with d. Furthermore, I_h^k should have the property that if

$$u_f = U|_{\Omega_f} \; \forall f \in \Delta(\mathcal{T}_h), \quad \text{where } U \in \mathcal{P}\Lambda^k(\Omega),$$

then $I_h^k u = U$. This will ensure that the opererator $\pi_h^k = I_h^k \circ P_h^k \circ \Phi_h^k$ is a projection.

It remains to construct the operator $I_h^k : \bigotimes_{f \in \Delta(\mathcal{T}_h)} \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}) \to \mathcal{P}\Lambda^k(\mathcal{T}_h)$ so that it is $H\Lambda$ bounded and it commutes with d. Furthermore, I_h^k should have the property that if

$$u_f = U|_{\Omega_f} \ \forall f \in \Delta(\mathcal{T}_h), \quad \text{where } U \in \mathcal{P}\Lambda^k(\Omega),$$

then $I_h^k u = U$. This will ensure that the opererator $\pi_h^k = I_h^k \circ P_h^k \circ \Phi_h^k$ is a projection.

The operator I_h^k will be of the form $I_h^k = \sum_f I_{f,h}^k$, where

$$I_{f,h}^k u \in \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}), \qquad S_{f,h}^k u := \operatorname{tr}_f I_{f,h}^k u \in \mathring{\mathcal{P}}\Lambda^k(f)$$

Each operator $I_{f,h}$ will be of the form $I_{f,h} = E_{f,h}^k \circ S_{f,h}^k$, where $E_{f,h}^k : \mathring{\mathcal{P}}\Lambda^k(f) \to \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$ is the harmonic extension.

Harmonic extension

For $\phi \in \mathring{\mathcal{P}}\Lambda^k(f)$ define $E^k_{f,h}\phi = u_{f,h} \in \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$, where $u_{f,h}$ solves

$$\begin{aligned} \langle u_{f,h}, d\tau \rangle &= 0, \quad \tau \in \bar{\mathcal{P}}\Lambda^{k-1}(\mathcal{T}_{f,h}), \\ \langle du_{f,h}, d\nu \rangle &= 0, \quad \nu \in \bar{\mathcal{P}}\Lambda^{k}(\mathcal{T}_{f,h}) \\ \operatorname{tr}_f u_{f,h} &= \phi. \end{aligned}$$

Here,

$$\bar{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}) = \{\nu \in \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}) \mid \operatorname{tr}_f \nu = 0\},$$

and all the L^2–inner products are with respect to the domain Ω_f. This extension operator commutes with d.

Harmonic extension

For $\phi \in \mathring{\mathcal{P}}\Lambda^k(f)$ define $E^k_{f,h}\phi = u_{f,h} \in \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h})$, where $u_{f,h}$ solves

$$\begin{aligned} \langle u_{f,h}, d\tau \rangle &= 0, \quad \tau \in \bar{\mathcal{P}}\Lambda^{k-1}(\mathcal{T}_{f,h}), \\ \langle du_{f,h}, d\nu \rangle &= 0, \quad \nu \in \bar{\mathcal{P}}\Lambda^{k}(\mathcal{T}_{f,h}) \\ \operatorname{tr}_f u_{f,h} &= \phi. \end{aligned}$$

Here,

$$\bar{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}) = \{\nu \in \mathring{\mathcal{P}}\Lambda^k(\mathcal{T}_{f,h}) \mid \operatorname{tr}_f \nu = 0\},$$

and all the L^2-inner products are with respect to the domain Ω_f. This extension operator commutes with d.

Hence, it remains to define the operator

$$S^k_{f,h} = \operatorname{tr}_f I^k_{f,h} : \bigotimes_{f \in \Delta(\mathcal{T}_h)} \mathcal{P}\Lambda^k(\mathcal{T}_{f,h}) \to \mathring{\mathcal{P}}\Lambda^k(f)$$

The operators $I^0_{f,h}$ and $S^0_{f,h} = \operatorname{tr}_f I^0_{f,h}$

We inductively require

$$\int_f S^0_{f,h} u \wedge v = \int_f \operatorname{tr}_f (u_f - \sum_{g \subsetneq f} I^0_{g,h} u) \wedge v, \quad v \in \mathcal{P}'(f, 0).$$

The operators $I^0_{f,h}$ and $S^0_{f,h} = \mathrm{tr}_f\, I^0_{f,h}$

We inductively require

$$\int_f S^0_{f,h} u \wedge v = \int_f \mathrm{tr}_f (u_f - \sum_{g \subsetneq f} I^0_{g,h} u) \wedge v, \quad v \in \mathcal{P}'(f, 0).$$

which implies that

$$\int_f \mathrm{tr}_f (I^0_h u - u_f) \wedge v = 0, \quad v \in \mathcal{P}'(f, 0).$$

The operators $I^0_{f,h}$ and $S^0_{f,h} = \mathrm{tr}_f\, I^0_{f,h}$

We inductively require

$$\int_f S^0_{f,h} u \wedge v = \int_f \mathrm{tr}_f (u_f - \sum_{g \subsetneq f} I^0_{g,h} u) \wedge v, \quad v \in \mathcal{P}'(f,0).$$

which implies that

$$\int_f \mathrm{tr}_f (I^0_h u - u_f) \wedge v = 0, \quad v \in \mathcal{P}'(f,0).$$

This ensures that π^0_h is a projection.

The operators $S^k_{f,h} = \mathrm{tr}_f\, I^k_{f,h}$

The operators $S^k_{f,h}$ are defined inductively by

$$\int_f S^k_{f,h} u \wedge \eta = \int_f \mathrm{tr}_f (dI^{k-1}_h R^k_f u + R^{k+1}_f du_f - \sum_{g \subsetneq f} I^k_{g,h} u) \wedge \nu,$$

for $\nu \in \mathcal{P}'(f,k)$.

The operators $S^k_{f,h} = \mathrm{tr}_f\, I^k_{f,h}$

The operators $S^k_{f,h}$ are defined inductively by

$$\int_f S^k_{f,h} u \wedge \eta = \int_f \mathrm{tr}_f (d I^{k-1}_h R^k_f u + R^{k+1}_f d u_f - \sum_{g \subsetneq f} I^k_{g,h} u) \wedge \nu,$$

for $\nu \in \mathcal{P}'(f,k)$. Here R^k_f is a Poincaré operator of the form

$$(R^k_f \nu)_x = \int_0^1 t^{k-1} \nu_{a+t(x-a)} \lrcorner (x-a)\, dt$$

where $a = a_f \in f$.

The operators $S^k_{f,h} = \mathrm{tr}_f\, I^k_{f,h}$

The operators $S^k_{f,h}$ are defined inductively by

$$\int_f S^k_{f,h} u \wedge \eta = \int_f \mathrm{tr}_f (dI^{k-1}_h R^k_f u + R^{k+1}_f du_f - \sum_{g \subsetneq f} I^k_{g,h} u) \wedge \nu,$$

for $\nu \in \mathcal{P}'(f,k)$. Here R^k_f is a Poincaré operator of the form

$$(R^k_f \nu)_x = \int_0^1 t^{k-1} \nu_{a+t(x-a)} \lrcorner (x-a)\, dt$$

where $a = a_f \in f$.
We have the identity

$$\nu = dR^k_f \nu + R^{k+1}_f d\nu,$$

for all $\nu \in H\Lambda^k(\Omega_g)$, $k \geq 1$ with $g \subset f$.

π_h^k is a projection

From the definition above we obtain that

$$\int_f \mathrm{tr}_f\, I_h^k u \wedge v = \int_f \mathrm{tr}_f (dI_h^{k-1} R_f^k u + R_f^{k+1} du_f) \wedge v, \quad v \in \mathcal{P}'(f,k).$$

π_h^k is a projection

From the definition above we obtain that

$$\int_f \operatorname{tr}_f I_h^k u \wedge v = \int_f \operatorname{tr}_f (dI_h^{k-1} R_f^k u + R_f^{k+1} du_f) \wedge v, \quad v \in \mathcal{P}'(f,k).$$

By an inductive argument this will imply that π_h^k is a projection.

π_h^k is a cochain projection

To show that I_h^k commutes with d we will use the fact that

$$\int_f dI_h^{k-1} du \wedge \nu = 0, \quad \nu \in \mathcal{P}'(f,k). \tag{3}$$

Here $u \in \otimes_g \mathcal{P}\Lambda^{k-2}(\mathcal{T}_{g,h})$.

π_h^k is a cochain projection

To show that I_h^k commutes with d we will use the fact that

$$\int_f dI_h^{k-1} du \wedge v = 0, \quad v \in \mathcal{P}'(f,k). \tag{3}$$

Here $u \in \otimes_g \mathcal{P}\Lambda^{k-2}(\mathcal{T}_{g,h})$. This will imply that

$$\int_f I_h^k du \wedge v = \int_f dI_h^{k-1} R_f^k du \wedge v$$
$$= \int_f dI_h^{k-1}(u - dR_f^{k-1}u) \wedge v = \int_f dI_h^{k-1}u, \; v \in \mathcal{P}'(f,k)$$

π_h^k is a cochain projection

To show that I_h^k commutes with d we will use the fact that

$$\int_f dI_h^{k-1} du \wedge v = 0, \quad v \in \mathcal{P}'(f,k). \tag{3}$$

Here $u \in \otimes_g \mathcal{P}\Lambda^{k-2}(\mathcal{T}_{g,h})$. This will imply that

$$\int_f I_h^k du \wedge v = \int_f dI_h^{k-1} R_f^k du \wedge v$$
$$= \int_f dI_h^{k-1}(u - dR_f^{k-1}u) \wedge v = \int_f dI_h^{k-1} u, \; v \in \mathcal{P}'(f,k)$$

Finally, the identity (3) can be established by induction on k.