Ranking Sets of Objects by Using Game Theory

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Summary

Preferences over sets

Properties that prevent the interaction

Alignment with regular semivalues

Interaction among objects

Specific **p**-aligned total prorder

Based on two papers

Stefano Moretti and Alexis Tsoukiàs. Ranking sets of possibly interacting objects using Shapley extensions. Proceedings of the 13th International Conference on Principles of Knowledge Representation and Reasoning (KR2012), June 10-14, 2012, Rome.

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Roberto Lucchetti, Stefano Moretti and Fioravante Patrone. Work in progress.

Central question(s)

How to derive a ranking over the set of all subsets of a finite set N "compatible" with a given ranking over the elements of N?

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- Most papers dealing with the first issue provide an axiomatic approach (Kannai and Peleg (1984), Barbera et al (2004), Bossert (1995), Fishburn (1992), Roth (1985) etc.)

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- Most papers dealing with the first issue provide an axiomatic approach (Kannai and Peleg (1984), Barbera et al (2004), Bossert (1995), Fishburn (1992), Roth (1985) etc.)

- **Extension axiom**: given total preorder $\succcurlyeq$ on N, a total preorder $\sqsupseteq$ on 2^N is an *extension* of $\succcurlyeq$ if for each $x, y \in N$,

$$\{x\} \sqsupseteq \{y\} \Leftrightarrow x \succcurlyeq y$$

Not needed in the second approach.

Example: max and min

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- For instance, let $N = \{1, 2, 3\}$ and $1 \succ 2 \succ 3$.
According to the max extension, for each $S, T \in 2^N \setminus \{\emptyset\}$, we have

$$(S \sqsupseteq^{\max} T) \Leftrightarrow (\text{ best}(S) \succcurlyeq \text{ best}(T))$$

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So the extension $\sqsupseteq^{\max}$ of $\succcurlyeq$ is:
$\{1, 2, 3\} \backsimeq^{\max} \{1, 3\} \backsimeq^{\max} \{1, 2\} \backsimeq^{\max} \{1\} \sqsupset^{\max} \{2\} \backsimeq^{\max} \{2, 3\} \sqsupset^{\max} \{3\}$

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Interactions are even more essential in the dual approach.

Which kind of interaction effects?

- Let $N = \{x, y, z\}$ and suppose that an agent's preference is such that $x \succcurlyeq y$, $x \succcurlyeq z$ and $y \succcurlyeq z$.

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- Trying to extend $\succcurlyeq$ to 2^N, one could guess that set $\{x, y\}$ is better than $\{y, z\}$, because the agent will receive both y and x instead of y and z (and x is preferred to z).

- However, in case of incompatibility among x and y, or complementarity effects between y and z, the relative ranking between the two sets $\{x, y\}$ and $\{y, z\}$ could be reversed.

Well-known extensions prevent interaction

Axiom [Responsiveness, RESP] A total preorder $\sqsupseteq$ on 2^N satisfies the *responsiveness* property if for all $S \in 2^N$ such that $i, j \notin S$

$$(S \cup \{i\}) \sqsupseteq (S \cup \{j\}) \Leftrightarrow \{i\} \sqsupseteq \{j\}$$

- This axiom was introduced by Roth (1985) studying colleges' preferences for the "college admissions problem" (see also Gale and Shapley (1962)).

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- Bossert (1995) used the same property for ranking sets of alternatives with a fixed cardinality and to characterize the class of *rank-ordered lexicographic* extensions.

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- many others...

Basic-Basic on coalitional games

A *coalitional game* is a pair (N, v), where N denotes the finite set of *players* and $v : 2^N \to \mathbb{R}$ is the *characteristic function*, with $v(\emptyset) = 0$.

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Given a game, a regular semivalue (Carreras and Freixas 1999; 2000) may be computed to convert information about the worth that coalitions can achieve into a personal attribution (of payoff) to each of the players:

$$\pi_i^{\mathbf{p}}(v) = \sum_{S \subset N : i \notin S} p_s\big(\ v(S \cup \{i\}) - v(S)\big)$$

for each $i \in N$, where p_s represents the probability that a coalition $S \in 2^N$ (of cardinality s) with $i \notin S$ forms. So coalitions of the same size have the same probability to form.

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(It must hold $\sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1$, requiring $p_s > 0$ for all s is the *regularity* of the semivalue.)

Shapley and Banzhaf regular semivalues

- The Shapley value (Shapley 1953) is the regular semivalue $\pi^{\hat{\mathbf{p}}}(v)$, such that

$$\hat{p}_s = \frac{1}{n\binom{n-1}{s}} = \frac{s!(n-s-1)!}{n!}$$

for each $s = 0, 1, \ldots, n-1$.

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for each $s = 0, 1, \ldots, n-1$.

- Another important semivalue is the Banzhaf power index (Banzhaf III 1964), defined as the regular semivalue $\pi^{\tilde{\mathbf{p}}}(v)$ such that

$$\tilde{p}_s = \frac{1}{2^{n-1}}$$

for each $s = 0, 1, \ldots, n-1$, (each coalition has an equal probability to be chosen)

p-aligned total preorders

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Denote by $V(\sqsupseteq)$ the class of coalitional games that numerically represent $\sqsupseteq$)

DEF. Let π^p be a regular semivalue. A total prorder $\sqsupseteq$ on 2^N is **p**-*aligned* if for each numerical representation $v \in V(\sqsupseteq)$ we have that

$$\{i\} \sqsupseteq \{j\} \Leftrightarrow \pi_i^{\hat{p}}(v) \geq \pi_j^{\hat{p}}(v)$$

for all $i, j \in N$. ■.

In other words, the ranking assigned by the semivalue to the players (objects) respects the initial ranking and does *not* depend from the utility function selected to represent the ordering on 2^N.

A basic formula

The following is a basic formula to calculate the ranking of the objects

$$\pi_i^{\mathbf{p}}(v) - \pi_j^{\mathbf{p}}(v) =$$
$$\sum_{S:i,j\notin S}(p_s + p_{s+1})\,[v(S\cup\{i\}) - v(S\cup\{j\})] =$$
$$\sum_{s=0}^{n-2}(p_s + p_{s+1})\left[\sum_{S:i,j\notin S,|S|=s}[v(S\cup\{i\}) - v(S\cup\{j\}]\right]$$

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$$\sum_{s=0}^{n-2}(p_s + p_{s+1})\left[\sum_{S:i,j\notin S,|S|=s}[v(S\cup\{i\}) - v(S\cup\{j\}]\right]$$

Write $x_s = p_s + p_{s+1}$ and

$$\left[\sum_{S:i,j\notin S,|S|=s}[v(S\cup\{i\}) - v(S\cup\{j\}]\right] = a_{s+1}^{ijv}.$$

Semivalues $\pi^{\mathbf{p}}$ aligned with $\sqsupseteq$ can be found by solving the semi-infinite system of linear inequalities:

$$a_1^{ijv}x_1 + a_2^{ijv}x_2 + \cdots + a_{n-1}^{ijv}x_{n-1} \geq 0, \quad v\in V(\sqsupseteq), \quad i \sqsupseteq j,$$
$$x_1 \geq 0, \ldots \quad x_{n-1} \geq 0 \quad x \neq 0$$

Example: Shapley-aligned total preorder...

For each coalitional game v, the Shapley value is denoted by $\phi(v) = \pi^{\hat{p}}(v)$.
Let $N = \{1, 2, 3\}$ and let $\sqsupseteq^a$ be a total preorder on N such that $\{1, 2, 3\} \sqsupset^a \{3\} \sqsupset^a \{2\} \sqsupset^a \{1, 3\} \sqsupset^a \{2, 3\} \sqsupset^a \{1\} \sqsupset^a \{1, 2\} \sqsupset^a \emptyset$.

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For every $v \in V(\sqsupseteq^a)$

$$\phi_2(v) - \phi_1(v) = \frac{1}{2}\big(v(2) - v(1)\big) + \frac{1}{2}\big(v(2, 3) - v(1, 3)\big) > 0$$

On the other hand

$$\phi_3(v) - \phi_2(v) = \frac{1}{2}\big(v(3) - v(2)\big) + \frac{1}{2}\big(v(1, 3) - v(1, 2)\big) > 0.$$

$\left(\frac{1}{2} = p_0 + p_1 = p_1 + p_2.\right)$

... **p**-aligned for other regular semivalues

Note that $\sqsupseteq^a$ is **p**-aligned for every regular semivalue such that $p_0 \geq p_2$:

$$\pi_2^p(v) - \pi_1^p(v) = (p_0 + p_1)\big(v(2) - v(1)\big) + (p_1 + p_2)\big(v(2,3) - v(1,3)\big) > 0$$

On the other hand

$$\pi_3^p(v) - \pi_2^p(v) = (p_0 + p_1)\big(v(3) - v(2)\big) + (p_1 + p_2)\big(v(1,3) - v(1,2)\big) > 0$$

for every $v \in V(\sqsupseteq^a)$.

Total preorder **p**-aligned for no regular semivalues

It is quite possible that for a given preorder there is no **p**-ordinal regular semivalue associated to it ($(1, 0, \ldots, 0)$ is always good). It is enough, for instance, to consider the case $N = \{1, 2, 3\}$ and the following total preorder:

$$N \sqsupset \{1, 2\} \sqsupset \{2, 3\} \sqsupset \{1\} \sqsupset \{1, 3\} \sqsupset \{2\} \sqsupset \{3\} \sqsupset \emptyset.$$

Then 1 and 2 cannot be ordered since, for every fixed semivalue **p** the quantity

$$(p_0 + p_1)(v(\{1\}) - v(\{2\})) + (p_1 + p_2)(v(\{1, 3\}) - v(\{2, 3\}))$$

can be made both positive and negative by suitable choices of v.

A geometric characterization of alignment

Theorem

Given a total order $\sqsupseteq$ on 2^N, the set of regular semivalues $\sqsupseteq$ is aligned with is either empty or at least two dimensional convex set.

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Also with $n > 4$, the set of regular semivalues for which a complete preorder is aligned can be exactly two dimensional.

EXAMPLE Let $N = \{1, 2, 3, 4, 5\}$. A trichotomous preorder such

$$VG = \{\{1\}, \{1,3\}, \{2\}, \{2,3,4\}, \{4,5\}, \{1,2,5\}, \{1,2,3,4\}\},$$

$$G = \{\{3\}, \{1,3,5\}, \{4\}, \{1,2,4,5\}, \{2,4\}, \{3,5\}\},$$

$$B = \{2^N \setminus \{VG \cup G\}\}.$$

Such a total preorder is aligned for every regular semivalue of the form

$$\mathbf{p} = (p_0, p_1, \frac{1 - p_0 - 2p_1}{2}, p_1, \frac{1 - p_0 - 2p_1}{2}).$$

Proposition Let $\sqsupseteq$ be a total preorder on 2^N. If $\sqsupseteq$ satisfies the RESP property, then it is p-aligned with every regular semivalue π^p. ■

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- All the extensions from the literature listed in the previous slide are **p**-aligned with all regular semivalues...

Axiom[Permutational Responsiveness, PR]

We denote by Σ_{ij}^{s} the set of all subsets of N of cardinality s which do not contain neither i nor j, i.e.
$\Sigma_{ij}^{s} = \{S \in 2^N : i, j \notin S, |S| = s\}$.

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$\Sigma_{ij}^{s} = \{S \in 2^N : i, j \notin S, |S| = s\}$.

Order the sets $S_1, S_2, \ldots, S_{n_s}$ in Σ_{ij}^{s} when you add i and j, respectively:

$$
\begin{array}{ccc}
S_1 \cup \{i\} & \sqsupseteq & S_{l(1)} \cup \{j\} \\
|\bigsqcup & & |\bigsqcup \\
S_2 \cup \{i\} & \sqsupseteq & S_{l(2)} \cup \{j\} \\
|\bigsqcup & & |\bigsqcup \\
\ldots & \sqsupseteq & \ldots \\
|\bigsqcup & & |\bigsqcup \\
S_{n_s} \cup \{i\} & \sqsupseteq & S_{l(n_s)} \cup \{j\}
\end{array}
$$

$$\Leftrightarrow \{i\} \sqsupseteq \{j\}$$

Again a sufficient condition...

Proposition Let $\sqsupseteq$ be a total preorder on 2^N. If $\sqsupseteq$ satisfies the PR property, then $\sqsupseteq$ is **p**-aligned with every semivalue. $\blacksquare$

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Proposition Let $\sqsupseteq$ be a total preorder on 2^N. If $\sqsupseteq$ satisfies the PR property, then $\sqsupseteq$ is **p**-aligned with every semivalue. ■

- $\{1,2,3,4\} \sqsupset^b \{2,3,4\} \sqsupset^b \{3,4\} \sqsupset^b \{4\} \sqsupset^b \{3\} \sqsupset^b \{2\} \sqsupset^b \{2,4\} \sqsupset^b \{1,4\} \sqsupset^b \{1,3\} \sqsupset^b \{2,3\} \sqsupset^b \{1,3,4\} \sqsupset^b \{1,2,4\} \sqsupset^b \{1,2,3\} \sqsupset^b \{1,2\} \sqsupset^b \{1\} \sqsupset^b \emptyset$ is **p**-aligned for all p but does not satisfy the PR property.

Axiom[Double Permutational Responsiveness, DPR]

Order the sets $S_1, S_2, \ldots, S_{n_s+n_{s-1}}$ in $\Sigma_{ij}^{s} \cup \Sigma_{ij}^{s-1}$ when you add i and j, respectively:

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\begin{array}{ccc}
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\ldots & \sqsupseteq & \ldots \\
|\bigsqcup & & |\bigsqcup \\
S_{n_s+n_{s-1}} \cup \{i\} & \sqsupseteq & S_{l(n_s+n_{s-1})} \cup \{j\}
\end{array}
$$

$$\Leftrightarrow \{i\} \sqsupseteq \{j\}$$

A characterization with possibility of interaction

Theorem

The following statements are equivalent:

1) $\sqsupseteq$ *fulfills the DPR property;*
2) $\sqsupseteq$ *is* $\mathbf{p}$*-aligned for all semivalues.*

- $\{1,2,3,4\} \sqsupset^b \{2,3,4\} \sqsupset^b \{3,4\} \sqsupset^b \{4\} \sqsupset^b \{3\} \sqsupset^b \{2\} \sqsupset^b \{2,4\} \sqsupset^b \{1,4\} \sqsupset^b \{1,3\} \sqsupset^b \{2,3\} \sqsupset^b \{1,3,4\} \sqsupset^b \{1,2,4\} \sqsupset^b \{1,2,3\} \sqsupset^b \{1,2\} \sqsupset^b \{1\} \sqsupset^b \emptyset$ is $\mathbf{p}$-aligned for all $\mathbf{p}$, is not PR, but it is DPR.

Finding semivalues aligned with $\sqsupseteq$

Let $\sqsupseteq$ be a total preorder on 2^N. For each $A \in 2^N$, let $\mathcal{P}^s_{ij}(\sqsupseteq, A)$ be the set of all subsets T containing neither i nor j and with cardinality s such that $T \cup \{i\}$ is weakly preferred to S, i.e. $\mathcal{P}^s_{ij}(\sqsupseteq, A) = \{S \in \Sigma^s_{ij} : S \cup \{i\} \sqsupseteq A\}$.

Theorem

Let $\sqsupseteq$ be a total preorder on 2^N and consider a semivalue $\mathbf{p} = (p_0, \ldots, p_{n-1})$. Then $\sqsupseteq$ is $\mathbf{p}$-aligned if and only if for all $i, j \in N$ and all $A \in 2^N$

$$\sum_{s=0}^{n-2} (p_s + p_{s+1}) \left(|\mathcal{P}^s_{ij}(\sqsupseteq, A)| - |\mathcal{P}^s_{ji}(\sqsupseteq, A)| \right) \geq 0 \Leftrightarrow \{i\} \sqsupseteq \{j\},$$

Finding semivalues aligned with $\sqsupseteq$ is transformed in a (almost) *classical system of linear inequalities.*

Axiom[Weighted Permutational Responsiveness, WPR]

Let $\mathbf{p}$ be a semivalue with rational coordinates and let $\mathbf{v}$ be a multiple of $\mathbf{p}$ in $\mathbb{N}^n$. Let $x_s = v_s + v_{s+1}$. Order all sets in decreasing order, with repetitions $S_1, S_2, \ldots, S_{2^{n-2}}$ in $2^{N\setminus\{i,j\}}$ when you add i and j, respectively:

$$
\begin{array}{ccccccc}
\text{repeated } x_{s_1} \text{ times} & \left\{\begin{array}{l} S_1 \cup \{i\} \\ \ldots \\ S_1 \cup \{i\} \\ \end{array}\right. & \begin{array}{c} \sqsupseteq \\ \ldots \\ \sqsupseteq \\ \end{array} & \left.\begin{array}{l} S_{l(1)} \cup \{j\} \\ \ldots \\ \ldots \\ S_{l(1)} \cup \{j\} \end{array}\right\} & \text{repeated } x_{s_{l(1)}} \text{ times} \\
 & \mid\sqcup & & \mid\sqcup & \\
 & \ldots & \sqsupseteq & \ldots & \\
 & \mid\sqcup & & \mid\sqcup & \\
\text{repeated } x_{s_{2n-2}} \text{ times} & \left\{\begin{array}{l} S_{2^{n-2}} \cup \{i\} \\ \ldots \\ \ldots \\ S_{2^{n-2}} \cup \{i\} \end{array}\right. & \begin{array}{c} \sqsupseteq \\ \ldots \\ \sqsupseteq \end{array} & \left.\begin{array}{l} S_{l(2^{n-2})} \cup \{j\} \\ \ldots \\ S_{l(2^{n-2})} \cup \{j\} \end{array}\right\} & \text{repeated } x_{s_{l(2^{n-2})}} \text{ times}
\end{array}
$$

$$\Leftrightarrow \{i\} \sqsupseteq \{j\}$$

Example

Let $N = \{1,2,3\}$ and consider the order

$$N \sqsupset \{1\} \sqsupset \{2,3\} \sqsupset \{1,3\} \sqsupset \{2\} \sqsupset \{1,2\} \sqsupset \{3\} \sqsupset \emptyset.$$

$\mathbf{v} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ $\mathbf{v} = (2,1,1)$. Then consider players 1 and 2

$$\begin{array}{ccc} \{1\} & \sqsupset & \{2,3\} \\ \{1\} & \sqsupset & \{2,3\} \\ \{1\} & \sqsupset & \{2\} \\ \{1,3\} & \sqsupset & \{2\} \\ \{1,3\} & \sqsupset & \{2\} \end{array}$$

A simple algorithm to check $\mathbf{p}$-alignment

Theorem

Let $\sqsupseteq$ be a total preorder on 2^N and consider a semivalue $\mathbf{p} = (p_0, \ldots, p_{n-1})$, with rational p. Then $\sqsupseteq$ is $\mathbf{p}$-aligned if and only if the property WPR holds.