Approssimazione di problemi iperbolici

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Convection diffusion equation

As usual. . . a one dimensional example

$$\begin{cases} -\varepsilon u''(x) + bu'(x) = 0 \qquad 0 < x < 1 \\ u(0) = 0,\ u(1) = 1 \end{cases}$$

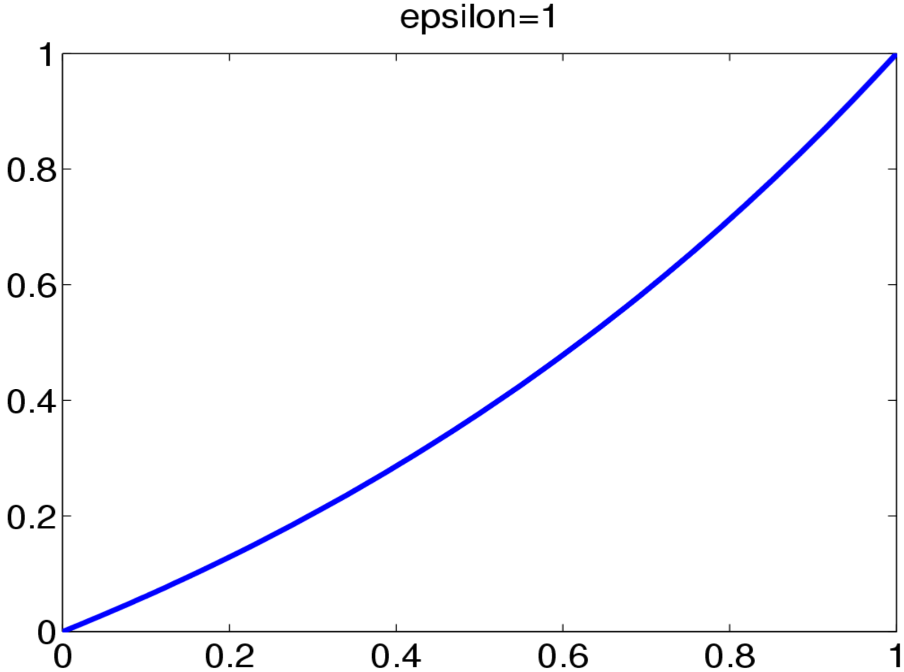
Non-homogeneous boundary conditions (!)
Péclet number $\mathbb{P} = |b|L/(2\varepsilon)$ ($L = 1$ in our case)

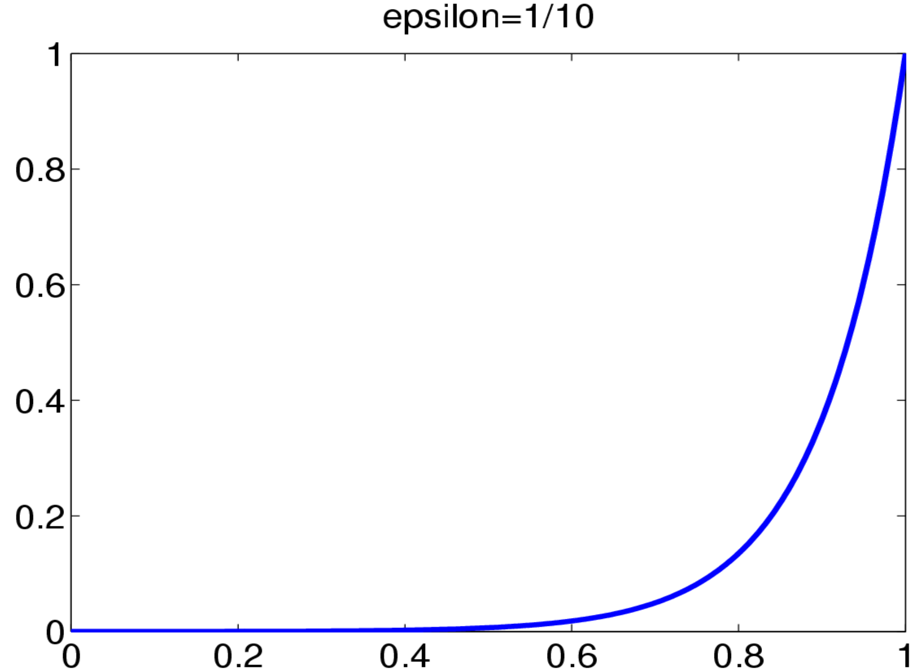
Closed form solution can be explicitly computed

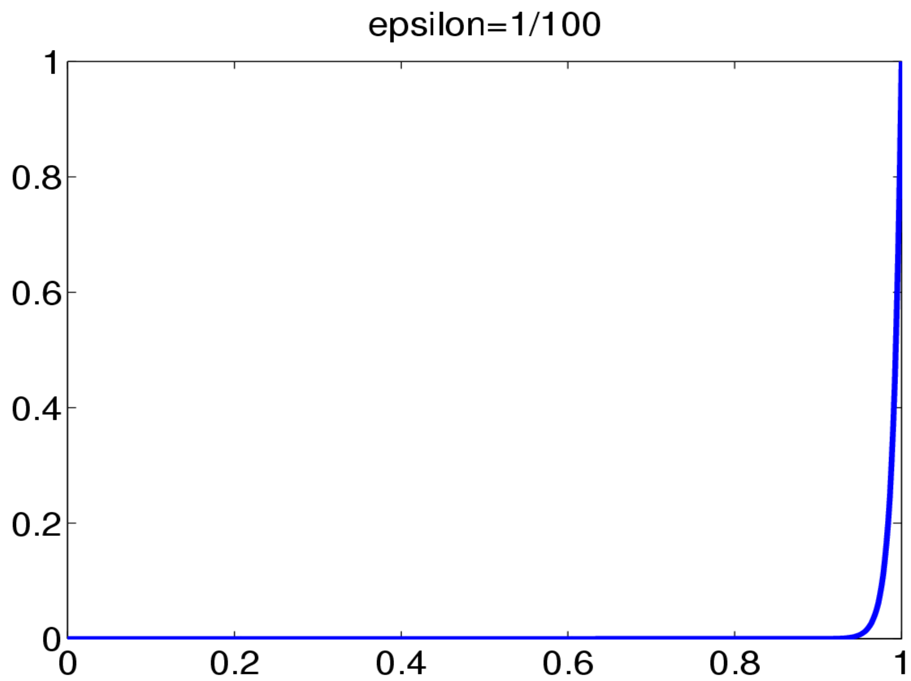
$$u(x) = \frac{\exp(bx/\varepsilon) - 1}{\exp(b/\varepsilon) - 1}$$

Convection diffusion equation (cont'ed)

$$u(x) = \frac{\exp(bx/\varepsilon) - 1}{\exp(b/\varepsilon) - 1}$$







If $b/\varepsilon \ll 1$ then $u(x) \simeq x$

If $b/\varepsilon \gg 1$ then $u(x) \simeq \exp(-b(1-x)/\varepsilon)$

In the second case, *boundary layer* of size $\mathcal{O}(\varepsilon/b)$

Convection diffusion equation (cont'ed)

Approximation by finite elements

$$a(u,v)=\int_0^1 \left(\varepsilon u'(x)v'(x)+bu'(x)v(x)\right)\,dx$$

After some computations. . . stiffness matrix is (uniform mesh):

$$\left(\frac{b}{2}-\frac{\varepsilon}{h}\right)u_{i+1}+\frac{2\varepsilon}{h}u_i+\left(-\frac{b}{2}-\frac{\varepsilon}{h}\right)u_{i-1}$$

Local (discrete) Péclet number is $\mathbb{P}(h)=|b|h/(2\varepsilon)$, so that our system has the structure

$$(\mathbb{P}(h)-1)u_{i+1}+2u_i-(\mathbb{P}(h)+1)u_{i-1}=0$$

Convection diffusion equation (cont'ed)

$$(\mathbb{P}(h) - 1)u_{i+1} + 2u_i - (\mathbb{P}(h) + 1)u_{i-1} = 0$$

General solution

$$u_i = \frac{1 - \left(\frac{1+\mathbb{P}(h)}{1-\mathbb{P}(h)}\right)^i}{1 - \left(\frac{1+\mathbb{P}(h)}{1-\mathbb{P}(h)}\right)^N} \quad i = 1, \dots, N$$

If $\mathbb{P}(h) > 1$ solution oscillates!

Stabilization techiniques

- Upwind (finite differences)
- Artificial viscosity, streamline diffusion (loosing consistency)
- Petrov–Galerkin, SUPG (strongly consistent)

Hyperbolic equations

Let's consider the model problem (one dimensional convection equation)

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0, & t > 0,\ x \in \mathbb{R} \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases}$$

Solution is a traveling wave $u(x,t) = u_0(x - at)$.

We consider a finite difference approximation.

Hyperbolic equations (cont'ed)

$u_j^n \simeq u(x_j, t_n)$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(h_{j+1/2}^n - h_{j-1/2}^n)$$

where $h_{j+1/2} = h(u_j, u_{j+1})$ is a *numerical flux*

Indeed,

$$\frac{\partial}{\partial t}U_j = -\left((au)(x_{j+1/2}) - (au)(x_{j-1/2})\right) \quad \text{with } U_j = \int_{x_{j-1/2}}^{x_{j+1/2}} u,\, dx$$

Hyperbolic equations (cont'd)

Courant–Friedrichs–Lewy (CFL) condition

$$\left| a \frac{\Delta t}{\Delta x} \right| \leq 1$$

Very clear geometrical interpretation (see also multidimensional extension and generalization to systems)

Remark: implicit schemes (in time) don't have restrictions, but add artificial diffusion