Approssimazione di problemi parabolici

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Parabolic problems

Heat equation

$$\frac{\partial u(t)}{\partial t} - \Delta u(t) = f(t)$$

Variational formulation: for each t, find $u(t) \in V = H_0^1(\Omega)$ s.t.

$$\left(\frac{\partial u(t)}{\partial t}, v\right) + a(u(t), v) = (f(t), v) \quad \forall v \in V$$

Space semidiscretization. Take $V_h \subset V$ and, for each t, look for $u_h(t) \in V_h$ such that

$$\left(\frac{\partial u_h(t)}{\partial t}, v_h\right) + a(u_h(t), v_h) = (f(t), v_h) \quad \forall v_h \in V_h$$

Parabolic problems (cont'ed)

Fully discretized problem

- Explicit Euler

$$\left(\frac{u_h^{n+1}-u_h^n}{\Delta t}, v_h\right) + a(u_h^n, v_h) = (f^n, v_h) \quad \forall v_h \in V_h$$

- Implicit Euler

$$\left(\frac{u_h^{n+1}-u_h^n}{\Delta t}, v_h\right) + a(u_h^{n+1}, v_h) = (f^{n+1}, v_h) \quad \forall v_h \in V_h$$

Parabolic problems (cont'ed)

θ-method (somewhat inbetween explicit and implicit)
$0 \leq \theta \leq 1$

$$\left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, v_h\right) + (1-\theta)a(u_h^n, v_h) + \theta a(u_h^{n+1}, v_h) =$$
$$(1-\theta)(f^n, v_h) + \theta(f^{n+1}, v_h) \quad \forall v_h \in V_h$$

In one space dimension (finite differences, and $f = 0$)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{h^2}(1-\theta)(u_{i+1}^n - 2u_i^n + u_{i-1}^n) +$$
$$\theta(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1})$$

Parabolic problems (cont'ed)

If $u(0) = \sin(\pi x)$ then the solution in $[0,1]$ with homogeneous Dirichlet boundary conditions is

$$u(t) = \sin(\pi x)\exp(-\pi^2 t)$$

In particular, it goes to zero as $t \to +\infty$

Study of discrete (absolute) stability

Discrete solution has the form

$$u_i^n = \alpha^n \sin(\pi i h)$$

Stability condition $|\alpha| \leq 1$

Parabolic problems (cont'ed)

$$u_i^{n+1} - u_i^n = \frac{k}{h^2}(1-\theta)(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \theta(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1})$$

$$u_i^n = \alpha^n \sin(\pi i h)$$

Some trigonometry

$$\sin\pi(i+1)h - 2\sin(\pi i h) + \sin\pi(i-1)h = 2\sin(\pi i h)\cos(\pi h) - 2\sin(\pi i h) = \sin(\pi i h)(-4\sin^2(\pi h/2))$$

Hence

$$\alpha - 1 = \frac{k}{h^2}((1-\theta) + \theta\alpha)(-4\sin^2(\pi h/2))$$

Parabolic problems (cont'ed)

Finally

$$\alpha = \frac{1-(1-\theta)w}{1+\theta w} = 1 - \frac{w}{1+\theta w}$$

with $w = 4\frac{k}{h^2}\sin^2(\pi h/2) \geq 0$
Condition $|\alpha| \leq 1$ equivalent to

$$w(1-2\theta) \leq 2$$

- $1/2 \leq \theta \leq 1$ inconditionally stable
- $0 \leq \theta < 1/2$ stability condition $\displaystyle\frac{k}{h^2} \leq \frac{1}{2(1-2\theta)}$