$\boxed{\text{Esercizio 1}}$

$$f(x,y) = \frac{1}{2}x^2y^2 - 2y^2 + \frac{1}{3}x^3$$

$$\frac{\partial f}{\partial x} = xy^2 + x^2 \qquad \nabla f = 0 \iff \begin{cases} xy^2 + x^2 = 0 \\ x^2y - 4y = 0 \end{cases} \quad \begin{cases} x(y^2+x) = 0 \\ y(x^2-4) = 0 \end{cases}$$

$$\frac{\partial f}{\partial y} = x^2y - 4y$$

$$\begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} y^2+2=0 \text{ imp.} \\ x=2 \end{cases} \vee \begin{cases} y = \pm\sqrt{2} \\ x=-2 \end{cases}$$

$P_1(0,0) \quad P_2(-2,\sqrt{2}) \quad P_3(-2,-\sqrt{3})$ punti stazionari

$$Hf(x,y) = \begin{pmatrix} y^2+2x & 2xy \\ 2xy & x^2-4 \end{pmatrix}$$

$$Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix} \quad \text{semid. neg.}$$

$$Hf(-2,\sqrt{2}) = \begin{pmatrix} -2 & -4\sqrt{2} \\ -4\sqrt{2} & 0 \end{pmatrix} \quad \det < 0 \Rightarrow \text{matrice indefinita}$$

$\Rightarrow P_2$ sella

$Hf(P_3)$ indef. $\Rightarrow P_3$ sella

Per $P$ non posso dire nulla. Ma guardando alla restrizione $f(x,0) = \frac{1}{3}x^3$, concludo che anche $P$ è punto di sella.

$\boxed{\text{Esercizio 2}}$

$$a_n = \frac{2^n(n+1)}{n^2+3}$$

$$\sqrt[n]{a_n} = \frac{2\sqrt[n]{n+1}}{\sqrt[n]{n^2+3}} \longrightarrow 2 \qquad \Rightarrow \qquad R = \frac{1}{2}$$

$$x = \frac{1}{2} \qquad \sum_{n=1}^{+\infty} \frac{\cancel{2^n}(n+1)}{n^2+3} \cdot \frac{1}{\cancel{2^n}}$$

$$\frac{n+1}{n^2+3} \sim \frac{1}{n} \quad \underset{\substack{\text{CR.}\\ \text{confronto}\\ \text{asint.}}}{\Longrightarrow} \quad \text{la serie diverge}$$

$$x = -\frac{1}{2} \qquad \sum_{n=1}^{+\infty} (-1)^n \frac{n+1}{n^2+3}$$

converge per il CR. di Leibniz

$$\left( b_n = \frac{n+1}{n^2+3} \text{ è infinitesima, decrescente} \right)$$

$\boxed{\text{Esercizio 3}}$

$$0 \leq |f(x,y)| \leq \frac{|x^2 y| + |3y^3|}{x^2+y^2} = \underbrace{\frac{x^2}{x^2+y^2}}_{\text{limitata}} \cdot \underbrace{|y|}_{\text{infinitesima}} + \overbrace{\frac{3y^2}{x^2+y^2}}^{\text{limitata}} \overbrace{|y|}^{\text{infinitesima}} \to 0$$

Per il cr. del confronto anche $\lim_{(x,y)\to(0,0)} f(x,y)=0$

da cui $f$ continua in $(0,0)$.

Oppure:

$$|f(\rho\cos\theta, \rho\sin\theta)| = |\rho\cos^2\theta\sin\theta - 3\rho\sin^3\theta| =$$

$$= \rho\,|\cos^2\theta\sin\theta - 3\sin^3\theta| \leq$$

$$\leq \rho\,(|\cos^2\theta\sin\theta| + 3|\sin^3\theta|)$$

$$\leq 4\rho \xrightarrow[\rho\to 0]{} 0$$

---

$$\frac{\partial f}{\partial x}(0,0) = \frac{d}{dx} f(x,0)\Big|_{x=0} = \frac{d}{dx}\, 0\,\Big|_{x=0} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \frac{d}{dy} f(0,y)\Big|_{y=0} = \frac{d}{dy}(-3y)\Big|_{y=0} = -3$$

$$\Rightarrow \nabla f(0,0) = (0,-3)$$

---

$f$ diff.$^{le}$ in $(0,0)$ $\Leftrightarrow$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\frac{x^2y-3y^3}{x^2+y^2}+3y}{\sqrt{x^2+y^2}} = 0$

$$\Leftrightarrow \lim_{(x,y)\to(0,0)} \frac{4x^2y}{(x^2+y^2)^{3/2}} = 0$$

Ma in coord. polari $\underbrace{\qquad\qquad}_{\mapsto} = 4\cos^2\theta\sin\theta \not\to 0$

Pertanto $f$ non è differenziabile in $(0,0)$.

ESERCIZIO 4

$$\gamma(t) = (\cos t, \sin t, e^{2t}) \qquad t \in [0, 2\pi]$$

$\gamma$ è di classe $C^1$ e $\gamma'(t) = (-\sin t, \cos t, 2e^{2t}) \neq (0,0,0)$
$\forall t \in [0, 2\pi]$ $\Rightarrow$ $\gamma$ è regolare

$\gamma(0) \neq \gamma(2\pi) \Rightarrow \gamma$ non è chiusa

$$\int_\gamma z^2 ds = \int_0^{2\pi} e^{4t}\sqrt{1+4e^{4t}}\, dt = \left.\frac{(1+4e^{4t})^{3/2}}{\frac{3}{2}\cdot 16}\right|_0^{2\pi} =$$

$$= \frac{1}{24}\left[(1+4e^{8\pi})^{3/2} - 5^{3/2}\right]$$

$$\int_\gamma \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (e^{2t}\cos t, e^{2t}\sin t, \sin t)\cdot(-\sin t, \cos t, 2e^{2t})\, dt$$

$$= \int_0^{2\pi} 2e^{2t}\sin t\, dt = I$$

Integrando per parti due volte:

$$I = -2e^{2t}\cos t\Big|_0^{2\pi} + \int_0^{2\pi} 4e^{2t}\cos t = -2e^{2t}\cos t\Big|_0^{2\pi} + 4e^{2t}\sin t\Big|_0^{2\pi} - \int_0^{2\pi} 8e^{2t}\sin t$$

$$= -2e^{4\pi} + 2 - 4I$$

$$\Rightarrow \quad I = \frac{2}{5}\left(1 - e^{4\pi}\right)$$

$\boxed{\text{ESERCIZIO 5}}$

La normale richiesta è $\hat{n} = \dfrac{(\partial_x f, \partial_y f, -1)}{\sqrt{1+|\nabla f|^2}}$

dove $f(x,y) = \sqrt{x^2+y^2}$.

Quindi
$$\hat{n} = \frac{\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right)}{\sqrt{1+|\nabla f|^2}}$$

$$\text{Flusso} = \iint_D \left(x, x^2+y^2, y^2\sqrt{x^2+y^2}\right) \cdot \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right) dx\,dy$$

$$D = \{1 < x^2+y^2 < 4\}$$

$$= \iint_D \left(\frac{x^2}{\sqrt{x^2+y^2}} + y\sqrt{x^2+y^2} - y^2\sqrt{x^2+y^2}\right) dx\,dy =$$

$$= \int_0^{2\pi}\int_1^2 \left(\rho^2\cos^2\theta + \rho^3\sin\theta - \rho^4\sin^2\theta\right) d\rho\, d\theta$$

$$= \frac{1}{3}\cdot 7 \cdot \pi + 0 - \frac{31}{5}\cdot \pi = -\frac{58}{15}\pi$$

Esercizio 6

$f(x,y) = \dfrac{|x^3 y|}{x^2+y^2}$ è pari in $x$ e in $y$

Pertanto, essendo $D$ simmetrico rispetto ad entrambi gli assi, abbiamo

$$\iint_D \frac{|x^3 y|}{x^2+y^2}\,dx\,dy = 4 \iint_{D^{++}} \frac{x^3 y}{x^2+y^2}\,dx\,dy$$

$$D^{++} = D \cap \{x>0\} \cap \{y>0\}$$

$$= 4 \int_0^{\frac{\pi}{2}} \int_1^3 \rho^3 \cos^3\theta \sin\theta \, d\rho\, d\theta =$$

$$= \cancel{4} \cdot \frac{80}{\cancel{4}} \cdot \left(-\frac{\cos^4\theta}{4}\right)\Bigg|_0^{\frac{\pi}{2}} = 20$$