Intrinsic Harnack estimates for some doubly nonlinear degenerate parabolic equations

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Abstract

We prove an intrinsic Harnack inequality for non-negative local weak solutions of a wide class of doubly nonlinear degenerate parabolic equations whose prototype is

\[ u_t - \text{div}(u^{m-1}|Du|^{p-2}Du) = 0, \quad p \geq 2, m \geq 1. \]

As a consequence, we get that such solutions are locally Hölder continuous.


1 Introduction

Consider an open set \( E \subset \mathbb{R}^N \) and \( T > 0 \) and quasi-linear parabolic differential equations

\[ u_t - \text{div} A(x, t, u, Du) = B(x, t, u, Du) \]

in \( E_T = E \times (0, T] \). The functions \( A : E_T \times \mathbb{R}^N \rightarrow \mathbb{R}^N \), \( B : E_T \times \mathbb{R}^N \rightarrow \mathbb{R} \) are assumed to be measurable and subject to the structure conditions

\[
\begin{align*}
A(x, t, u, \eta) \cdot \eta & \geq C_0 \Phi(|u|)|\eta|^p - C^p, \\
|A(x, t, u, \eta)| & \leq C_1 \Phi(|u|)|\eta|^{p-1} + C_{p-1} \Phi(|u|)^\frac{1}{p}, \\
|B(x, t, u, \eta)| & \leq C_2 \Phi(|u|)|\eta|^{p-1} + C_2 C_{p-1} \Phi(|u|)^\frac{1}{p}
\end{align*}
\]

for almost all \((x, t) \in E_T\), for all \( u \in \mathbb{R} \) and \( \eta \in \mathbb{R}^N \), with \( p \geq 2, C_0, C_1 \) positive constants and \( C_2, C \) non-negative constants. The function \( \Phi : [0, +\infty] \rightarrow [0, +\infty] \) is assumed to be continuous and such that

\[ \gamma_1 s^\alpha \leq \Phi(s) \leq \gamma_2 s^\alpha, \]

for all \( s \geq 0 \), for some \( \alpha \geq 0 \) and \( \gamma_1, \gamma_2 > 0 \). The prototype of equations (1.1)–(1.2) is

\[ u_t - \text{div}(|u|^{m-1}|Du|^{p-2}Du) = 0, \quad m \geq 1, \]

which models the filtration of a polytropic non-newtonian fluid in a porous medium. In [9], a one dimensional version of (1.4) has been proposed to describe the filtration of water in porous building materials, such as bricks.

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Equations of this type are classified as doubly nonlinear and include the standard porous media equation \((p = 2)\), and the parabolic \(p\)-Laplacian \((m = 1)\). From a theoretical point of view, it is interesting to see how much of the regularity properties of solutions of the two model equations is preserved in this more general case.

The aim of our paper consists in proving that any non-negative local weak solution of (1.1) satisfies an intrinsic Harnack inequality (Theorem 2.1).

The first parabolic version of the Harnack inequality was proved independently by Hadamard [8] and Pini [14] for non-negative local solutions of the heat equation, by means of explicit representations of solutions in terms of heat potentials. Later on, in [13], Moser showed that the Harnack inequality continues to hold for non-negative weak solutions of linear parabolic equations in divergence form \(u_t = \text{div}(A(x,t)Du)\), with bounded and measurable coefficients, satisfying a uniform ellipticity condition. As in Moser’s proof the linearity is immaterial, following the ideas of [13], a further contribution was given by [1, 16], where quasi-linear parabolic equations (1.1)–(1.2), with \(\Phi \equiv 1\) and \(p = 2\) are considered.

For a long time the case \(p \neq 2\) remained unsolved. This was not simply a matter of technique, as the Harnack inequality in the classical formulation fails for \(p \neq 2\) (see [3] for a counterexample). Some progress was made in [4], where it was noticed that a Harnack estimate can be obtained for non-negative weak solutions of the \(p\)-Laplacian, \(p > 2\), by introducing a new geometrical setting, intrinsic to the solution itself and related to the degeneracy exhibited by the equation. The proof relies upon the Hölder continuity of the weak solutions and uses the comparison principle with explicit solutions. Consequently, it cannot be extended to more general equations.

A very interesting approach has been recently developed in [6] to prove the intrinsic Harnack inequality for a wide class of degenerate, quasilinear, parabolic equations, including equations of the \(p\)-Laplacian and porous medium type. The main novelty of [6] consists in producing a proof based on measure-theoretical arguments, that never employ the Hölder continuity of the solutions. On the contrary, the intrinsic Harnack inequality can be used to establish the Hölder continuity, according to Moser’s original ideas.

In the present paper, we adapt the technique of [6] to our situation, and show that a local Hölder continuity condition can be deduced for non-negative weak solutions of (1.1).

Concerning the literature, one can find several papers dealing with local and global Hölder estimates for weak solutions of general doubly non-linear degenerate parabolic equations. We cite [10, 11, 15, 17] and the references therein. In [18] the author is concerned with Harnack inequalities for (1.4), also in the singular case. His approach uses Hölder regularity results, the maximum principle and the possibility to construct suitable explicit comparison functions.

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2 Notation and main result

A function \(u : E_T \rightarrow \mathbb{R}\) is said to be a local weak solution of (1.1) if

\[
u \in C(0,T;L^2_{\text{loc}}(E)), \quad \Phi(|u|) \frac{1}{p} |Du| \in L^p_{\text{loc}}(E_T),
\]

\(u\) is locally bounded, and

\[
\int_K \psi dx \bigg|_{t_2}^{t_1} + \int_{t_1}^{t_2} \int_K [-u \psi_t + A(x,t,u,Du) \cdot D\psi] dx dt = \int_{t_1}^{t_2} \int_K B(x,t,u,Du) \psi dx dt,
\] (2.1)
for every compact set $K \subset E$, for every sub-interval $[t_1, t_2] \subset (0, T]$ and for every test function
$$\psi \in W^{1,2}_{\text{loc}}(0, T; L^2(K)) \cap L^p_{\text{loc}}(0, T; W^{1,p}_0(K)).$$

We denote by $K_\rho(y)$ the cube of $\mathbb{R}^N$ centered at $y$ with edge $2\rho$. If $y = 0$, we simply write $K_\rho$ instead of $K_\rho(0)$. For $\theta > 0$, we set
$$Q_\rho^- (\theta) = K_\rho \times (-\theta \rho^p, 0],$$
$$Q_\rho^+ (\theta) = K_\rho \times (0, \theta \rho^p].$$

We will prove the following main result.

**Theorem 2.1** Let $u$ be a continuous, non-negative, local weak solution to (1.1) in $E_T$. Let $(x_0, t_0) \in E_T$ be such that $u(x_0, t_0) > 0$. Then there exist constants $c, \gamma > 0$ and $\kappa > 1$, depending only upon the data, such that for all cylinders $(x_0, t_0) + Q_\rho^\pm(\theta) \subset E_T$ either $u(x_0, t_0) \leq \gamma(C\rho)^{\frac{p}{p+\alpha}}$ or

$$u(x_0, t_0) \leq \kappa \inf_{K_\rho(x_0)} u(x, t_0 + \theta \rho^p), \quad \theta = \left(\frac{c}{u(x_0, t_0)}\right)^{p+\alpha-2}. \quad (2.2)$$

Actually, the continuity assumption in the theorem above can be removed and (2.2) continues to hold for a.e. $(x_0, t_0) \in E_T$.

In the following we denote by $\gamma$ positive constants which depend only on the data, namely $N, p, C_0, C_1$. We will not distinguish these constants by subscripts, but provide that they can be enlarged without invalidating the inequalities considered.

Finally, throughout the paper, $u$ denotes a non-negative local weak solution to (1.1) and, if $k \in \mathbb{R}_+$, we set
$$(u - k)_+ = \max\{u - k, 0\}, \quad (u - k)_- = \max\{-u - k, 0\}.$$

### 3 Energy estimates

**Proposition 3.1** There exist two positive constants $\varpi, \tilde{\gamma}$, depending only on $N, p, C_0, C_1$, such that for every cylinder $(y, s) + Q_\rho^-(\theta) \subset E_T$, $k \in \mathbb{R}_+$ and every piecewise smooth cutoff function $\zeta$ vanishing on the boundary of $K_\rho(y)$, with $\zeta_t \geq 0$, it holds

$$\sup_{s-\theta \rho^p < t \leq s} \int_{K_\rho(y)} (u - k)_{+}^2 \zeta^p(x, t)dx - \int_{K_\rho(y)} (u - k)_{+}^2 \zeta^p(x, s - \theta \rho^p)dx$$
$$+ \varpi \int_{(y, s) + Q_\rho^- (\theta)} \Phi(u)|D(u - k)_{\pm}|^p \zeta^p dx dt$$
$$\leq \tilde{\gamma} \left( \int_{(y, s) + Q_\rho^- (\theta)} (u - k)_{+}^2 \zeta^{p-1}_t \zeta dx dt + \int_{(y, s) + Q_\rho^- (\theta)} \Phi(u)(u - k)_{+}^p |D\zeta|^p dx dt \right)$$
$$+ \varpi \int_{(y, s) + Q_\rho^- (\theta)} \left( C^p_2 \Phi(u)(u - k)_{+}^p + C^p \chi_{((u - k)_{+}>0)} \right) \zeta^p dx dt. \quad (3.1)$$

Analogous estimates hold in the cylinder $(y, s) + Q_\rho^+(\theta) \subset E_T$. 


Proof. We prove (3.1) for \((u - k)_-\). We proceed formally, multiplying both sides of (1.1) by \(-(u - k)_-\) and integrating on \(K_p(y) \times (s - \theta \rho^p, \tau]\), where \(s - \theta \rho^p < \tau \leq s\). As in general \(u_t\) does not make sense for a weak solution, to give a rigorous proof of (3.1) we need to introduce the Steklov averages of \(u\). We refer the reader to Proposition 3.1 of Chapter II in [5] for details. We obtain

\[
- \int_{K_p(y) \times (s - \theta \rho^p, \tau]} u_t(u - k)_- \zeta^p \, dx dt
\]

Concerning the left hand side we have

\[
- \int_{K_p(y) \times (s - \theta \rho^p, \tau]} u_t(u - k)_- \zeta^p \, dx dt = \frac{1}{2} \int_{K_p(y)} \int_s^\tau [(u - k)_-^2]_t \zeta^p \, dt \, dx
\]

\[
\geq \frac{1}{2} \int_{K_p(y)} (u - k)_-^2 \zeta^p(x, \tau) \, dx - \frac{1}{2} \int_{K_p(y)} (u - k)_-^2 \zeta^p(x, s - \theta \rho^p) \, dx
\]

On the other hand, from the first condition in (1.2) it follows that

\[
- \int_{K_p(y) \times (s - \theta \rho^p, \tau]} A(x, t, u, Du) \cdot Du \chi_{\{u - k\}_- > 0} \zeta^p \, dx dt
\]

\[
\leq - C_0 \int_{K_p(y) \times (s - \theta \rho^p, \tau]} \Phi(u) |D(u - k)_-|^p \zeta^p \, dx dt
\]

\[
+ C^p \int_{K_p(y) \times (s - \theta \rho^p, \tau]} \zeta^p \chi_{\{u - k\}_- > 0} \, dx dt
\]

and from the second condition in (1.2) and Young inequality it follows that

\[
\int_{K_p(y) \times (s - \theta \rho^p, \tau]} |A(x, t, u, Du)| |D\zeta| (u - k)_- \zeta^{p-1}
\]

\[
\leq C_1 \int_{K_p(y) \times (s - \theta \rho^p, \tau]} \Phi(u) |D(u - k)_-|^{p-1} \zeta^{p-1} |D\zeta|(u - k)_-
\]

\[
+ C^{p-1} \int_{K_p(y) \times (s - \theta \rho^p, \tau]} \Phi(u) \frac{1}{2} \zeta^{p-1} |D\zeta|(u - k)_-
\]

\[
\leq \varepsilon \int_{K_p(y) \times (s - \theta \rho^p, \tau]} \Phi(u) |D(u - k)_-|^p \zeta^p + C_p^p C \int_{K_p(y) \times (s - \theta \rho^p, \tau]} \Phi(u) |D\zeta|^p (u - k)_-
\]

\[
+ \int_{K_p(y) \times (s - \theta \rho^p, \tau]} \Phi(u) |D\zeta|^p (u - k)_- C^p \int_{K_p(y) \times (s - \theta \rho^p, \tau]} \zeta^p \chi_{\{u - k\}_- > 0}.
\]
Finally, the third condition of (1.2) implies
\[
\iint_{K_{e}(y) \times (s - \theta \rho^p, s)} |B(x, t, u, Du)| (u - k)_- \, \zeta^p \, dx \, dt
\]
\[
\leq C^2 \varepsilon \iint_{K_{e}(y) \times (s - \theta \rho^p, s)} \Phi(u)(u - k)^p \, \zeta^p \, dx \, dt
\]
\[+ \varepsilon \iint_{K_{e}(y) \times (s - \theta \rho^p, s)} \Phi(u)|D(u - k)_-|^p \zeta^p \, dx \, dt
\]
\[+ C^2 \iint_{K_{e}(y) \times (s - \theta \rho^p, s)} \Phi(u)(u - k)^p \, \zeta^p \, dx \, dt
\]
\[+ C^p \iint_{K_{e}(y) \times (s - \theta \rho^p, s)} \zeta^p \chi_{\{|u - k|_+ > 0\}} \, dx \, dt.
\]
Combining all the estimates so far, choosing \( \varepsilon \) small enough, and then taking the supremum over \( \tau \) we obtain (3.1). By the same argument, we deduce estimate (3.1) with \((u - k)_+\) instead of \((u - k)_-\).

**Remark 3.2** By a simple computation, it is possible to rewrite estimate (3.1) with a slight change in the third integral on the left hand side, namely
\[
\sup_{s - \theta \rho^p < t \leq s} \int_{K_{e}(y)} (u - k)^2 \zeta^p(x, t) \, dx - \int_{K_{e}(y)} (u - k)^2 \zeta^p(x, s - \theta \rho^p) \, dx
\]
\[+ \varepsilon \int_{(y, s) + Q_+^-(\theta)} \Phi(u)|D[(u - k)_\pm]|^p \, dx \, dt
\]
\[\leq \gamma \left( \iint_{(y, s) + Q_+^-(\theta)} (u - k)^2 \zeta^{p-1} \, dx \, dt + \iint_{(y, s) + Q_+^-(\theta)} \Phi(u)(u - k)^p \chi_{\{|\zeta|_+ > 0\}} \, dx \, dt \right)
\]
\[+ \gamma \iint_{(y, s) + Q_+^-(\theta)} \left( C^2 \Phi(u)(u - k)^p \chi_{\{|u - k|_+ > 0\}} + C^p \chi_{\{|u - k|_+ > 0\}} \right) \zeta^p \, dx \, dt.
\]
possibly for different values of the constants \( \gamma, \varepsilon \).

The next result is a crucial point in our strategy, as it allows to obtain pointwise estimates for \( u \) starting from estimates of the measure of its level sets. It is a version of De Giorgi’s result (see [2, 12]), suitably adapted to our situation.

**Lemma 3.3** Let \((y, s) + Q_{2\rho}^-(\theta)\) be a cylinder contained in \(E_T\) and let \(\mu_+, \omega\) be two constants such that
\[
\mu_+ \geq \sup_{(y, s) + Q_{2\rho}^-} u, \quad \omega \geq \mu_+.
\]
Assume that \(C_{2\rho} \leq 1\). Finally, let \(\xi, a \in (0, 1)\). Then, the following two assertions hold.

(i) There exists \(\nu_-\), depending upon \(\theta, \omega, \xi, a\) and the data, such that, if
\[
\{|u \leq \xi \omega\} \cap \{(y, s) + Q_{2\rho}^-\} \leq \nu_-|Q_{2\rho}^-|\]
then either \((C\rho)^p > (\xi \omega)^{p + \alpha}\) or
\[
u \geq a \xi \omega \quad \text{in} \quad (y, s) + Q_{\rho}^-.
\]

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(ii) There exists $\nu_+$, depending upon $\mu_+, \omega, \theta, \xi, a$ and the data, such that, if
$$
\{u \geq \mu_+ - \xi \omega\} \cap \{(y, s) + Q_{2p}^-(\theta)\} \subseteq \nu_+ |Q_{2p}^-(\theta)|
$$
then either $(C\nu)^p > \mu_+^a(\xi \omega)^p$ or
$$
u = \mu_+ - a \xi \omega \quad \text{in } (y, s) + Q_{2p}^-(\theta).
$$

**Proof.** We limit ourselves to proving (i) in the case when $(y, s) = (0, 0)$. This is always possible by using a translation. To keep $u$ away from 0, we define
$$
v = \max\{u, a \xi \omega\}.
$$
We set
$$
\rho_n = \rho + \frac{\rho}{2^n}, \quad K_n = K_{\rho_n}, \quad Q_n = K_n \times (-\theta \rho_n, 0],
$$
for $n = 0, 1, 2, \ldots$ and we choose $\zeta(x, t) = \zeta_1(x)\zeta_2(t)$ as a cutoff function on $Q_n$ such that
$$
\zeta_1 = \left\{ \begin{array}{ll}
1 & \text{in } K_{n+1} \cap K_n, \\
0 & \text{in } \mathbb{R}^N \setminus K_n
\end{array} \right.,
$$
and
$$
\zeta_2 = \left\{ \begin{array}{ll}
0 & \text{if } t \leq -\theta \rho_n^p \\
1 & \text{if } t \geq -\theta \rho_{n+1}^p
\end{array} \right., \quad 0 \leq (\zeta_2)_{t} \leq \frac{2^{p(n+1)} \rho}{\theta \rho^p}.
$$
We apply the energy estimates (3.2) on $Q_n$, for $(u - k_n)_-$ and $\zeta$ defined as above, getting
$$
\sup_{-\theta \rho_n^p < t < 0} J_{K_n} (u - k_n)^2 \zeta^p (x, t) dx + \int_{Q_n} \Phi(u) |D((u - k_n)_- \zeta)|^p dx dt
$$
$$
\leq \gamma \int_{Q_n} (u - k_n)^2 \zeta^p (x, t) dx dt + \gamma \int_{Q_n} \Phi(u) (u - k_n)_-^p |D\zeta|^p dx dt
$$
$$
+ \gamma \int_{Q_n} \left( C_{2p}^p \Phi(u) (u - k_n)_-^p + C_{p,1}^p (u - k_n)_- (u - k_n)_- > 0 \right) \zeta^p dx dt.
$$
(3.5)
At this point, we need to estimate the left hand side from below and the right hand side from above. As $\xi \omega \geq (u - k_n)_- \geq (v - k_n)_-$ and $p \geq 2$, we easily have
$$
\int_{K_n} (u - k_n)^2 \zeta^p (x, t) dx \geq \int_{K_n} (v - k_n)^2 \zeta^p (x, t) dx \geq (\xi \omega)^2 \int_{K_n} (v - k_n)^p \zeta^p (x, t) dx.
$$
Moreover,
$$
\gamma_1 (a \xi \omega)^p \int_{Q_n} |D((u - k_n)_- \zeta)|^p dx dt \leq \int_{Q_n} \Phi(v) |D((u - k_n)_- \zeta)|^p dx dt
$$
$$
= \int_{Q_n \cap \{u = v\}} \Phi(u) |D((u - k_n)_- \zeta)|^p dx dt
$$
$$
+ \int_{Q_n \cap \{u < v\}} \Phi(a \xi \omega)(v - k_n)_-^p |D\zeta|^p dx dt
$$
$$
\leq \int_{Q_n} \Phi(u) |D((u - k_n)_- \zeta)|^p dx dt
$$
$$
+ \int_{Q_n} \gamma_2 (a \xi \omega)^p (u - k_n)_-^p |D\zeta|^p dx dt.
$$
Using (3.3), (3.4) and noticing that $u \leq \xi \omega$, when $u - k_n > 0$, by (3.5) and the previous estimates we get

$$\sup_{-\theta \rho < t \leq 0} (\xi \omega)^{2-n} \int_{K_n} (v - k_n)_{x,t}^{p}(x,t) dx + \omega \gamma_1(a \xi \omega)^{p} \int_{Q_n} |D[(v - k_n)_{-\zeta}]|^{p} dx d\tau \leq \gamma \frac{2^{p(n+1)}}{\rho^p} (\xi \omega)^{p} |A_n| \left( \frac{1}{\theta(\xi \omega)^{p-2}} + (\xi \omega)^{\alpha} + (C_2 \rho)^{p} (\xi \omega)^{\alpha} + \frac{(C \rho)^{p}}{(\xi \omega)^{p}} \right),$$

where $A_n \overset{\text{def}}{=} \{u < k_n\} \cap Q_n$. Note that, by the definition of $\nu$, there holds $A_n = \{v < k_n\} \cap Q_n$, for every $n$. Assuming $(C \rho)^{p} \leq (\xi \omega)^{p+\alpha}$, and recalling that $C_2 \rho \leq 1$, we can estimate

$$\sup_{-\theta \rho < t \leq 0} (\xi \omega)^{2-n} \int_{K_n} (v - k_n)_{x,t}^{p}(x,t) dx + \omega \gamma_1(a \xi \omega)^{p} \int_{Q_n} |D[(v - k_n)_{-\zeta}]|^{p} dx d\tau \leq \gamma \frac{2^{p(n+1)}}{\rho^p} (\xi \omega)^{p} |A_n| \left( \frac{1}{\theta(\xi \omega)^{p-2}} + (\xi \omega)^{\alpha} \right). \quad (3.6)$$

Applying Hölder inequality and recalling that $\zeta = 1$ on $Q_{n+1}$, it turns out that

$$\left( \frac{1-a}{2n+1} \right)^{p} (\xi \omega)^{p} |A_{n+1}| \leq \int_{Q_{n+1}} (v - k_n)_{x,t}^{p} dx d\tau \leq \left( \int_{Q_n} [(v - k_n)_{-\zeta}]^{p \frac{N+p}{N}} dx d\tau \right)^{\frac{N}{N+p}} |A_n|^{\frac{p}{N+p}}. \quad (3.7)$$

From Proposition 3.1 of Chapter I in [5], it follows that the right hand side of (3.7) can be estimated by

$$\gamma \left( \int_{Q_n} |D[(v - k_n)_{-\zeta}]|^{p} dx d\tau \right)^{\frac{N}{N+p}} \times \left( \sup_{-\theta \rho < t \leq 0} \int_{K_n} [(v - k_n)_{-\zeta}]^{p}(x,t) dx \right)^{\frac{p}{N+p}} |A_n|^{\frac{p}{N+p}}, \quad (3.8)$$

where $\gamma$ depends only upon $N,p$. Now, combining estimates (3.7), (3.8) and (3.6) we find

$$|A_{n+1}| \leq \gamma \frac{2^{2np}}{(1-a)^{p} \rho^p} a^{\frac{N}{N+p}} (\xi \omega)^{-a \frac{(p-2)}{N+p}} \left( \frac{1}{\theta(\xi \omega)^{p-2}} + (\xi \omega)^{\alpha} \right) |A_n|^{1+ \frac{p}{N+p}}.$$

Setting $Y_n = |A_n|$, the last inequality can be rewritten as

$$Y_{n+1} \leq \gamma \frac{2^{2np}}{a^{\frac{N}{N+p}} (1-a)^{p} \theta(\xi \omega)^{p-2}} \frac{\left( \frac{1}{\theta(\xi \omega)^{p-2}} + (\xi \omega)^{\alpha} \right)^{\frac{N}{N+p}}}{\left( \frac{1}{\theta(\xi \omega)^{p-2}} + (\xi \omega)^{\alpha} \right)^{\frac{N}{N+p}} Y_n^{1+ \frac{p}{N+p}}}.$$

If $|A_0| \leq \nu_- |Q_0|$, where

$$\nu_- = \frac{a^{\frac{N}{N+p}} (1-a)^{p} \left( \frac{1}{\theta(\xi \omega)^{p-2}} + (\xi \omega)^{\alpha} \right)^{\frac{N}{N+p}}}{{1 + \frac{1}{\theta(\xi \omega)^{p-2}} + (\xi \omega)^{\alpha}}^{\frac{N}{N+p}}}, \quad (3.9)$$

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then Lemma 4.1 of Chapter I in [5] implies that $Y_n \to 0$. This means that

$$ u \geq a \xi \omega \quad \text{in } Q_p^- (\theta), $$

which is the thesis.

The second part of the statement can be proved more easily, since we do not need to introduce any truncation of $u$. It suffices to replace $k_n$ with $\tilde{k}_n = \mu_+ - \xi_\omega$ and to apply the energy estimates (3.2) for $(u - \tilde{k}_n)_+$. For the sequel, we just need to know the explicit expression of $\nu_+$ which is given by

$$\nu_+ = \gamma (1 - a)^{N + p} \frac{\theta (\xi \omega)^{p - 2} (\mu_+ - \xi \omega)^{\alpha}}{(1 + \theta (\xi \omega)^{p - 2} \mu_+^{\alpha})^{\frac{N + p}{p}}}.$$  

(3.10)

## 4 Expansion of positivity

The main result of the present section is given in the following proposition, which asserts that if $u$ is strictly positive on a cube $K_\rho(y)$ at some level $s$, then it remains strictly positive in a larger cube in a proper time interval, depending on the amount of positivity of $u$ itself.

**Proposition 4.1** Assume that

$$ u(x,s) \geq \xi M, \quad x \in K_{2\rho}(y), $$

(4.1)

for some $(y,s) \in E_T$, $M > 0$, $\xi \in (0,1]$ with $C_2 \rho \leq 1$. Then there exist positive constants $\gamma$, $b$ and $\eta$, with $\eta \in (0,1)$, depending only on the data, such that either $(C \rho)^p > \gamma (\xi M)^{p + \alpha}$ or

$$ u(x,t) \geq \eta (\xi M), $$

for $x \in K_{4\rho}(y)$ and every $t$ such that

$$ s + \left( \frac{b}{\eta \xi M} \right)^{p + \alpha - 2} (16^p - 4^p) \rho^p \leq t \leq s + \left( \frac{b}{\eta \xi M} \right)^{p + \alpha - 2} (16\rho)^p. $$

We point out that the time interval where the solution remains strictly positive is definitely contained in the domain $E_T$ since, in Theorem 2.1, we require that a large working cylinder is contained in $E_T$.

In order to simplify the notation we suppose, without loss of generality, that $(y,s) = (0,0)$. In the sequel, without mentioning it explicitly, we assume that the hypotheses of Proposition 4.1 are fulfilled. Moreover, we deal only with the case where $p + \alpha - 2 > 0$, which is more significant. Arguing as in [6], one can see that the constants involved in the proposition above are stable as $p + \alpha - 2 \to 0^+$.

**Lemma 4.2** Let $a \in (0,1)$. Then there exists $\delta \in (0,1)$, depending only upon $a$ and the data, such that, if $Q_{2\rho}^- (\theta) \subset E_T$ and

$$ |\{u \leq \xi M \} \cap Q_{2\rho}^+ (\theta)| \leq \frac{\delta}{\theta (\xi M)^{p + \alpha - 2}} |Q_{2\rho}^+ (\theta)|, $$

(4.2)

then either $(C \rho)^p > (\xi M)^{p + \alpha}$ or

$$ u \geq a \xi M \quad \text{in } K_{\rho} \times (0,\theta(2\rho)^p). $$

(4.3)
Proof. Let us consider
\[ \rho_n = \rho + \frac{\rho}{2^n}, \quad K_n = K_{\rho_n}, \quad \tilde{Q}_n = K_n \times (0, \theta(2\rho)^p], \quad \xi_n = a\xi + \frac{1-a}{2^n}\xi, \]
and a cutoff function \( \zeta(x,t) = \zeta(x) \) independent of \( t \) and satisfying (3.3). We define
\[ \tilde{v} = \max\{a\xi M, u\}. \]
From the energy estimates (3.2) for \( (u - \xi_n M)_- \), \( \tilde{Q}_n \) and \( \zeta \), arguing as in Lemma 3.3, it follows that
\[
\begin{align*}
\sup_{0 < t \leq \theta(2\rho)^p} (\xi M)^{2-p} & \int_{K_n} (\tilde{v} - \xi_n M)_- (x,t) \zeta^p(x)dx + \varpi_1 (a\xi M)^{\alpha} \sum_{Q_n} |D[(\tilde{v} - \xi_n M)_- \zeta]|^p dxd\tau \\
& \leq \gamma \frac{2^{p(n+1)}}{\rho^p} (\xi M)^p \left( (\xi M)^{\alpha} + C_\rho^p \rho^p (\xi M)^{\alpha} + \frac{C\rho^p}{(\xi M)^p} \right) |\tilde{A}_n|,
\end{align*}
\]
where \( \tilde{A}_n \equiv \{u < \xi_n M\} \cap \tilde{Q}_n = \{\tilde{v} < \xi_n M\} \cap \tilde{Q}_n, n = 0, 1, 2, \ldots \) Note that the integral on the lower side of the cylinder \( \tilde{Q}_n \) vanishes as a consequence of assumption (4.1) and the fact that \( \xi_n \leq \xi \). At this point, assuming \( (C\rho)^p \leq (\xi M)^{p+\alpha} \), we get
\[
\sup_{0 < t \leq \theta(2\rho)^p} (\xi M)^{2-p} \int_{K_n} (\tilde{v} - \xi_n M)_- (x,t) \zeta^p(x)dx \\
+ \varpi_1 (a\xi M)^{\alpha} \sum_{Q_n} |D[(\tilde{v} - \xi_n M)_- \zeta]|^p dxd\tau \\
\leq \gamma \frac{2^{p(n+1)}}{\rho^p} (\xi M)^{p+\alpha} |\tilde{A}_n|.
\]
After some computations, which are completely similar to those in the proof of Lemma 3.3, we obtain
\[
|\tilde{A}_{n+1}| \leq \gamma \frac{2^{2np}}{a^{\frac{\alpha}{p+\alpha}}(1-a)^p \rho^p} (\theta(\xi M)^{p+\alpha-2}) \frac{\rho^p}{\alpha N} \tilde{Y}_n^{1+\frac{p}{p+\alpha}}. \tag{4.4}
\]
Setting \( \tilde{Y}_n = |\tilde{A}_n|/|Q_n| \), (4.4) turns out to be
\[
\tilde{Y}_{n+1} \leq \gamma \frac{2^{2np}}{a^{\frac{\alpha}{p+\alpha}}(1-a)^p} (\theta(\xi M)^{p+\alpha-2}) \frac{\rho^p}{\alpha N} \tilde{Y}_n^{1+\frac{p}{p+\alpha}}.
\]
The thesis follows from Lemma 4.1 of Chapter I in [5], provided \( \tilde{Y}_0 \leq \nu \), with
\[
\nu = \frac{\delta}{\theta(\xi M)^{p+\alpha-2}},
\]
where
\[
\delta = \gamma a^{\frac{2N}{p+\alpha}} (1-a)^{N+p}. \ ■
\]
From now on we assume that \( (C\rho)^p \leq (\xi M)^{p+\alpha} \) and \( p + \alpha - 2 > 0 \). As a consequence of Lemma 4.2, we observe that choosing \( \theta = \delta(\xi M)^{2-p-\alpha} \), hypothesis (4.2) is obviously true and therefore (4.3) gives, in particular,
\[
u \left( x, \frac{\delta \rho^p}{(\xi M)^{p+\alpha-2}} \right) \geq a\xi M \quad \text{in } K_\rho. \tag{4.5}
\]
For every $\tau \geq 0$ we set 
\[
\xi_{\tau} = \frac{\xi}{f(\tau)}, \quad \text{where} \quad f(\tau) = e^{\frac{\tau+p}{\gamma_1}},
\]
Since $\xi_{\tau} \leq \xi$, one still has $u(x,0) \geq \xi_{\tau}M$ in $K_{2\rho}$, by (4.1), and hence, replacing $\xi$ by $\xi_{\tau}$ in (4.5) we obtain 
\[
u(x, \frac{f(\tau)}{\xi M})^{p+\alpha-2} \geq a \xi_{\tau}M, \quad \text{for all } x \in K_{2\rho} \text{ and every } \tau \geq 0.
\]
Defining 
\[
w(x, \tau) = \frac{f(\tau)}{\xi M} (\delta \rho^p)^{\frac{1}{p+\alpha-2}} u(x, \left(\frac{f(\tau)}{\xi M}\right)^{p+\alpha-2} \delta \rho^p),
\]
and fixing $a = 1/2$, we have 
\[
w(x, \tau) \geq \frac{1}{2} (\delta \rho^p)^{\frac{1}{p+\alpha-2}} \overset{\text{def}}{=} k_0
\]
for every $\tau \geq 0$ and all $x \in K_{\rho}$. Moreover, recalling that $u \geq 0$, by formal computations it is easily seen that 
\[
w_{\tau} \geq \text{div} \tilde{A}(x,\tau,w,Dw) + \tilde{B}(x,\tau,w,Dw),
\]
where 
\[
\tilde{A}(x,\tau,w,Dw) = \psi(\tau)^{p+\alpha-1} A(x, \psi(\tau)^{p+\alpha-2}, \psi^{-1}w, \psi^{-1}Dw) \\
\tilde{B}(x,\tau,w,Dw) = \psi(\tau)^{p+\alpha-1} B(x, \psi(\tau)^{p+\alpha-2}, \psi^{-1}w, \psi^{-1}Dw)
\]
and 
\[
\psi(\tau) = \frac{f(\tau)}{\xi M} (\delta \rho^p)^{\frac{1}{p+\alpha-2}}.
\]
Such a formal differential inequality can be made rigorous starting from the weak formulation (2.1), performing the corresponding change of variables from $t$ into $\tau$ and taking positive test functions. The new functions $\tilde{A}, \tilde{B}$ preserve the structure conditions (1.2). Indeed, it is easily checked that 
\[
\begin{align*}
\tilde{A}(x,\tau,w,\eta) \cdot \eta & \geq C_0^1 \Phi(w) |\eta|^p - \tilde{C}(\tau)^p, \\
|\tilde{A}(x,\tau,w,\eta)| & \leq C_1^1 \Phi(w) |\eta|^{p-1} + \tilde{C}(\tau)^{p-1} \Phi(w) \frac{1}{p}, \\
|\tilde{B}(x,\tau,w,\eta)| & \leq C_2^1 \Phi(w) |\eta|^{p-1} + C_2^1 \tilde{C}(\tau)^{p-1} \Phi(w) \frac{1}{p}
\end{align*}
\]
with 
\[
\tilde{C}(\tau) = C \psi(\tau)^{1+\frac{p}{\gamma_1}},
\]
and 
\[
\Phi(s) = \psi(\tau)^{\alpha} \Phi(\psi^{-1}(\tau)s).
\]
As a consequence of (1.3) 
\[
\gamma_1 s^{\alpha} \leq \Phi(s) \leq \gamma_2 s^{\alpha},
\]
uniformly in \( \tau \), for all \( s \geq 0 \). At this point, the energy estimates that we need for \( w \) are the following

\[
\sup_{0 < \tau \leq \theta(16\rho)^p} \int_{K_{16\rho}} (w - k)^2 \zeta^p(x, \tau) dx + \infty \int_{Q_{16\rho}^*(\theta)} \tilde{\Phi}(w)|D[(w - k)\zeta]|^p dx ds
\]

\[
\leq \gamma \int_{Q_{16\rho}^*(\theta)} (w - k)^2 \zeta \zeta dx ds + \gamma \int_{Q_{16\rho}^*(\theta)} \tilde{\Phi}(w)(w - k)^p |D\zeta|^p dx ds
\]

\[
+ \gamma \int_{Q_{16\rho}^*(\theta)} \left( C_2^p \tilde{\Phi}(w)(w - k)^p + \tilde{C}^p(s)\chi_{\{w(k) > 0\}}\right) \zeta^p dx ds, \quad (4.7)
\]

where \( \zeta \) is a piecewise smooth cutoff function in the cylinder \( Q_{16\rho}^*(\theta) \) vanishing on the parabolic boundary of \( Q_{16\rho}^*(\theta) \) and such that \( 0 \leq \zeta \leq 1, \zeta_\tau \geq 0 \).

Our aim now consists in proving the “expansion of positivity” for \( w \). Namely we are going to extend (4.6) to \( K_{2\rho} \) when \( \tau \) is sufficiently large.

**Proposition 4.3**

Set

\[
Q_{3\rho}(\theta) = K_{3\rho} \times ((16\rho)^p\theta - (8\rho)^p\theta, (16\rho)^p\theta].
\]

Then, for every \( \nu > 0 \) there exist \( \sigma \in (0,1) \), depending upon the data and \( \nu \), \( \gamma > 1 \) depending on the data and \( \sigma \), such that either \( (Cp)^p > \gamma(\xi M)^p + \alpha \) or

\[
\{w < \sigma k_0 \} \cap Q_{3\rho}(\theta) \leq \nu|Q_{3\rho}(\theta)|,
\]

with \( \theta_\sigma = (\sigma k_0)^{p+\alpha-2} \) and \( k_0 \) given in (4.6).

**Proof.** Introduce the levels

\[
k_j = \frac{k_0}{2^j}, \quad j = 0, 1, \ldots, j_*,
\]

with \( j_* \in \mathbb{N}, j_*>1 \). Fix \( j \in \{0, \ldots, j_* - 2\} \) and set

\[
v_* = \max\{k_{j+2}, w\}.
\]

By writing the energy estimates (4.7) for \((w - k)_-\) and choosing a test function \( \zeta \) such that

\[
\zeta = 1 \quad \text{in} \quad Q_{3\rho}(\theta), \quad |D\zeta| \leq \frac{1}{8\rho}, \quad 0 \leq \zeta \leq \frac{1}{\theta(8\rho)^p},
\]

we obtain

\[
\int_{Q_{3\rho}(\theta)} \tilde{\Phi}(w)|D(w - k)_-|^p dx ds \leq \gamma \left( \frac{k_j^2}{\theta(8\rho)^p} + \frac{k_j^{\alpha + \alpha}}{(8\rho)^p} + C_2^p k_j^{p + \alpha} + \left[ \tilde{C}((16\rho)^p\theta) \right]^p \right)|Q_{3\rho}(\theta)|.
\]

It is immediate to see that

\[
\int_{Q_{3\rho}(\theta)} \tilde{\Phi}(w)|D(w - k)_-|^p dx ds \geq \int_{Q_{3\rho}(\theta) \cap \{v_* = w\}} \tilde{\Phi}(v_*)|D(v_* - k_j)_-|^p dx ds
\]

\[
\geq \gamma_1 k_j^{\alpha + 2} \int_{Q_{3\rho}(\theta)} |D(v_* - k_j)_-|^p dx ds.
\]
Setting $\theta = \theta_\ast = k_j^{2-p-\alpha}$, by means of the last two inequalities, it turns out that

$$
\iint_{Q_{8\rho}(\theta_\ast)} |D(v_\ast - k_j)_-|^p dx ds \leq \gamma \left( \frac{k_j^p}{(8\rho)^p} \right) \left( \frac{k_j^{2-p} k_j^\alpha}{\theta_\ast} + \frac{k_j^\alpha}{k_j^{j+2}} + \frac{C_0^p k_j^\alpha (8\rho)^p}{k_j^{j+2}} \right) + \left[ \tilde{C}((16\rho)^p \theta_\ast)^p (8\rho)^p \right] \left( \frac{\theta_\ast}{k_j^\alpha} \right) |Q_{8\rho}(\theta_\ast)|.
$$

It is easily seen that

$$
\frac{k_j^{2-p} k_j^\alpha}{\theta_\ast} \leq \gamma (\text{data}), \quad \frac{k_j^\alpha}{k_j^{j+2}} \leq \gamma (\text{data}).
$$

Moreover, recalling the definitions of $\tilde{C}$ and $k_0$, we obtain

$$
\left[ \tilde{C}((16\rho)^p \theta_\ast)^p (8\rho)^p \right] \left( \frac{\theta_\ast}{k_j^\alpha} \right) \leq \gamma (\text{data}; j_\ast),
$$

as $\rho \theta_\ast = \gamma (\text{data}; j_\ast)$. Thus, assuming that

$$
(C\rho)^p \leq \gamma^{-1} (\text{data}; j_\ast) (\xi M)^{p+\alpha}
$$

we have

$$
\iint_{Q_{8\rho}(\theta_\ast)} |D(v_\ast - k_j)_-|^p dx ds \leq \gamma \left( \frac{k_j^p}{(8\rho)^p} \right) |Q_{8\rho}(\theta_\ast)|, \quad (4.8)
$$

with $\gamma$ depending only on the data. Now, set

$$
A_j(\tau) = \{ v_\ast (\cdot, \tau) < k_j \} \cap K_{8\rho}, \quad A_j = \{ v_\ast < k_j \} \cap Q_{8\rho}(\theta_\ast),
$$

and notice that $A_j(\tau) = \{ w(\cdot, \tau) < k_j \} \cap K_{8\rho}, A_j = \{ w < k_j \} \cap Q_{8\rho}(\theta_\ast)$, and the same holds true with $j + 1$ replacing $j$, due to the choice of $v_\ast$. Moreover

$$
|A_j| = \int_{\theta_\ast ((16\rho)^p}^{\theta_\ast (8\rho)^p} |A_j(\tau)| d\tau.
$$

From Lemma 2.2 of Chapter I in [5] it follows that

$$
(k_j - k_{j+1}) |A_{j+1}(\tau)| \leq \frac{\gamma \rho^{N+1}}{|K_{8\rho} \setminus A_j(\tau)|} \int_{A_j(\tau) \setminus A_{j+1}(\tau)} |Dv_\ast| dx, \quad (4.9)
$$

for every $\tau \in (\theta_\ast (16\rho)^p - \theta_\ast (8\rho)^p, \theta_\ast (16\rho)^p]$. On the other hand, by (4.6), we have

$$
|K_{8\rho} \setminus A_j(\tau)| \geq |K_\rho| = \rho^N
$$
and, consequently, (4.9) gives
\[ \frac{1}{2} k_j |A_{j+1}| \leq \gamma \rho \int_{A_j \setminus A_{j+1}} |Dv_*| dx. \]
Integrating both sides of the above inequality with respect to \( \tau \) running in the interval \((\theta_* \rho^p(16\rho^p - 8\rho), \theta_* (16\rho)^p]\), applying Hölder inequality and using (4.8), we get
\[
\frac{1}{2} k_j |A_{j+1}| \leq \gamma \rho \left( \int_{A_j \setminus A_{j+1}} |Dv_*|^p dx d\tau \right)^{\frac{1}{p}} |A_j \setminus A_{j+1}|^{\frac{p-1}{p}} \leq \gamma \rho \left( \frac{k_j^p}{(8\rho)^p} |Q_{\kappa_0}(\theta_*)| \right)^{\frac{1}{p}} |A_j \setminus A_{j+1}|^{\frac{p-1}{p}} = \gamma k_j |Q_{\kappa_0}(\theta_*)|^{\frac{1}{p}} |A_j \setminus A_{j+1}|^{\frac{p-1}{p}}.
\]
Summing over \( j \) from 0 to \( j_* - 2 \) leads to
\[
\sum_{j=0}^{j_*-2} |A_{j+1}|^{\frac{p}{p-1}} \leq \gamma |Q_{\kappa_0}(\theta_*)|^{\frac{1}{p}} \sum_{j=0}^{j_*-2} |A_j \setminus A_{j+1}|.
\]
Finally, since \( A_{j+1} \subset A_j \subset A_0 \subset Q_{\kappa_0}(\theta_*) \) for every \( j \), we easily deduce that
\[
(j_* - 1) |A_{j_*-1}|^{\frac{p}{p-1}} \leq \gamma |Q_{\kappa_0}(\theta_*)|^{\frac{1}{p}} \sum_{j=0}^{j_*-2} (|A_j| - |A_{j+1}|) \leq \gamma |Q_{\kappa_0}(\theta_*)|^{\frac{1}{p}}.
\]
Thus, we have established that
\[
|A_{j_*-1}| \leq \left( \frac{\gamma}{j_* - 1} \right)^{\frac{p}{p-1}} |Q_{\kappa_0}(\theta_*)|.
\]
At this point the statement follows immediately. Indeed, for any \( \nu > 0 \), we can choose \( j_* \) large enough to have \( \left( \frac{\gamma}{j_* - 1} \right)^{\frac{p}{p-1}} \leq \nu \). Setting \( \sigma = 1/2^{j_*-1} \in (0,1) \) we conclude that
\[
|\{ w < \sigma k_0 \} \cap Q_{\kappa_0}(\theta_*)| = |A_{j_*-1}| \leq \nu |Q_{\kappa_0}(\theta_*)|.
\]

**Proposition 4.4** There exist \( \sigma \in (0,1) \) and \( \gamma > 1 \), depending only upon the data, such that either \( (C\rho)p > \gamma (\xi M)^{p+\alpha} \) or
\[
|w(x,\tau) \geq \frac{1}{2} \sigma k_0 \ \text{ in } \ K_{\xi_\rho} \times \left[ \frac{(16\rho^p - 4\rho)^p}{\sigma k_0} + \frac{(16\rho^p)^p}{\xi_\rho^p} \right].
\]
**Proof.** We first observe that \( Q_{\kappa_0}(\theta_*) = (0,\tau_*) + Q^-_{\kappa_0}(\theta_*) \), where \( \tau_* = \theta_*(16\rho)^p \). Then, applying Lemma 3.3 (i) to the function \( w \) over the cylinder \((0,\tau_*) + Q^-_{\kappa_0}(\theta_*)\) with the choice \( a = \frac{1}{2} \) and \( \xi_\rho \) replaced by \( \sigma k_0 \), we find that if
\[
\frac{|\{ w < \sigma k_0 \} \cap (0,\tau_*) + Q^-_{\kappa_0}(\theta_*)|}{|Q_{\kappa_0}(\theta_*)|} \leq \gamma \frac{[\theta_*(\sigma k_0)^{p+\alpha-2}]^{\frac{p}{p-1}}}{[1 + \theta_*(\sigma k_0)^{p+\alpha-2}]^{\frac{p}{p-1}}} = \delta_*,
\]

(4.10)
with $\gamma$ depending only on the data, then either $C_p \rho^p > \gamma(\xi M)^{p+\alpha}$ or

\[ w(x, \tau) \geq \frac{1}{2} \sigma k_0 \quad \text{in} \quad (0, \tau_*) + Q^\pm_{4\rho}(\theta_*) \]

Note that $\delta_*$ depends only on the data since we have $\theta_*(\sigma k_0)^{p+\alpha-2} = 1$, by definition of $\theta_*$. Applying Proposition 4.3 with $\nu = \delta_*$, we ensure condition (4.10) and hence the assertion is proved. ■

**Proof of Proposition 4.1.** To prove the claim, it suffices to translate Proposition 4.4 in the original variables. As $\tau$ ranges over the interval

\[ \left( \frac{(16\rho - 4)^{p} \rho}{(\sigma k_0)^{p+\alpha-2}}, \frac{(16\rho)^{p}}{(\sigma k_0)^{p+\alpha-2}} \right) \]

then, recalling the definition of $k_0$, we find that

\[ b_1 \overset{\text{def}}{=} \exp \left\{ \frac{2p+\alpha-2 (16\rho - 4)^{p}}{p + \alpha - 2} \sigma k_0 \right\} < \begin{cases} f(\tau) \leq \exp \left\{ \frac{2p+\alpha-2 16\rho}{p + \alpha - 2} \sigma k_0 \right\} \overset{\text{def}}{=} b_2, \end{cases} \]

where $\sigma, \delta$ are given by Proposition 4.4 and Lemma 4.2, respectively. It is worth observing that $b_1$ and $b_2$ depend only upon the data and are independent of $\rho, M$ and $u$. Concerning $u$ we obtain

\[ u(x, t) \geq \frac{\sigma \xi M}{4b_2} \overset{\text{def}}{=} \eta \xi M \quad (4.11) \]

for all $x \in K_{4\rho}$ and every $t$ such that

\[ \left( \frac{b_1}{\xi M} \right)^{p+\alpha-2} \delta \rho^p < t \leq \left( \frac{b_2}{\xi M} \right)^{p+\alpha-2} \delta \rho^p, \]

or, equivalently,

\[ \left( \frac{b_1 \sigma}{4b_2 \eta \xi M} \right)^{p+\alpha-2} \delta \rho^p < t \leq \left( \frac{\sigma}{4\eta \xi M} \right)^{p+\alpha-2} \delta \rho^p. \]

Choosing $b$ with the following property

\[ \frac{b_1 \sigma}{4b_2} \left( \frac{\delta}{16\rho - 4} \right)^{\frac{1}{p+\alpha-2}} < b \leq \frac{\sigma}{4} \left( \frac{\delta}{16\rho} \right)^{\frac{1}{p+\alpha-2}}, \]

we infer that (4.11) holds true in $K_{4\rho}$ for every $t$ such that

\[ \left( \frac{b}{\eta \xi M} \right)^{p+\alpha-2} (16\rho - 4)^{p} \rho^p \leq t \leq \left( \frac{b}{\eta \xi M} \right)^{p+\alpha-2} (16\rho)^{p}. \]

**5 Intrinsic Harnack inequality**

Let us fix a point $(x_0, t_0) \in E_T$ with $u(x_0, t_0) > 0$. Let us consider the intrinsic cylinders

\[ (x_0, t_0) + Q^\pm_{4\rho}(\theta), \quad \theta = \left( \frac{c}{u(x_0, t_0)} \right)^{p+\alpha-2}, \]

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where $c$ has to be determined. By operating the following change of variables

$$x' = \frac{x - x_0}{\rho}, \quad t' = u(x_0, t_0)^{p+\alpha-2} \frac{t - t_0}{\rho^{p}}.$$ 

these cylinders become

$$Q^+ = K_4 \times (0, 4^p e^{p+\alpha-2} [0,\infty), \quad Q^- = K_4 \times (-4^p e^{p+\alpha-2}, 0].$$

Moreover, the rescaled function

$$v(x', t') = \frac{1}{u(x_0, t_0)} u \left( x_0 + \rho x', t_0 + \frac{t' \rho^p}{u(x_0, t_0)^{p+\alpha-2}} \right)$$

satisfies $v(0, 0) = 1$ and is a non-negative, local weak solution of

$$v_t' - \text{div}_{x'} A(x', t', v, D_{x'} v) = B(x', t', v, D_{x'} v)$$

with

$$A(x', t', v, D_{x'} v) = \frac{\rho^{p-1}}{u(x_0, t_0)^{p+\alpha-1}} \times$$

$$A \left( x_0 + \rho x', t_0 + \frac{t' \rho^p}{u(x_0, t_0)^{p+\alpha-2}}, u(x_0, t_0)v, \frac{u(x_0, t_0)}{\rho} D_{x'} v \right)$$

$$B(x', t', v, D_{x'} v) = \frac{\rho^p}{u(x_0, t_0)^{p+\alpha-1}} \times$$

$$B \left( x_0 + \rho x', t_0 + \frac{t' \rho^p}{u(x_0, t_0)^{p+\alpha-2}}, u(x_0, t_0)v, \frac{u(x_0, t_0)}{\rho} D_{x'} v \right).$$

One can check the following structure conditions

$$\left\{ \begin{array}{c} A(x', t', v, \eta) \cdot \eta \geq C_0 \Psi(v) |\eta|^p - \bar{C}^p, \\ |A(x', t', v, \eta)| \leq C_1 \Psi(v) |\eta|^{p-1} + \bar{C}^{p-1} \Phi(v)^{\frac{1}{p}} \\ |B(x', t', v, \eta)| \leq \bar{C}_2 \Psi(v) |\eta|^{p-1} + \bar{C}_2 \bar{C}^{p-1} \Psi(v)^{\frac{1}{p}}, \end{array} \right.$$ 

with

$$\bar{C}_2 = C_2 \rho, \quad \bar{C} = \frac{C \rho}{u(x_0, t_0)^{1+\frac{\alpha}{p}}},$$

and

$$\Psi(s) = \frac{\Phi(u(x_0, t_0)s)}{u(x_0, t_0)^{\alpha}}.$$ 

Note that, as a consequence of (1.3), we have

$$\gamma_1 s^\alpha \leq \Psi(s) \leq \gamma_2 s^\alpha,$$

for all $s \geq 0$.

In order to keep the notation simple, from now on we will write $(x, t)$ instead of $(x', t')$. Moreover, we suppose that $\bar{C}_2 \leq 1$. Establishing Theorem 2.1 is equivalent to proving the following theorem.
Theorem 5.1 There exist constants \( \gamma, \gamma_0 > 0 \) and \( \gamma_1 > 1 \) which depend only upon the data, such that either \( u(x_0, t_0)^{p+\alpha} < \gamma(C\rho)^p \) or \( v(x, \gamma_1) \geq \gamma_0 \) a.e. in \( K_1 \).

We split the proof of Theorem 5.1 into three simpler steps.

**First Step.** Let us introduce the family of nested cylinders \( \{Q_\tau\}, \tau \in [0,1) \) defined by
\[
Q_\tau = Q^{-\tau}_{(1)} = K_\tau \times (-\tau^p, 0],
\]
and the families of non-negative numbers \( \{m_\tau\} \) and \( \{n_\tau\} \) given by
\[
m_\tau = \sup_{Q_\tau} v, \quad n_\tau = (1-\tau)^{-\beta},
\]
where \( \beta > 0 \) is a parameter to be chosen. We point out that the choice of \( \beta \) will involve only the data. Therefore, all the subsequent quantities depending on \( \beta \) will depend on the data as soon as \( \beta \) will be fixed.

Let \( \tau_0 \) be the largest root of the equation \( m_\tau = n_\tau \). It exists because \( m_0 = n_0 = 1 \) and \( n_\tau \to +\infty \) as \( \tau \to 1^- \), while \( m_\tau \) remains bounded. Since \( v \) is continuous, there exists \( (\bar{x}, \bar{t}) \in Q_{\tau_0} \) such that
\[
v(\bar{x}, \bar{t}) = n_{\tau_0} = (1-\tau_0)^{-\beta}.
\]
(5.2)

Moreover \( (\bar{x}, \bar{t}) + Q_{1-\tau_0} \subset Q_{1+\tau_0} \subset Q_1 \), so we have
\[
\sup_{(\bar{x}, \bar{t}) + Q_{1-\tau_0}} v \leq \sup_{Q_{1+\tau_0}} v < 2^\beta (1-\tau_0)^{-\beta}.
\]
(5.3)

Let us consider the cylinder \( (\bar{x}, \bar{t}) + Q_{R_0}(\theta_0) \), with
\[
R_0 = \frac{1-\tau_0}{2}, \quad \theta_0 = M_0^{2-\alpha-p}, \quad M_0 = 2^\beta (1-\tau_0)^{-\beta},
\]

In order to employ the “expansion of positivity” (Proposition 4.1), we need to find a time level at which the function \( v \) is strictly positive over a whole cube. This is done in the next step, by using a measure-theoretical argument.

**Second Step.** We need the following technical lemma.

**Lemma 5.2** Assume that
\[
\int_{Q_1} |Dw|^p dx dt \leq \alpha, \quad \left| \left\{ w > \frac{1}{2} \right\} \cap Q_1 \right| > \mu.
\]

Then, there exists \( \bar{s} \in (-1, -\mu/4] \) such that
\[
\int_{K_1} |Dw(\cdot, \bar{s})|^p dx \leq \frac{2\alpha}{\mu} \quad \text{and} \quad \left| \left\{ w(\cdot, \bar{s}) \geq \frac{1}{2} \right\} \cap K_1 \right| \geq \frac{\mu}{2}.
\]
(5.4)

**Proof** See [6, Lemma 9.1].
Proposition 5.3 One has either \( C > 1 \) or

\[
|\{v \geq 2^{-(\beta + 1)} M_0\} \cap \{(\bar{x}, \bar{t}) + Q_{\frac{\tau_0}{2}}(\theta_0)\}| > \nu|Q_{\frac{\tau_0}{2}}(\theta_0)|, \tag{5.5}
\]

where \( \nu \) is defined by (3.10) with the choices \( \xi = 1 - 2^{-\beta - 1}, a = \xi^{-1}(1 - 3/2^{3+2}) \), and \( \mu_+ = \omega = M_0, \theta = \theta_0 \). Note that \( \nu \) depends on the data and \( \beta \).

Proof. If \(|\{v \geq 2^{-(\beta + 1)} M_0\} \cap \{(\bar{x}, \bar{t}) + Q_{\frac{\tau_0}{2}}(\theta_0)\}| \leq \nu|Q_{\frac{\tau_0}{2}}(\theta_0)|\), and \( \bar{C} \leq 1 \) then, by Lemma 3.3 (ii), with the indicated choice of the involved parameters, one gets

\[
v(\bar{x}, \bar{t}) \leq \frac{3}{4}(1 - \tau_0)^{-\beta},
\]

which would be a contradiction of (5.2). \( \blacksquare \)

From now on we assume that \( \bar{C} \leq 1 \). It follows that (5.5) is true, hence the set where \( v \) is bounded away from a given quantity occupies a sizable portion of the cylinder \((\bar{x}, \bar{t}) + Q_{\frac{\tau_0}{2}}(\theta_0)\). The next proposition asserts that there exists at least one subcylinder such that \( v \) remains large in any arbitrarily prefixed large portion of the subcylinder.

Proposition 5.4 For every \( \lambda_0 \in (0, 1) \) and for every \( \nu_0 \in (0, 1) \), there exist \((y, s) \in (\bar{x}, \bar{t}) + Q_{\frac{\tau_0}{2}}(\theta_0)\), a constant \( \eta_0 \in (0, 1) \), depending only upon the data, \( \nu_0, \lambda_0, \beta \), such that

\[
(y, s) + Q_{\frac{\tau_0}{2}R_0}(\theta_0) \subset (\bar{x}, \bar{t}) + Q_{\frac{\tau_0}{2}}(\theta_0),
\]

and

\[
|\{v < \lambda_02^{-(\beta + 1)} M_0\} \cap \{(y, s) + Q_{\frac{\tau_0}{2}R_0}(\theta_0)\}| \leq \nu_0|Q_{\frac{\tau_0}{2}R_0}(\theta_0)|. \tag{5.6}
\]

Proof. We set

\[
k = \frac{1}{2}(1 - \tau_0)^{-\beta} = 2^{-(\beta + 1)} M_0,
\]

and we consider the cylinders \((\bar{x}, \bar{t}) + Q_{\frac{\tau_0}{2}}(\theta_0) \subset (\bar{x}, \bar{t}) + Q_{\frac{\tau_0}{2}R_0}(\theta_0) \subset Q_{\frac{\tau_0}{4}R_0} \). We write the energy estimates for \((v - k) + (\bar{x}, \bar{t}) + Q_{\frac{\tau_0}{2}R_0}(\theta_0)\), with the choice of a cutoff function \( \zeta \) such that

\[
0 \leq \zeta \leq \frac{4^p}{\theta_0^p R_0^p}, \quad |D \zeta| \leq \frac{4}{R_0} \quad \text{in} \ (\bar{x}, \bar{t}) + Q_{\frac{\tau_0}{2}}(\theta_0)
\]

\[
\zeta = 0 \quad \text{on the parabolic boundary of} \ (\bar{x}, \bar{t}) + Q_{\frac{\tau_0}{2}}(\theta_0).
\]

Since \((v - k) + (\bar{x}, \bar{t}) + Q_{\frac{\tau_0}{2}R_0}(\theta_0)\), due to (5.3), discarding the term containing the essential supremum and using the fact that \( R_0 \leq 1, 2k > 1 \), \( C, \bar{C} \leq 1 \), we
\[
\gamma_1 \left( \frac{k}{2} \right)^{\alpha} \int_{(\bar{x}, \bar{t})+Q_{R_0}(\theta_0)} |D (v - k/2)_+|^p \zeta^p \, dx \, dt \\
\leq \gamma \int_{(\bar{x}, \bar{t})+Q_{R_0}(\theta_0)} \left( \Psi(v) (v - k/2)_+ |D \zeta|^p + (v - k/2)^2 \zeta \right) \, dx \, dt \\
+ \gamma \left( \int_{(\bar{x}, \bar{t})+Q_{R_0}(\theta_0)} \left( C^p \Psi(v)(v - k/2)_+ + C^p \chi_{\{(v-k/2)_+ > 0\}} \right) \zeta^p \, dx \, dt \right) \\
\leq \gamma \frac{k^{p+\alpha}}{R_0^p} |Q_{R_0}(\theta_0)|,
\]

where \( \gamma \) is a constant depending upon the data and \( \beta \). It follows, in particular, that

\[
\int_{(\bar{x}, \bar{t})+Q_{R_0}(\theta_0)} |D (v - k/2)_+|^p \zeta^p \, dx \, dt \leq \gamma \frac{k^{p+\alpha}}{R_0^p} |Q_{R_0}(\theta_0)|.
\]  

(5.7)

Now, with respect to the new coordinates

\[
x' = \frac{2(x - \bar{x})}{R_0}, \quad t' = \frac{2^p(t - \bar{t})}{\theta_0 R_0^{2p}},
\]

the cylinder

\[
(\bar{x}, \bar{t}) + Q_{R_0}(\theta_0)
\]

becomes

\[
Q_1 = K_1 \times (-1, 0].
\]

Moreover, by (5.5) and (5.7), the function

\[
w(x', t') = \frac{(v(x, t) - \frac{k}{2})_+}{k}
\]

satisfies

\[
\left\{ w \geq \frac{1}{2} \right\} \cap Q_1 \succ \nu \quad \text{and} \quad \int_{Q_1} |Dw|^p \leq \gamma,
\]

respectively. Then, Lemma 5.2 applies and we get the existence of \( \bar{s} \in (-1, -\nu/4) \) such that (5.4) is satisfied. At this point, by the result in [7] we find that for every \( \lambda, \bar{\nu} \in (0, 1) \) there exist \( \bar{\nu_1} \in K_1 \) and \( \bar{\varepsilon} \in (0, 1) \), which can be determined a priori only in terms of \( N, p, \bar{\nu}, \lambda, \gamma \) and \( \nu \), such that

\[
K_{\varepsilon}(\bar{\nu_1}) \subset K_1 \quad \text{and} \quad \left\{ w(\cdot, \bar{s}) > \frac{\lambda}{2} \right\} \cap K_{\varepsilon}(\bar{\nu_1}) \succ (1 - \bar{\nu})|K_{\varepsilon}|.
\]

Returning to the original variables and the original function \( v \), we find that there exist \( \bar{s} \in (\bar{t} - \theta_0(R_0/2)^p, \bar{t} - \theta_0(\nu/4)(R_0/2)^p), \bar{\nu_1} \in K_{\varepsilon_0}(\bar{x}) \) and \( \varepsilon \in (0, 1) \) such that \( K_{\varepsilon_0}(\bar{\nu_1}) \subset K_{\varepsilon_0}(\bar{x}) \) and

\[
\left\{ v(\cdot, \bar{s}) < \frac{\lambda + 1}{2} \right\} \cap K_{\varepsilon_0}(\bar{\nu_1}) \succ \bar{\nu} \left| K_{\varepsilon_0}\right|.
\]  

(5.8)
In order to extend the previous inequality to a cylinder, we consider
\[ s = \hat{s} + \hat{\theta} \left( \frac{\varepsilon R_0}{2} \right)^p, \quad \text{with} \quad \hat{\theta} = \hat{\nu}^p \theta_0, \]
and we write the energy estimates for \((v - \lambda k)_-\), where \(\lambda = \frac{\lambda + 1}{2}\), over the cylinders
\[ (\hat{y}, s) + Q_{\frac{\varepsilon R_0}{2}}^-(\hat{\theta}) \subset (\hat{y}, s) + Q_{\frac{\varepsilon R_0}{2}}^-(\hat{\theta}). \]
The cutoff function \(\zeta\) is chosen independent of \(t\) with \(\zeta = 1\) on \(K_{\varepsilon R_0}(\hat{y})\), \(\zeta = 0\) on the boundary of \(K_{\varepsilon R_0}(\hat{y})\) and such that \(0 \leq \zeta \leq 1\), \(|D\zeta| \leq 4(\varepsilon R_0)^{-1}\). Discarding the term containing \(|Dv|\) we obtain
\[ \int_{K_{\varepsilon R_0}(\hat{y})} (v - \lambda k)^2 (x, t) dx \leq \int_{K_{\varepsilon R_0}(\hat{y})} (v - \lambda k)^2 (x, \hat{s}) dx + \frac{\gamma k^{p+\alpha}}{(\varepsilon R_0)^p} \left| Q_{\frac{\varepsilon R_0}{2}}^- (\hat{\theta}) \right| \quad (5.9) \]
for every \(t\) such that \(s - \hat{\theta} \left( \frac{\varepsilon R_0}{2} \right)^p \leq t \leq s\). Since \(\hat{\lambda} < \lambda\), we can estimate the left hand side from below as follows
\[ \int_{K_{\varepsilon R_0}(\hat{y})} (v - \lambda k)^2 (x, \hat{s}) dx \geq \frac{1}{4} (1 - \hat{\lambda})^2 k^2 \left| \{v(\cdot, t) < \hat{\lambda} k\} \cap K_{\varepsilon R_0}(\hat{y}) \right| \]
for every \(t\) such that \(s - \hat{\theta} \left( \frac{\varepsilon R_0}{2} \right)^p \leq t \leq s\). Concerning the right hand side of (5.9), by (5.8) we have
\[ \int_{K_{\varepsilon R_0/2}(\hat{y})} (v - \lambda k)^2 (x, \hat{s}) dx \leq (\lambda k)^2 \left| \{v(\cdot, \hat{s}) < \lambda k\} \cap K_{\varepsilon R_0/2}(\hat{y}) \right| \leq \gamma k^2 \bar{v} |K_{\varepsilon R_0/4}|. \]
moreover, referring to the definitions of \(\hat{\theta}, \theta_0, k\), we get
\[ \frac{\gamma k^{p+\alpha}}{(\varepsilon R_0)^p} \left| Q_{\frac{\varepsilon R_0}{2}}^- (\hat{\theta}) \right| = \frac{\gamma k^{p+\alpha}}{(\varepsilon R_0)^p} \left( \frac{\varepsilon R_0}{2} \right)^p \left| K_{\varepsilon R_0} \right| \leq \gamma k^2 \bar{v} \left| K_{\varepsilon R_0/4} \right|. \quad (5.10) \]
Combining (5.9)–(5.10) we obtain
\[ \left| \{v(\cdot, t) < \hat{\lambda} k\} \cap K_{\varepsilon R_0}(\hat{y}) \right| < \frac{\gamma \bar{v}}{(1 - \hat{\lambda})^2} \left| K_{\varepsilon R_0/4} \right| \quad (5.11) \]
for every \(t\) such that \(s - \hat{\theta} \left( \frac{\varepsilon R_0}{2} \right)^p \leq t \leq s\).

Finally, we are ready to prove the thesis. Let us fix \(\lambda_0 \in (0, 1)\) and \(\nu_0 \in (0, 1)\). Choose \(\hat{\lambda} = \lambda_0\) and \(\hat{\nu} \in (0, 1)\) such that \(\frac{\hat{\nu}^p}{(1 - \hat{\lambda})^2} \leq \nu_0\). Without loss of generality, we may suppose that \(\hat{\nu}^{-1}\) is an integer. Let \(\hat{y}, \varepsilon\) be determined as above. We consider a partition of the cube \(K_{\varepsilon R_0}(\hat{y})\), up to a set of measure zero, into \(\hat{\nu}^{-N}\) pairwise disjoint cubes congruent to \(K_{\varepsilon R_0}(\hat{y})\). For \(j = 1, \ldots, \hat{\nu}^{-N}\), let \(y_j\) be the centers of such cubes. Up to a set of measure zero, the collection of cylinders
\[ (y_j, s) + Q_{\frac{\varepsilon R_0}{2}}^-(\theta_0), \quad j = 1, \ldots, \hat{\nu}^{-N}, \quad \text{where} \quad \gamma_0 = \frac{\hat{\nu}^p}{4}, \]
is a partition of the cylinder \((\hat{y}, s) + Q_{\frac{\varepsilon R_0}{2}}^- (\hat{\theta})\) into \(\hat{\nu}^{-N}\) sub-cylinders, each congruent to \(Q_{\frac{\varepsilon R_0}{2}}^-(\theta_0)\). Since we proved (5.11), (5.6) holds true for at least one of these cylinders. \(\blacksquare\)
Corollary 5.5 There exist \((y, s) \in (\bar{x}, \bar{t}) + Q_{\frac{9}{2}}(\theta_0)\) and \(\eta_0 \in (0, 1)\), such that either \(C > 1\) or
\[
v(x, s) \geq \frac{1}{8} (1 - \tau_0)^{-\beta} \quad \forall x \in K_r(y),
\] (5.12)
with
\[
r = \frac{\eta_0 R_0}{2} = \frac{1}{4} \eta_0 (1 - \tau_0).
\]
The constant \(\eta_0\) depends only upon \(\beta\) and the data.

Proof. Let \(\nu_0\) be determined by (3.9) for the choices \(\mu_\ast = 0\), \(\omega = M_0\), \(\xi = 2^{-(\beta+2)}\), \(a = \frac{1}{2}\) and \(\theta = \theta_0\). Note that \(\nu_0\) depends on the data and on \(\beta\). Let us fix \(\lambda_0 = \frac{1}{4}\). By Proposition 5.4 we obtain that the cylinder \((y, s) + Q_{2\mu_0 R_0}(\theta)\) satisfies (5.6). We conclude the proof by means of Lemma 3.3 (i).

Third Step. Now, combining all the results, we can conclude the proof of Theorem 5.1. Assuming (5.12), we apply Proposition 4.1 to the weak solution \(v\), defined by (5.1), for the choices \(\xi M = \frac{1}{8} (1 - \tau_0)^{-\beta}\) and \(2p = r\). We have either \(C^p r^p > \gamma \left(\frac{1}{8} (1 - \tau_0)^{-\beta}\right)^{p+\alpha}\) or
\[
v(x, t) \geq \eta \xi M
\]
for all \(x \in K_{2r}(y)\) and for every \(t\) in the interval
\[
s_1 \overset{\text{def}}{=} s + \left(\frac{b}{\eta \xi M}\right)^{p+\alpha-2} (8^p - 2^p) r^p \leq t \leq s + \left(\frac{b}{\eta \xi M}\right)^{p+\alpha-2} 8^p r^p \overset{\text{def}}{=} t_1.
\]
In the second case we infer, in particular, that \(v(x, s_1) \geq \eta \xi M\) and \(v(x, t_1) \geq \eta \xi M\) for \(x \in K_{2r}(y)\). By applying again the same Proposition, we get that either \(C^p (2r)^p > \gamma \left(\frac{1}{8} \eta (1 - \tau_0)^{-\beta}\right)^{p+\alpha}\) or
\[
v(x, t) \geq \eta^2 \xi M
\]
for all \(x \in K_{4r}(y)\) and for every \(t\) in the interval
\[
s_1 + \left(\frac{b}{\eta^2 \xi M}\right)^{p+\alpha-2} (8^p - 2^p) (2r)^p \leq t \leq t_1 + \left(\frac{b}{\eta^2 \xi M}\right)^{p+\alpha-2} 8^p (2r)^p.
\]
By iteration, we get either \(C^p (2^{k-1} r)^p > \gamma \left(\frac{1}{8} \eta^{k-1} (1 - \tau_0)^{-\beta}\right)^{p+\alpha}\) or
\[
v(x, t) \geq \eta^k \xi M
\]
for all \(x \in K_{2^{k-1} r}(y)\) and for every \(t\) in the interval \([s_k, t_k]\) with
\[
s_k = s + \left(\frac{b}{\xi M}\right)^{p+\alpha-2} (8^p - 2^p) r^p \sum_{j=1}^{k} \frac{2^{p(j-1)}}{\eta^{p+\alpha-2j}}
\]
\[
t_k = s + \left(\frac{b}{\xi M}\right)^{p+\alpha-2} 8^p r^p \sum_{j=1}^{k} \frac{2^{p(j-1)}}{\eta^{p+\alpha-2j}}
\]
for all \(k = 1, 2, \ldots\).
Now, fix \( n \in \mathbb{N} \) such that
\[
2^{-n} \leq \frac{1}{8}\eta_0(1-\tau_0) < 2^{-n+1}.
\]
It follows that
\[
2 \leq 2^n r < 4 \tag{5.13}
\]
and
\[
2^{-3\beta-3}\eta_0^\beta(2^\beta \eta)^n \geq \eta^n \xi M > 2^{-4\beta-3}\eta_0^\beta(2^\beta \eta)^n.
\]
We choose \( \beta \) such that \( 2^\beta \eta = 1 \). Once \( \beta \) is fixed (depending only on the data), also \( \eta_0 \) turns out to depend only on the data. Then, in particular,
\[
\eta^n \xi M > 2^{-4\beta-3}\eta_0^\beta \overset{\text{def}}{=} \gamma_0 \in (0, 1). \tag{5.14}
\]

Now, we have to distinguish two cases.

**First Case:** there exists \( k \leq n \) such that \( \bar{C} p (2^k - 1) r > \gamma \left( \frac{1}{8} \eta^{n-1}(1-\tau_0)^{-\beta} \right)^{p+\alpha} \). Then we have also \( \bar{C} p (2^n r)^p > \gamma \left( \frac{1}{8} \eta^n (1-\tau_0)^{-\beta} \right)^{p+\alpha} \). From (5.13) and (5.14) it follows that
\[
\bar{C} p 4^p \geq \bar{C} p (2^n r)^p > \gamma \left( \frac{1}{8} \eta^n (1-\tau_0)^{-\beta} \right)^{p+\alpha} \geq \gamma(\text{data}).
\]
Recalling the definition of \( \bar{C} \), this is equivalent to saying that
\[
C \rho \geq \gamma u(x_0, t_0)^{1 + \frac{\alpha}{p}},
\]
with \( \gamma = \gamma(\text{data}) \).

**Second Case:** we have \( \bar{C} p (2^n - 1) r \leq \gamma \left( \frac{1}{8} \eta^{n-1}(1-\tau_0)^{-\beta} \right)^{p+\alpha} \). Then
\[
v(x, t) \geq \eta^n \xi M
\]
for all \( x \in K_{2^n r}(y) \) and for every \( t \) in the interval \([s_n, t_n]\). Taking (5.14) and (5.13) into account we infer that
\[
v(x, t) \geq \gamma_0.
\]
for every \( x \in K_1 \subset K_2(y) \subset K_{2^n r}(y) \) and \( t \in [s_n, t_n] \). It remains to estimate the time interval. Using (5.13) and (5.14) we have
\[
t_n \geq s + \left( \frac{b}{\xi M} \right)^{p+\alpha-2} r^p \frac{2^{p(n+2)}}{\eta_0^{p+\alpha-2}} \geq -1 + 8^p \left( \frac{8^{\beta+1} b}{\eta_0^\beta} \right)^{p+\alpha-2}.
\]
If the right hand side is larger than 1, we are done. Otherwise, we iterate the procedure \( k \) times more until \( t_{n+k} > 1 \). Note that
\[
t_{n+k} \geq -1 + 8^p \left( \frac{8^{\beta+1} b}{\eta_0^\beta} \right)^{p+\alpha-2} \frac{2^{p k}}{\eta^{k(p+\alpha-2)}},
\]
so that the choice of \( k \) is independent of \( u \) and depends only on the data. It follows that there exists \( t = \gamma_1 > 1 \) such that
\[
v(x, \gamma_1) \geq \gamma_0 \quad \text{for all } x \in K_1.
\]
Thus, Theorem 5.1 is proved. Recalling (5.1), we can write the previous inequality in terms of $u$ and we obtain
\[ u(x_0, t_0) \leq \frac{1}{\gamma_0} \inf_{K_{u(x_0)}} u(x, t_0 + \theta^p) \]
with
\[ c = \gamma_1^{\frac{1}{p + \alpha - 2}} \quad \text{and} \quad \theta = \left( \frac{c}{u(x_0, t_0)} \right)^{p + \alpha - 2}. \]

6 Hölder continuity

The aim of the present section is to show that the intrinsic Harnack inequality implies a local Hölder continuity condition. Up to a translation, assume that the initial cylinder $Q_{R_0} = K_{R_0} \times (-R_0^p, \xi, 0]$, with $0 \leq \xi < \min\{p, p + \alpha - 2\}$, is contained in the domain of $u$, which is a non-negative, local weak solution to (1.1). Set
\[ \omega_0 = \text{osc}_{Q_{R_0}} u = \sup_{Q_{R_0}} u. \]

Let us define the intrinsic cylinder
\[ Q_0 = K_{R_0} \times (-\theta_0 R_0^p, 0], \quad \theta_0 = \left( \frac{c}{\omega_0} \right)^{p + \alpha - 2} \]
where $c > 0$ has to be determined only in dependence of the data. If $\omega_0 \leq c R_0^{\frac{\xi}{p + \alpha - 2}}$ for every cylinder as $Q_{R_0}$ (keeping the same constant $c$), then it turns out that $u$ is locally Hölder continuous (see [12]). Thus, assume that there exists $R_0$ such that $\omega_0 > c R_0^{\frac{\xi}{p + \alpha - 2}}$. In this case, we have that $Q_0 \subset Q_{R_0}$ and, consequently,
\[ \text{osc}_{Q_0} u \leq \omega_0. \]

The aim of the next theorem is to show that we can construct a sequence of nested and shrinking intrinsic cylinders \( \{Q_n\} \) with the same vertex, such that the oscillation of $u$ in $Q_n$ tends to zero, as $n \to \infty$, in a way that can be quantitatively determined by means of the structure conditions (1.2). We point out that the proof of such a result is a little bit more involved than the one given in [6] since, in general, $\omega_0 - u$ is not a solution of (1.1). This fact is clear in the case of the model equation (1.4). The Hölder continuity will then follow from [12].

**Theorem 6.1** There exist positive constants $c, \gamma$ and $\delta, \varepsilon \in (0, 1)$, that can be quantitatively determined only in terms of the data such that, setting
\[ R_n = \varepsilon R_{n-1}, \quad \omega_n = \max\left\{ \delta \omega_{n-1}, \gamma (C R_{n-1})^{\frac{p}{p + \alpha - 2}} \right\}, \quad \theta_n = \left( \frac{c}{\omega_n} \right)^{p + \alpha - 2}, \quad Q_n = Q_{R_n}(\theta_n), \]
for $n \in \mathbb{N}$, there hold $Q_{n+1} \subset Q_n$ and
\[ \text{osc}_{Q_n} u \leq \omega_n. \]

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Theorem 6.1 can be proved by using an iterative argument. For the sake of simplicity, we limit ourselves to the first iteration.

Let $P_0 = (0, -\theta_0 R_0^p / 2)$ be the mid point of $Q_0$. We distinguish two cases.

### 6.1 First case

Assume first that $u(P_0) \geq \frac{1}{8} \omega_0$. By Theorem 2.1 there exist $c, \kappa, \gamma > 0$, depending only upon the data, such that either

$$\gamma u(P_0) p^{+\alpha} \leq C^p R_0^p \tag{6.1}$$

or

$$\frac{1}{8\kappa} \omega_0 \leq \inf_{Q_1^R(\theta_0)} u(x,t). \tag{6.2}$$

Note that we have used an equivalent formulation of (2.2). Setting

$$\delta = 1 - \frac{1}{8\kappa}, \quad \varepsilon = \frac{\omega_0^{p+\alpha-2}}{4}, \quad R_1 = \varepsilon R_0,$$

$$\omega_1 = \max \left\{ \delta \omega_0, \gamma_1 (CR_0)^{\frac{p}{p+\alpha}} \right\}, \quad \theta_1 = \left( \frac{c}{\omega_1} \right)^{p+\alpha-2},$$

with $\gamma_1 = 8 \gamma^{-p\alpha p-\alpha}$, it is easily seen that the cylinder $Q_1 = K_{R_1} \times (-\theta_1 R_1^p, 0]$ is contained in $Q_1^R(\theta_0)$. If (6.1) holds, then

$$\text{osc}_{Q_1} u \leq \omega_0 \leq \gamma_1 (CR_0)^{\frac{p}{p+\alpha}} \leq \omega_1.$$

If (6.2) is true then

$$\text{osc}_{Q_1} u \leq \delta \omega_0 \leq \omega_1.$$

In any case

$$\text{osc}_{Q_1} u \leq \omega_1. \tag{6.3}$$

### 6.2 Second case

Assume now that $u(P_0) < \frac{1}{8} \omega_0$. We are going to show that, also in this case, we can fix (possibly different) values of the constants $\varepsilon, \delta, c, \gamma$, dependent only on the data, such that (6.3) continues to hold.

Let us consider $Q_1^- = P_0 + Q_{R_0^-} \left( \frac{\theta_0}{2} \right)$. If $C^p R_0^p > \left( \frac{1}{2} \omega_0 \right)^{p+\alpha}$ then we can restart as from (6.1). From now on assume that $C^p R_0^p \leq \left( \frac{1}{2} \omega_0 \right)^{p+\alpha}$. Then

$$\left| \left\{ u \leq \frac{1}{2} \omega_0 \right\} \cap Q_1^- \right| \geq \nu |Q_1^-|, \tag{6.4}$$

where $\nu$ is determined by (3.9), for the choices $a = \xi = 1/2, \omega = \mu_+ = \omega_0, \theta = \theta_0/2$. Indeed, if (6.4) were not true, then Lemma 3.3 would imply that

$$u(x,t) \geq \frac{1}{4} \omega_0, \quad \text{in } P_0 + Q_{R_0^-} \left( \frac{\theta_0}{2} \right).$$
In particular, \( u(\mathcal{P}_0) \geq \frac{1}{2} \omega_0 \), which is impossible. Thus (6.4) is established. Note that \( \nu \) depends only on the data (once \( c \) will be fixed in dependence only of the data). As for Proposition 5.4, one can see that the following lemma holds.

**Lemma 6.2** For every \( \lambda > 1 \) and \( \eta \in (0,1) \), there exist \( (y,s) \in Q_0^- \) and \( \delta \in (0,1) \) such that

\[
(y,s) + Q_{2\delta R_0} \left( \frac{\theta_0}{2} \right) \subset Q_0^-
\]

Now, fix \( \lambda = \frac{3}{2} \) and \( \eta = \nu \), where \( \nu \) is obtained by (3.10) when \( \mu_+ = \omega = \omega_0 \), \( a = 1/2 \), \( \xi = 1/4 \) and \( \theta = \theta_0/2 \). It follows that there are \( (\bar{y}, \bar{s}) \in Q_0^- \) and \( \bar{\delta} \in (0,1) \) such that

\[
(y,s) + Q_{2\bar{\delta} R_0} \left( \frac{\theta_0}{2} \right) \subset Q_0^-
\]

Hence, Lemma 3.3 (ii) yields either \( (C \bar{\delta} R_0)^p > 4^{-p} \omega_0^{\beta-\alpha} \) (and in this case we finish the proof, as before), or

\[
u \leq \frac{7}{8} \omega_0 \quad \text{in} \quad (\bar{y}, \bar{s}) + Q_{\bar{\rho}} \left( \frac{\theta_0}{2} \right),
\]

where we have set

\[
\bar{\rho} = \frac{\bar{\delta} R_0}{2}.
\]

At this point, we change the time variable by

\[
t' = \omega_0 \alpha t,
\]

and set

\[
w(x,t') = u(x,t), \quad v(x,t') = \omega_0 - w(x,t').
\]

It turns out that \( v \) is a local weak solution of

\[
v_t = \text{div} A'(x,t',w,Dv) + B(x,t',w,Dv)
\]

with

\[
A'(x,t',w,Dv) = -\omega_0^{-\alpha} A(x,\omega_0^{-\alpha} t',w,-Dv)
\]

\[
B'(x,t',w,Dv) = -\omega_0^{-\alpha} B(x,\omega_0^{-\alpha} t',w,-Dv).
\]

The structure conditions for the new coefficients are the following

\[
\begin{align*}
A'(x,t',w,\eta) \cdot \eta & \geq \omega_0^{-\alpha} C_0 \Phi(w) |\eta|^p - \omega_0^{-\alpha} C^p, \\
|A'(x,t',w,\eta)| & \leq \omega_0^{-\alpha} C_1 \Phi(w) |\eta|^{p-1} + \omega_0^{-\alpha} C^{p-1} \Phi(w)^{\frac{1}{p}} \\
|B'(x,t',w,\eta)| & \leq \omega_0^{-\alpha} C_2 \Phi(w) |\eta|^{p-1} + \omega_0^{-\alpha} C_2 C^{p-1} \Phi(w)^{\frac{1}{p}}.
\end{align*}
\]
To simplify the notation, from now on we write $t$ instead of $t'$. The corresponding energy estimates are

$$
\sup_{s - \theta R^p < t \leq s} \int_{K_R(y)} (v - k)^2 \zeta^p(x,t) dx - \int_{K_R(y)} (v - k)^2 \zeta^p(x,s - \theta R^p) dx \\
+ \omega_0 \int_{(y,s) + Q_R^-} \Phi(w) |D(v - k)_-|^p \zeta^p dx dt \\
\leq \gamma \left( \int_{(y,s) + Q_R^-} (v - k)^2 \zeta dx dt + \omega_0^\alpha \int_{(y,s) + Q_R^-} \Phi(v)(v - k)^p |D\zeta|^p dx dt \right) \\
+ \gamma \int_{(y,s) + Q_R^-} \left( C^p_0 \omega_0^\alpha \Phi(w)(v - k)^p + \omega_0^\alpha C^p \chi_{(v - k)_- > 0} \right) \zeta^p dx dt. \tag{6.6}
$$

If we choose levels

$$
k \leq \frac{1}{8} \omega_0,
$$

then $w \geq \frac{7}{8} \omega_0$, whenever $(v - k)_- > 0$. On the other hand, $w \leq \omega_0$ in $K_{R_0} \times (-\theta_0 \omega_0^\alpha R^p_0, 0]$, so that (6.6) gives

$$
\sup_{s - \theta R^p < t \leq s} \int_{K_R(y)} (v - k)^2 \zeta^p(x,t) dx - \int_{K_R(y)} (v - k)^2 \zeta^p(x,s - \theta R^p) dx \\
+ \omega_0 \int_{(y,s) + Q_R^-} |D(v - k)_-|^p \zeta^p dx dt \\
\leq \gamma \left( \int_{(y,s) + Q_R^-} (v - k)^2 \zeta dx dt + \omega_0^\alpha \int_{(y,s) + Q_R^-} (v - k)^p |D\zeta|^p dx dt \right) \\
+ \gamma \int_{(y,s) + Q_R^-} \left( C^p_0 (v - k)^p + \omega_0^\alpha C^p \chi_{(v - k)_- > 0} \right) \zeta^p dx dt \tag{6.7}
$$

for every cylinder $(y, s) + Q_R^- \subset K_{R_0} \times (-\theta_0 \omega_0^\alpha R^p_0, 0]$. Moreover, condition (6.5) leads to

$$
v(x,s) \geq \frac{1}{8} \omega_0,
$$

for $(x,s) \in (y, \omega_0^\alpha s) + Q_R^- \left( \frac{\omega_0^\alpha}{\sigma} \right)$. In particular, we have

$$
v(s, 0) \geq \frac{1}{8} \omega_0, \quad \text{in } K_p(y), \quad \text{with } s_0 = \omega_0^\alpha s.
$$

Now, arguing as in [6], (note that the energy estimate (6.7) are the same as those considered there), one can check the following proposition.

**Proposition 6.3** Let $n \in \mathbb{N}$ be fixed. Then, there exist $\gamma, \sigma, b_1, b_2$, depending only on the data and $n$, such that either $C^p R^p_0 > \gamma \omega_0^{p + \alpha}$ or

$$
v(x,t) \geq \frac{\sigma \omega_0}{16 b_2},
$$

in $K_{2n - 1}(\tilde{y}) \times (s_0 + t_1, s_0 + t_2)$, where

$$
t_i = \left( \frac{8 b_1}{\omega_0} \right)^{p - 2} \delta_0 \rho^p, \quad i = 1, 2.
$$
Going now back to the function $u$, we find
\[ u(x, t) \leq (1 - \eta)\omega_0, \] (6.8)
for every $x \in K_{2n-1, p}(\bar{y})$ and $t \in (\bar{s} + \omega_0^{-\alpha} t_1, \bar{s} + \omega_0^{-\alpha} t_2)$, where
\[ \eta = \frac{\sigma}{16b_2}. \]

Recalling that \( \bar{s} \in (-\theta_0 R_{01}^p, -\theta_0 R_{01}^p/2] \) and choosing $n$ large enough, one can see that it is possible to find a positive constant $c$ satisfying $\bar{s} + \omega_0^{-\alpha} t_2 > 0$ and $\bar{s} + \omega_0^{-\alpha} t_1 < 0$ and depending only on the data. Now,
\[
\bar{s} + \omega_0^{-\alpha} t_1 \leq \frac{-\theta_0 R_{01}^p}{2} + \omega_0^{-\alpha} t_1 
= R_{01}^p \frac{p + \alpha - 2}{\alpha p - \alpha - 2} \left( \frac{-1}{2} + \frac{(8b_1)^{p-2} \delta_0 \delta^p}{\alpha p - \alpha - 2} \right)
= -\theta R_{11}^p \frac{(1 - \eta)^{p + \alpha - 2}}{2^p} \left( \frac{1}{2} - \frac{(8b_1)^{p-2} \delta_0 \delta^p}{c^{p + \alpha - 2} \delta^p} \right),
\]
where
\[ \theta = \left( \frac{c}{(1 - \eta)\omega_0} \right)^{p + \alpha - 2}, \quad R_1 = \varepsilon R_0, \]
with $\varepsilon > 0$ to be determined. We require that
\[
\left( \frac{1 - \eta)^{p + \alpha - 2}}{2^p} \left( \frac{1}{2} - \frac{(8b_1)^{p-2} \delta_0 \delta^p}{c^{p + \alpha - 2} \delta^p} \right) \right) \geq 1
\]
in order to conclude, by the previous estimate, that $\bar{s} + \omega_0^{-\alpha} t_1 \leq -\theta R_{11}^p$. This means that (6.8) is true in $K_{2n-1, p}(\bar{y}) \times (-\theta R_{11}^p, 0]$ and, a fortiori, in $K_{2n-1, p}(\bar{y}) \times (-\theta R_{11}^p, 0]$, being
\[ \theta_1 = \left( \frac{c}{\omega_1} \right)^{p + \alpha - 2}, \quad \omega_1 = \max\{(1 - \eta)\omega_0, \gamma(CR_0)^{p+\alpha}\}. \]
Finally, by choosing a possibly larger value of $n$ (or a possibly smaller value of $\varepsilon$), we can ensure that $K_{2n-1, p}(\bar{y}) \supset K_{R_1}$. We have then established that
\[ \operatorname{osc} u \leq \omega_1, \]
where $Q_1 = Q^{-}_{R_1}(\theta_1)$, and the second case is concluded as before.

References


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