The effect of a thin layer of heterogeneities in an elastic structure.

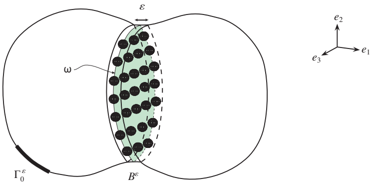
Giuseppe Geymonat

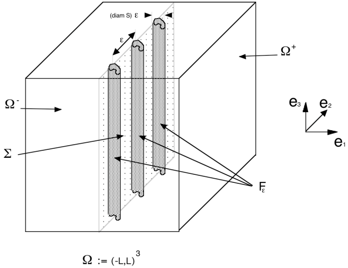
LMS
École Polytechnique

To Gianni Gilardi with best wishes

Joint work with M. Bellieud, S. Hendili, F. Krasucki, P. E. Steltzig, G. Michaille, M. Vidrascu

Examples of thin layers of heterogeneities in a structure (of elastic material)





The heterogeneities can be holes, elastic material, rigid inclusions

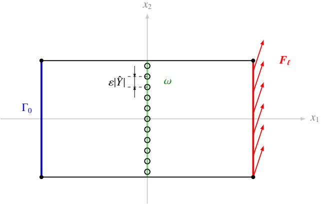
Problem peculiarities:

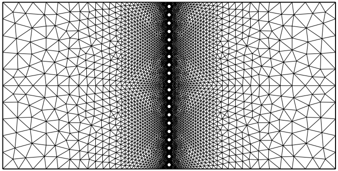
- a thin layer of very small heterogeneities with highly contrasted materials (the materials characteristics of the structure and the heterogeneity can be very different)
- a large number of very small heterogeneities periodically distributed in the layer
- small deformations

Problem peculiarities:

- a thin layer of very small heterogeneities with highly contrasted materials (the materials characteristics of the structure and the heterogeneity can be very different)
- a large number of very small heterogeneities periodically distributed in the layer
- small deformations

Computational difficulties:

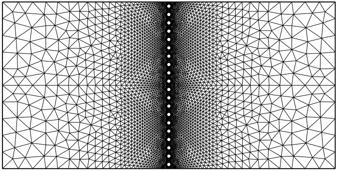


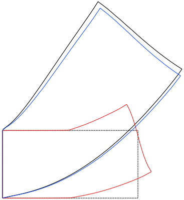


- The computational cost increases with the number of heterogeneities
- It can be difficult to obtain a correct mesh

How much can the heterogeneities be important?

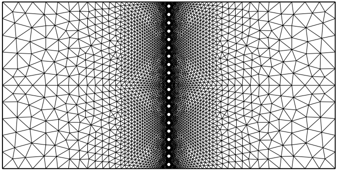
How much can the heterogeneities be important?

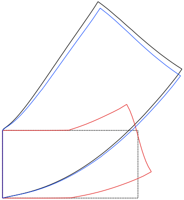




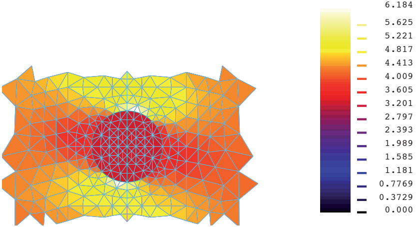
The influence of the heterogeneities on the deformed shape

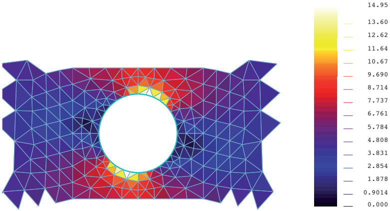
How much can the heterogeneities be important?





The influence of the heterogeneities on the deformed shape





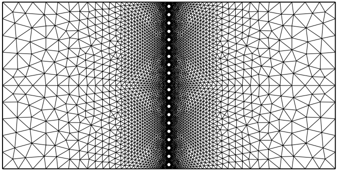
The influence of the heterogeneities on the stresses

The methods

i) Matched asymptotic expansions (S. Hendili, F. Krasucki, M. Vidrascu)
ii) Variational convergence (M. Bellieud, F. Krasucki, G. Michalle; F. Krasucki, P. E. Steltzig)

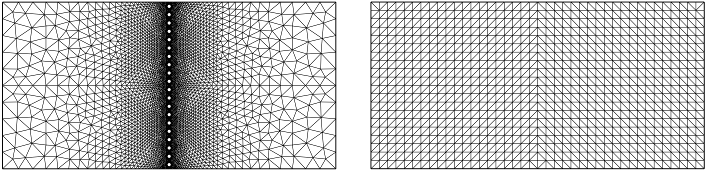
Goals of the matched asymptotic expansions method

- obtain a precise macroscopic behaviour replacing the layer by a surface (low cost)

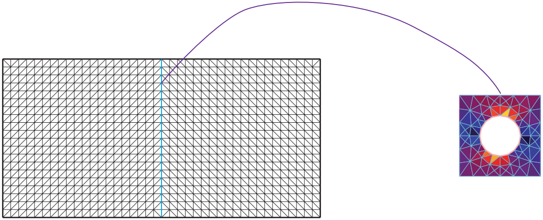


Goals of the matched asymptotic expansions method

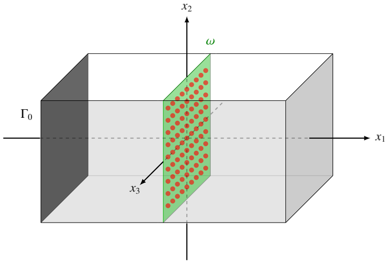
- obtain a precise macroscopic behaviour replacing the layer by a surface (low cost)

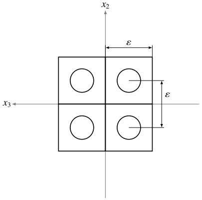


- **and still** obtain precise informations on the local fields near the heterogeneities (important for the applications)



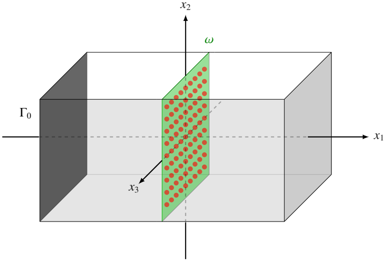
The problem (the unknown field solution is $\mathbf{u}^{\varepsilon}$)

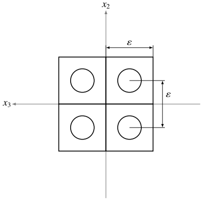




holes, elastic inclusions, rigid inclusions

The problem (the unknown field solution is $\mathbf{u}^\varepsilon$)

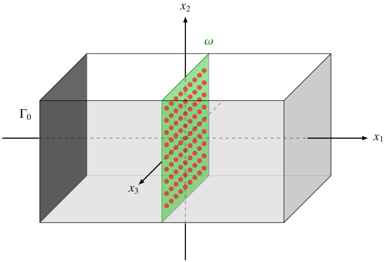


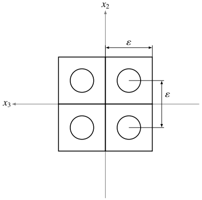


holes, elastic inclusions, rigid inclusions

What happens for $\varepsilon \to 0$?

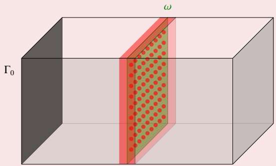
The problem (the unknown field solution is $\mathbf{u}^{\varepsilon}$)



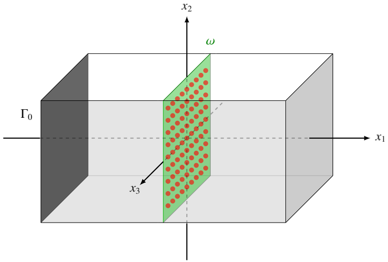


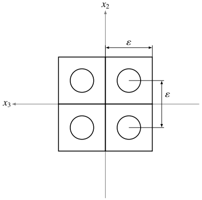
holes, elastic inclusions, rigid inclusions

What happens for $\varepsilon \to 0$?



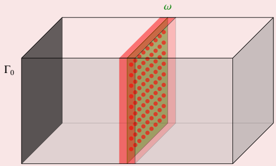
The problem (the unknown field solution is $\mathbf{u}^\varepsilon$)

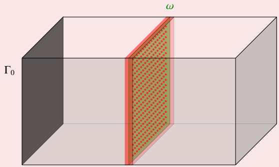




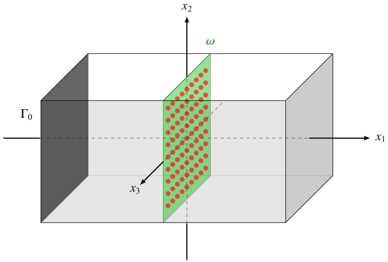
holes, elastic inclusions, rigid inclusions

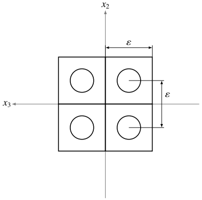
What happens for $\varepsilon \to 0$?





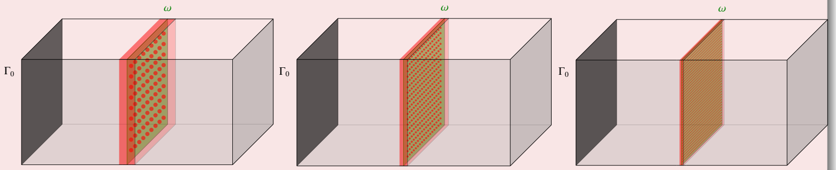
The problem (the unknown field solution is $\mathbf{u}^{\varepsilon}$)



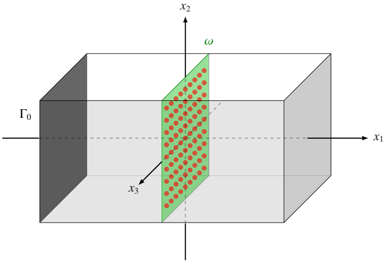


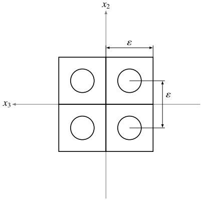
holes, elastic inclusions, rigid inclusions

What happens for $\varepsilon \to 0$?



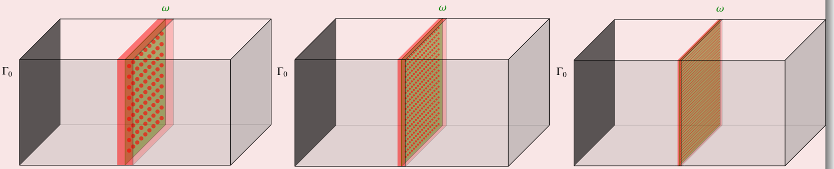
The problem (the unknown field solution is $\mathbf{u}^\varepsilon$)





holes, elastic inclusions, rigid inclusions

What happens for $\varepsilon \to 0$?



A two-scale internal boundary layer

The main steps of the matched asymptotic expansions method

The main steps of the matched asymptotic expansions method

- ⋆ Decomposition of the domain and scale separation
- ⋆ Associated asymptotic developments of the solution $\mathbf{u}^\varepsilon$.
- ⋆ Matching conditions

The main steps of the matched asymptotic expansions method

- ⋆ Decomposition of the domain and scale separation
- ⋆ Associated asymptotic developments of the solution $\mathbf{u}^\varepsilon$.
- ⋆ Matching conditions

$\Longrightarrow$ Boundary value problems at the different orders

$\Longrightarrow$ Construction of the approximate solutions

The main steps of the matched asymptotic expansions method

- ⋆ Decomposition of the domain and scale separation
- ⋆ Associated asymptotic developments of the solution $\mathbf{u}^{\varepsilon}$.
- ⋆ Matching conditions

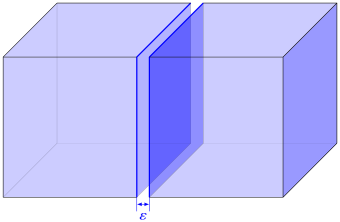
⟹ Boundary value problems at the different orders

⟹ Construction of the approximate solutions

Some references

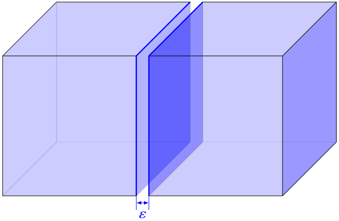
- ⋆ *Van Dyke* (1964).
- ⋆ *Nguetseng, Sanchez-Palencia* (1986).
- ⋆ *Abdelmoula, Marigo* (2000).
- ⋆ *G., Hendili, Krasucki, Vidrascu* (2011).
- ⋆ *David, Marigo, Pideri* (2012).

Domain decomposition

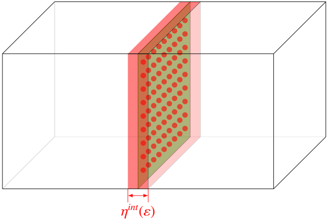


Outer domain: $\Omega^{out}(\varepsilon) := \left\{ \mathbf{x} \in \Omega \; ; \; |x_1| > \dfrac{\varepsilon}{2} \right\}$

Domain decomposition



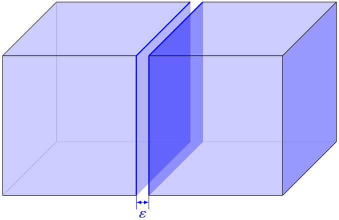
Outer domain: $\Omega^{out}(\varepsilon) := \left\{ \mathbf{x} \in \Omega \; ; \; |x_1| > \frac{\varepsilon}{2} \right\}$



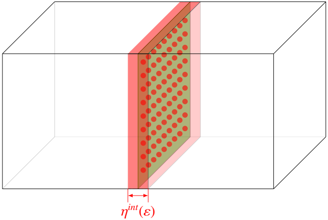
Inner domain: $\Omega^{int}(\varepsilon) := \left\{ \mathbf{x} \in \Omega \; ; \; |x_1| < \frac{\eta(\varepsilon)}{2} \right\}$

with $\lim_{\varepsilon \to 0} \eta(\varepsilon) = 0$ and $\lim_{\varepsilon \to 0} \frac{\eta(\varepsilon)}{\varepsilon} = \infty$

Domain decomposition

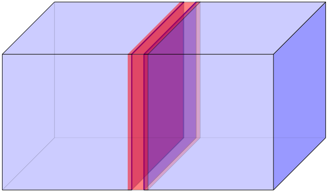


Outer domain: $\Omega^{out}(\varepsilon) := \left\{ \mathbf{x} \in \Omega \; ; \; |x_1| > \frac{\varepsilon}{2} \right\}$



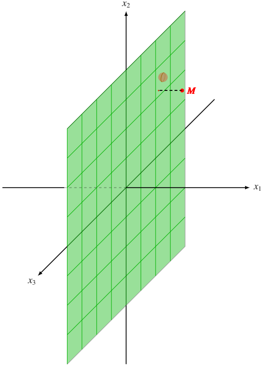
Inner domain: $\Omega^{int}(\varepsilon) := \left\{ \mathbf{x} \in \Omega \; ; \; |x_1| < \frac{\eta(\varepsilon)}{2} \right\}$

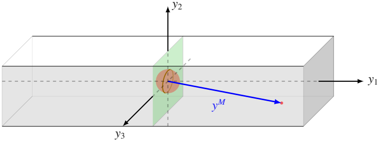
with $\lim_{\varepsilon \to 0} \eta(\varepsilon) = 0$ and $\lim_{\varepsilon \to 0} \frac{\eta(\varepsilon)}{\varepsilon} = \infty$



Overlapping region: $\frac{\varepsilon}{2} < |x_1| < \frac{\eta(\varepsilon)}{2}$

Scale separation in the inner domain





$$M(\mathbf{x}^M) \in \Omega^{int}(\varepsilon) = \left\{ \mathbf{x} \in \Omega \ ; \ |x_1| < \frac{\eta(\varepsilon)}{2} \right\}$$

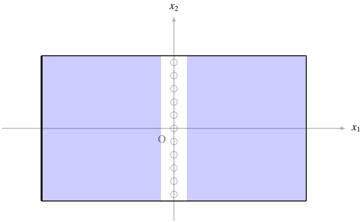
$$\mathbf{x}^M = \mathbf{x}^I + \varepsilon \mathbf{y}^M \Leftrightarrow \begin{cases} x_1^M = \varepsilon y_1^M \\ x_2^M = x_2^I + \varepsilon y_2^M \\ x_3^M = x_3^I + \varepsilon y_3^M \end{cases}$$

Consequences when $\varepsilon \to 0$

- $M(\hat{\mathbf{x}}, \mathbf{y}^M)$ with $\hat{\mathbf{x}} = (x_2, x_3)$
- the periodic cell is infinite in the direction y_1
- the periodic cell is bounded in the directions y_2 and y_3

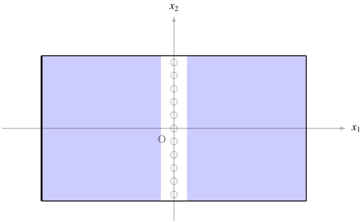
- Outer development : far from ω

$$\mathbf{u}^{\varepsilon}(x_1, x_2, x_3) = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{u}^i(x_1, x_2, x_3)$$



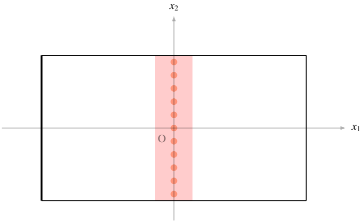
- Outer development : far from ω

$$\mathbf{u}^{\varepsilon}(x_1, x_2, x_3) = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{u}^i(x_1, x_2, x_3)$$

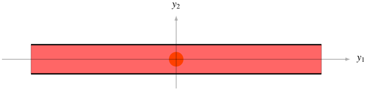


- Inner development : near ω

$$\mathbf{u}^{\varepsilon}(x_1, x_2, x_3) = \sum_{i=0}^{\infty} \varepsilon^i \mathbf{v}^i(\hat{\mathbf{x}}, y_1, \hat{\mathbf{y}})$$

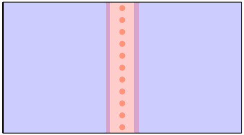


$$y_1 = \frac{x_1}{\varepsilon}$$
$$\hat{\mathbf{y}} = \left(\frac{x_2 - x_2^l}{\varepsilon}, \frac{x_3 - x_3^l}{\varepsilon}\right)$$
$\mathbf{v}^i$ $\hat{\mathbf{y}}$-periodic



Matching conditions

Matching conditions



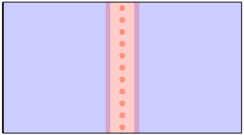
Overlapping region:

$\frac{\varepsilon}{2} < |x_1| < \frac{\eta(\varepsilon)}{2}$ or

$\frac{1}{2} < |y_1| = \frac{|x_1|}{\varepsilon} < \frac{\eta(\varepsilon)}{2\varepsilon}$

Matching conditions

In the overlapping region both expansions are valid



Overlapping region:

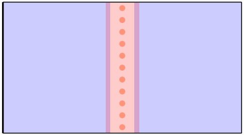
$\frac{\varepsilon}{2} < |x_1| < \frac{\eta(\varepsilon)}{2}$ or

$\frac{1}{2} < |y_1| = \frac{|x_1|}{\varepsilon} < \frac{\eta(\varepsilon)}{2\varepsilon}$

Matching conditions

In the overlapping region both expansions are valid

- For every $\mathbf{u}^i$ one has for $0 < x_1 < \frac{\eta(\varepsilon)}{2\varepsilon}$:
  $\mathbf{u}^i(\mathbf{x}) = \mathbf{u}^i(0+, \hat{\mathbf{x}}) + x_1 \frac{\partial \mathbf{u}^i}{\partial x_1}(0+, \hat{\mathbf{x}}) + \dots$

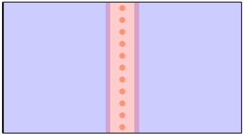


Overlapping region:

$\frac{\varepsilon}{2} < |x_1| < \frac{\eta(\varepsilon)}{2}$ or

$\frac{1}{2} < |y_1| = \frac{|x_1|}{\varepsilon} < \frac{\eta(\varepsilon)}{2\varepsilon}$

Matching conditions



Overlapping region:

$\frac{\varepsilon}{2} < |x_1| < \frac{\eta(\varepsilon)}{2}$ or

$\frac{1}{2} < |y_1| = \frac{|x_1|}{\varepsilon} < \frac{\eta(\varepsilon)}{2\varepsilon}$

In the overlapping region both expansions are valid

- For every $\mathbf{u}^i$ one has for $0 < x_1 < \frac{\eta(\varepsilon)}{2\varepsilon}$:
  $\mathbf{u}^i(\mathbf{x}) = \mathbf{u}^i(0+, \hat{\mathbf{x}}) + x_1 \frac{\partial \mathbf{u}^i}{\partial x_1}(0+, \hat{\mathbf{x}}) + \dots$
- hence since $x_1 = \varepsilon y_1$:

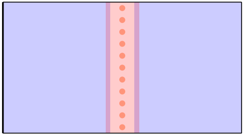
$$\begin{aligned} \mathbf{u}^\varepsilon(x_1, \hat{\mathbf{x}}) &= \mathbf{u}^0(0+, \hat{\mathbf{x}}) + \varepsilon \left(\mathbf{u}^1(0+, \hat{\mathbf{x}}) + y_1 \frac{\partial \mathbf{u}^0}{\partial x_1}(0+, \hat{\mathbf{x}}) \right) + \dots \\ &= \mathbf{v}^0(\hat{\mathbf{x}}, y_1, \hat{\mathbf{y}}) + \varepsilon \mathbf{v}^1(\hat{\mathbf{x}}, y_1, \hat{\mathbf{y}}) + \dots\dots \end{aligned}$$

Matching conditions

In the overlapping region both expansions are valid

- For every $\mathbf{u}^i$ one has for $0 < x_1 < \frac{\eta(\varepsilon)}{2\varepsilon}$:
  $\mathbf{u}^i(\mathbf{x}) = \mathbf{u}^i(0+,\hat{\mathbf{x}}) + x_1 \frac{\partial \mathbf{u}^i}{\partial x_1}(0+,\hat{\mathbf{x}}) + \ldots$
- hence since $x_1 = \varepsilon y_1$:

$$\begin{aligned} \mathbf{u}^\varepsilon(x_1,\hat{\mathbf{x}}) &= \mathbf{u}^0(0+,\hat{\mathbf{x}}) + \varepsilon\left(\mathbf{u}^1(0+,\hat{\mathbf{x}}) + y_1 \frac{\partial \mathbf{u}^0}{\partial x_1}(0+,\hat{\mathbf{x}})\right) + \ldots \\ &= \mathbf{v}^0(\hat{\mathbf{x}},y_1,\hat{\mathbf{y}}) + \varepsilon \mathbf{v}^1(\hat{\mathbf{x}},y_1,\hat{\mathbf{y}}) + \ldots.. \end{aligned}$$



Overlapping region:

$\frac{\varepsilon}{2} < |x_1| < \frac{\eta(\varepsilon)}{2}$ or

$\frac{1}{2} < |y_1| = \frac{|x_1|}{\varepsilon} < \frac{\eta(\varepsilon)}{2\varepsilon}$

- take the limit for $\varepsilon \longrightarrow 0$ with **fixed** x_1:

$$\lim_{y_1\to\pm\infty} \left(\mathbf{v}^0(\hat{\mathbf{x}},y_1,\hat{\mathbf{y}}) - \mathbf{u}^0(0\pm,\hat{\mathbf{x}})\right) = \mathbf{0}$$

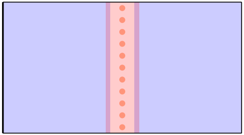
$$\lim_{y_1\to\pm\infty} \left(\mathbf{v}^1(\hat{\mathbf{x}},y_1,\hat{\mathbf{y}}) - \left(\mathbf{u}^1(0\pm,\hat{\mathbf{x}}) + y_1 \frac{\partial \mathbf{u}^0}{\partial x_1}(0\pm,\hat{\mathbf{x}})\right)\right) = \mathbf{0}$$

Matching conditions

In the overlapping region both expansions are valid

- For every $\mathbf{u}^i$ one has for $0 < x_1 < \frac{\eta(\varepsilon)}{2\varepsilon}$:
  $\mathbf{u}^i(\mathbf{x}) = \mathbf{u}^i(0+, \hat{\mathbf{x}}) + x_1 \frac{\partial \mathbf{u}^i}{\partial x_1}(0+, \hat{\mathbf{x}}) + \ldots$
- hence since $x_1 = \varepsilon y_1$:

$$\begin{aligned} \mathbf{u}^\varepsilon(x_1, \hat{\mathbf{x}}) &= \mathbf{u}^0(0+, \hat{\mathbf{x}}) + \varepsilon \left(\mathbf{u}^1(0+, \hat{\mathbf{x}}) + y_1 \frac{\partial \mathbf{u}^0}{\partial x_1}(0+, \hat{\mathbf{x}}) \right) + \ldots \\ &= \mathbf{v}^0(\hat{\mathbf{x}}, y_1, \hat{\mathbf{y}}) + \varepsilon \mathbf{v}^1(\hat{\mathbf{x}}, y_1, \hat{\mathbf{y}}) + \ldots.. \end{aligned}$$



Overlapping region:

$\frac{\varepsilon}{2} < |x_1| < \frac{\eta(\varepsilon)}{2}$ or

$\frac{1}{2} < |y_1| = \frac{|x_1|}{\varepsilon} < \frac{\eta(\varepsilon)}{2\varepsilon}$

- take the limit for $\varepsilon \longrightarrow 0$ with **fixed** x_1:

$$\lim_{y_1 \to \pm\infty} \left(\mathbf{v}^0(\hat{\mathbf{x}}, y_1, \hat{\mathbf{y}}) - \mathbf{u}^0(0\pm, \hat{\mathbf{x}}) \right) = \mathbf{0}$$

$$\lim_{y_1 \to \pm\infty} \left(\mathbf{v}^1(\hat{\mathbf{x}}, y_1, \hat{\mathbf{y}}) - \left(\mathbf{u}^1(0\pm, \hat{\mathbf{x}}) + y_1 \frac{\partial \mathbf{u}^0}{\partial x_1}(0\pm, \hat{\mathbf{x}}) \right) \right) = \mathbf{0}$$

Analogous matching conditions hold for the stresses.

Study of the problems at the different orders

Study of the problems at the different orders

Order $i = 0$

- Outer approximation:

$$\begin{cases} \mathbf{div}\boldsymbol{\sigma}^0 = \mathbf{0} & \text{in } \Omega \\ \boldsymbol{\sigma}^0 = \mathbf{A}\gamma(\mathbf{u}^0) & \text{in } \Omega \\ \boldsymbol{\sigma}^0\mathbf{n} = \mathbf{F} & \text{on } \Gamma_F \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \Gamma_0 \end{cases}$$

- Inner approximation

$$\mathbf{v}^0(\hat{\mathbf{x}}) = \mathbf{u}^0(0, \hat{\mathbf{x}})$$

Study of the problems at the different orders

Order $i = 0$

- Outer approximation:

$$\begin{cases} \mathbf{div}\boldsymbol{\sigma}^0 = \mathbf{0} & \text{in } \Omega \\ \boldsymbol{\sigma}^0 = \mathbf{A}\gamma(\mathbf{u}^0) & \text{in } \Omega \\ \boldsymbol{\sigma}^0\mathbf{n} = \mathbf{F} & \text{on } \Gamma_F \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \Gamma_0 \end{cases}$$

- Inner approximation

$$\mathbf{v}^0(\hat{\mathbf{x}}) = \mathbf{u}^0(0, \hat{\mathbf{x}})$$

- The heterogeneities disappear at the order 0

Study of the problems at the different orders

Order $i = 0$

- Outer approximation:

$$\begin{cases} \mathbf{div}\boldsymbol{\sigma}^0 = \mathbf{0} & \text{in } \Omega \\ \boldsymbol{\sigma}^0 = \mathbf{A}\gamma(\mathbf{u}^0) & \text{in } \Omega \\ \boldsymbol{\sigma}^0\mathbf{n} = \mathbf{F} & \text{on } \Gamma_F \\ \mathbf{u}^0 = \mathbf{0} & \text{on } \Gamma_0 \end{cases}$$

- Inner approximation

$$\mathbf{v}^0(\hat{\mathbf{x}}) = \mathbf{u}^0(0, \hat{\mathbf{x}})$$

- The heterogeneities disappear at the order 0
- This result does not depend on the type of heterogeneity

Order $i = 1$

$$\begin{cases} \mathbf{div}\boldsymbol{\sigma}^1 = \mathbf{0} & \text{in } \Omega\backslash\omega \\ \boldsymbol{\sigma}^1 = \mathbf{A}\gamma(\mathbf{u}^1) & \text{in } \Omega\backslash\omega \\ \boldsymbol{\sigma}^1\mathbf{n} = \mathbf{0} & \text{on } \Gamma_F \\ \mathbf{u}^1 = \mathbf{0} & \text{on } \Gamma_0 \end{cases}$$

Order $i = 1$

$$\begin{cases} \mathbf{div}\boldsymbol{\sigma}^1 = \mathbf{0} & \text{in } \Omega\backslash\omega \\ \boldsymbol{\sigma}^1 = \mathbf{A}\gamma(\mathbf{u}^1) & \text{in } \Omega\backslash\omega \\ \boldsymbol{\sigma}^1\mathbf{n} = \mathbf{0} & \text{on } \Gamma_F \\ \mathbf{u}^1 = \mathbf{0} & \text{on } \Gamma_0 \end{cases}$$

- Transmission conditions on ω :

$$\begin{cases} \left[\mathbf{u}^1\right](\hat{\mathbf{x}}) = u^0_{i,j}(0,\hat{\mathbf{x}})a_{i,j} \\ \left[\boldsymbol{\sigma}^1\mathbf{e}_1\right](\hat{\mathbf{x}}) = |Y|\mathbf{div}_x\left(\mathbf{A}\left(\frac{\partial \mathbf{u}^0}{\partial x_2}(0,\hat{\mathbf{x}})\otimes_S \mathbf{e}_2 + \frac{\partial \mathbf{u}^0}{\partial x_3}(0,\hat{\mathbf{x}})\otimes_S \mathbf{e}_3\right)\right) \\ -\mathbf{div}_x\left(\frac{\partial u^0_i}{\partial x_j}(0.\hat{\mathbf{x}})b_{i,j}\right) \end{cases}$$

- $\mathbf{v}^1 = u^0_{i,j}(0,\hat{\mathbf{x}})c_{i,j}$

Order $i = 1$

$$\begin{cases} \mathbf{div}\boldsymbol{\sigma}^1 = \mathbf{0} & \text{in } \Omega\backslash\omega \\ \boldsymbol{\sigma}^1 = \mathbf{A}\gamma(\mathbf{u}^1) & \text{in } \Omega\backslash\omega \\ \boldsymbol{\sigma}^1\mathbf{n} = \mathbf{0} & \text{on } \Gamma_F \\ \mathbf{u}^1 = \mathbf{0} & \text{on } \Gamma_0 \end{cases}$$

- Transmission conditions on ω :

$$\begin{cases} \left[\mathbf{u}^1\right](\hat{\mathbf{x}}) = u^0_{i,j}(0,\hat{\mathbf{x}})a_{i,j} \\ \left[\boldsymbol{\sigma}^1\mathbf{e}_1\right](\hat{\mathbf{x}}) = |Y|\mathbf{div}_x\left(\mathbf{A}\left(\frac{\partial \mathbf{u}^0}{\partial x_2}(0,\hat{\mathbf{x}}) \otimes_S \mathbf{e}_2 + \frac{\partial \mathbf{u}^0}{\partial x_3}(0,\hat{\mathbf{x}}) \otimes_S \mathbf{e}_3\right)\right) \\ -\mathbf{div}_x\left(\frac{\partial u^0_i}{\partial x_j}(0.\hat{\mathbf{x}})b_{i,j}\right) \end{cases}$$

- $\mathbf{v}^1 = u^0_{i,j}(0,\hat{\mathbf{x}})c_{i,j}$
- $|Y|$ is the volume of the heterogeneity
- the coefficients $(a_{i,j}, b_{i,j}, c_{i,j})$ are obtained from the solutions of some elementary problems that only depend on the heterogeneity (as in homogenization).

Conclusion

- The layer of heterogeneities is replaced by the internal surface ω
- $\Longrightarrow$ the macroscopic behaviour is computed on a domain without heterogeneities
- The coefficients $(a_{i,j}, b_{i,j}, c_{i,j})$ are computed only once
- The microscopic behaviour is computed for only one heterogeneity Y

Conclusion

- The layer of heterogeneities is replaced by the internal surface ω
- $\Longrightarrow$ the macroscopic behaviour is computed on a domain without heterogeneities
- The coefficients $(a_{i,j}, b_{i,j}, c_{i,j})$ are computed only once
- The microscopic behaviour is computed for only one heterogeneity Y

Drawbacks

- Only a formal method
- The transmission conditions on the exterior problem of order $i = 1$ are non usual
- The microscopic scale for the interior domain leads to an unbounded domain.

Conclusion

- The layer of heterogeneities is replaced by the internal surface ω
- $\Longrightarrow$ the macroscopic behaviour is computed on a domain without heterogeneities
- The coefficients $(a_{i,j}, b_{i,j}, c_{i,j})$ are computed only once
- The microscopic behaviour is computed for only one heterogeneity Y

Drawbacks

- Only a formal method
- The transmission conditions on the exterior problem of order $i = 1$ are non usual
- The microscopic scale for the interior domain leads to an unbounded domain.

A last (?) comment

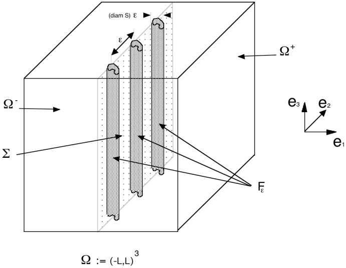
The method has been developed in a particular situation (the geometry of the heterogeneity is of the type εY); however it might be adapted for other situations

Goals of the variational convergence

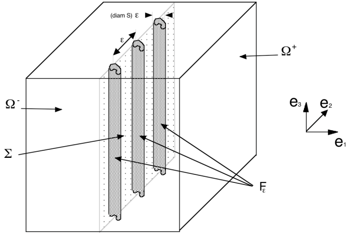
- To give a rigorous mathematical proof of the convergence of the solution $\mathbf{u}^\varepsilon$ to $\mathbf{u}^0$
- To characterize the problem whose solution is $\mathbf{u}^0$.
- To obtain error estimates and/or a first corrector

A joint work with M. Bellieud, F. Krasucki, G. Michaille

A joint work with M. Bellieud, F. Krasucki, G. Michaille



A joint work with M. Bellieud, F. Krasucki, G. Michaille



Notations

$\Sigma := \Omega \cap \{x_1 = 0\}$, $T \subset \mathbf{R}^2$ bdd, Lipschitz
$F_\varepsilon = \bigcup_{i \in I_\varepsilon} \varepsilon i \mathbf{e}_2 + \varepsilon T \times (0, L)$
$I_\varepsilon := \{i \in \mathbf{Z},\ \varepsilon i \mathbf{e}_2 + \varepsilon T \times (-L, L) \subset \Omega\}$

$$\begin{cases} \mathbf{div}\boldsymbol{\sigma}^\varepsilon = \mathbf{f} & \text{in } \Omega \\ \boldsymbol{\sigma}^\varepsilon = \lambda_\varepsilon tr(\gamma(\mathbf{u}^\varepsilon))\mathbf{I}_3 + 2\mu_\varepsilon \gamma(\mathbf{u}^\varepsilon) & \text{in } \Omega \\ \mathbf{u}^\varepsilon = \mathbf{0} & \text{on } \Gamma \end{cases}$$

$$\mu_\varepsilon = \mu_0 1_{\Omega \setminus F_\varepsilon} + k_\varepsilon \mu_1 1_{F_\varepsilon}$$
$$\lambda_\varepsilon = \lambda_0 1_{\Omega \setminus F_\varepsilon} + k_\varepsilon \lambda_1 1_{F_\varepsilon}.$$

$3\lambda_0 + 2\mu_0 > 0,\ \mu_0 > 0,$
$3\lambda_1 + 2\mu_1 > 0,\ \mu_1 > 0$

$k_\varepsilon = \frac{1}{\varepsilon^p}$ with $p > 0$

Variational formulation

$$\inf_{\mathbf{u}\in H_0^1(\Omega;\mathbf{R}^3)} \Phi_\varepsilon(\mathbf{u})$$

where

$$\Phi_\varepsilon(\mathbf{u}) := \{\frac{1}{2}\int_\Omega \{\lambda_\varepsilon(tr\boldsymbol{\gamma}(\mathbf{u}))^2 + \mu_\varepsilon \boldsymbol{\gamma}(\mathbf{u}) : \boldsymbol{\gamma}(\mathbf{u})\}dx - \int_\Omega \mathbf{f}\mathbf{u}dx\}$$

Variational formulation

$$\inf_{\mathbf{u}\in H_0^1(\Omega;\mathbf{R}^3)} \Phi_\varepsilon(\mathbf{u})$$

where

$$\Phi_\varepsilon(\mathbf{u}) := \{\frac{1}{2}\int_\Omega \{\lambda_\varepsilon(tr\gamma(\mathbf{u}))^2 + \mu_\varepsilon\gamma(\mathbf{u}) : \gamma(\mathbf{u})\}dx - \int_\Omega \mathbf{f}\mathbf{u}dx\}$$

- a priori estimate + Korn'inequality $\Longrightarrow$ existence and uniqueness of the solution $\mathbf{u}^\varepsilon$ with:

$$\int_\Omega \mathbf{u}^\varepsilon : \mathbf{u}^\varepsilon dx + \int_{\Omega\setminus F_\varepsilon} \gamma(\mathbf{u}^\varepsilon) : \gamma(\mathbf{u}^\varepsilon)dx + k_\varepsilon \int_{F_\varepsilon} \gamma(\mathbf{u}^\varepsilon) : \gamma(\mathbf{u}^\varepsilon)dx \leq C$$

Variational formulation

$$\inf_{\mathbf{u}\in H_0^1(\Omega;\mathbf{R}^3)} \Phi_\varepsilon(\mathbf{u})$$

where

$$\Phi_\varepsilon(\mathbf{u}) := \{\frac{1}{2}\int_\Omega \{\lambda_\varepsilon(tr\gamma(\mathbf{u}))^2 + \mu_\varepsilon \gamma(\mathbf{u}) : \gamma(\mathbf{u})\}dx - \int_\Omega \mathbf{f}\mathbf{u}dx\}$$

- a priori estimate + Korn'inequality $\Longrightarrow$ existence and uniqueness of the solution $\mathbf{u}^\varepsilon$ with:

$$\int_\Omega \mathbf{u}^\varepsilon : \mathbf{u}^\varepsilon dx + \int_{\Omega\setminus F_\varepsilon} \gamma(\mathbf{u}^\varepsilon) : \gamma(\mathbf{u}^\varepsilon)dx + k_\varepsilon \int_{F_\varepsilon} \gamma(\mathbf{u}^\varepsilon) : \gamma(\mathbf{u}^\varepsilon)dx \leq C$$

- $\mathbf{u}^\varepsilon$ converges weakly in $H_0^1(\Omega;\mathbf{R}^3)$ to $\mathbf{u}^0$

Variational formulation

$$\inf_{\mathbf{u}\in H_0^1(\Omega;\mathbf{R}^3)} \Phi_\varepsilon(\mathbf{u})$$

where

$$\Phi_\varepsilon(\mathbf{u}) := \{\frac{1}{2}\int_\Omega \{\lambda_\varepsilon(tr\gamma(\mathbf{u}))^2 + \mu_\varepsilon\gamma(\mathbf{u}) : \gamma(\mathbf{u})\}dx - \int_\Omega \mathbf{f}\mathbf{u}dx\}$$

- a priori estimate + Korn'inequality $\Longrightarrow$ existence and uniqueness of the solution $\mathbf{u}^\varepsilon$ with:

$$\int_\Omega \mathbf{u}^\varepsilon : \mathbf{u}^\varepsilon dx + \int_{\Omega\setminus F_\varepsilon} \gamma(\mathbf{u}^\varepsilon) : \gamma(\mathbf{u}^\varepsilon)dx + k_\varepsilon \int_{F_\varepsilon} \gamma(\mathbf{u}^\varepsilon) : \gamma(\mathbf{u}^\varepsilon)dx \leq C$$

- $\mathbf{u}^\varepsilon$ converges weakly in $H_0^1(\Omega;\mathbf{R}^3)$ to $\mathbf{u}^0$

Goal

Find the problem whose solution is $\mathbf{u}^0$, i.e. the Γ-limit of Φ_ε

Answer

The Γ-limit of Φ_ε is:

$$\Phi_0(\mathbf{u}) := \frac{1}{2}\int_\Omega \{\lambda_0(tr\gamma(\mathbf{u}))^2 + \mu_0\gamma(\mathbf{u}) : \gamma(\mathbf{u})\}dx - \int_\Omega \mathbf{f}\mathbf{u}dx + \Psi_F(\mathbf{u})$$

Answer

The Γ-limit of Φ_ε is:

$$\Phi_0(\mathbf{u}) := \frac{1}{2}\int_\Omega \{\lambda_0(tr\gamma(\mathbf{u}))^2 + \mu_0\gamma(\mathbf{u}) : \gamma(\mathbf{u})\}dx - \int_\Omega \mathbf{f}\mathbf{u}dx + \Psi_F(\mathbf{u})$$

A similar situation has been studied by A.L. Bessoud, F. Krasucki, G. Michaille (2009) when F_ε is the **full layer** $L_\varepsilon := (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \times \Sigma$. They found

$$\Phi_0(\mathbf{u}) := \frac{1}{2}\int_\Omega \{\lambda_0(tr\gamma(\mathbf{u}))^2 + \mu_0\gamma(\mathbf{u}) : \gamma(\mathbf{u})\}dx - \int_\Omega \mathbf{f}\mathbf{u}dx + \Psi_L(\mathbf{u})$$

with two significant cases: $p = 1$ and $p = 3$.

The case $p = 1$

full layer L_ε

$$\Psi_L(\mathbf{u}) = \int_\Sigma \{ \frac{2\lambda_1 mu_1}{\lambda_1 + 2mu_1} (tr\hat{\gamma}(\mathbf{u}))^2 + 2\mu_1 \hat{\gamma}(\mathbf{u}) : \hat{\gamma}(\mathbf{u}) \} d\Sigma$$

Remark: plate membrane energy

The case $p = 1$

full layer L_ε

$$\Psi_L(\mathbf{u}) = \int_\Sigma \{ \frac{2\lambda_1 mu_1}{\lambda_1 + 2mu_1} (tr\hat{\gamma}(\mathbf{u}))^2 + 2\mu_1 \hat{\gamma}(\mathbf{u}) : \hat{\gamma}(\mathbf{u}) \} d\Sigma$$

Remark: plate membrane energy

present situation

$$\Psi_F(\mathbf{u}) = k^2 \mu_1 \frac{3\lambda_1 + 2\mu_1}{2(\lambda_1 + \mu_1)} \int_\Sigma |\frac{\partial u_3}{\partial x_3}|^2 d\Sigma$$

Remark: extensional strain energy of the fibers

The case $p = 3$

full layer L_ε

$$\Psi_L(\mathbf{u}) = \frac{1}{3}\int_\Sigma \{ \frac{2\lambda_1 mu_1}{\lambda_1 + 2mu_1} (\hat{\Delta} u_1)^2 + 2\mu_1 \frac{\partial^2 u_1}{\partial x_\alpha \partial x_\beta} \frac{\partial^2 u_1}{\partial x_\alpha \partial x_\beta} \} d\Sigma$$

Remark: plate bending energy

The case $p = 3$

full layer L_ε

$$\Psi_L(\mathbf{u}) = \frac{1}{3}\int_\Sigma \{\frac{2\lambda_1 mu_1}{\lambda_1 + 2mu_1}(\hat{\Delta} u_1)^2 + 2\mu_1 \frac{\partial^2 u_1}{\partial x_\alpha \partial x_\beta}\frac{\partial^2 u_1}{\partial x_\alpha \partial x_\beta}\} d\Sigma$$

Remark: plate bending energy

present situation

$$\Psi_F(\mathbf{u}) = \sum_{\alpha,\beta=1}^{2} \mu_1 \frac{3\lambda_1 + 2\mu_1}{2(\lambda_1 + \mu_1)} k_{\alpha\beta} \int_\Sigma \frac{\partial^2 u_\alpha}{\partial x_3^2}\frac{\partial^2 u_\beta}{\partial x_3^2} d\Sigma$$

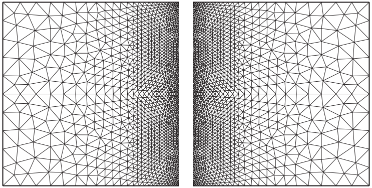
Remark: bending energy of the fibers

Proofs:

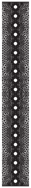
- a priori estimates
- choice of the good spaces : subspaces of $H_0^1(\Omega; \mathbf{R}^3)$ where $\Psi_F(\mathbf{u})$ has a meaning for $p = 1$, resp. $p = 3$
- ....

Thank you !

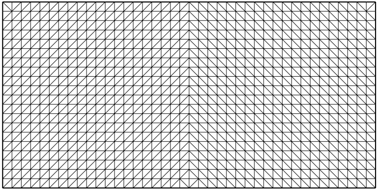
tanti auguri Gianni!!



$\mathcal{T}_h(\Omega^{ext}(\varepsilon))$



$\mathcal{T}_h(\Omega^{int}(\varepsilon))$



$\mathcal{T}_h(\Omega_0^{ext})$

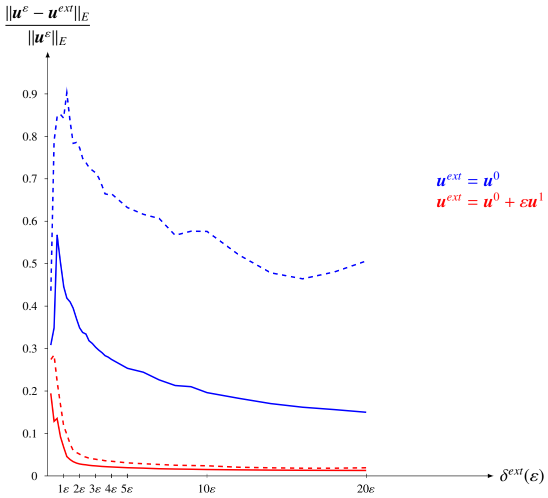


Figure: Holes $\varepsilon = \frac{1}{20}$ and $\varepsilon = \frac{1}{80}$

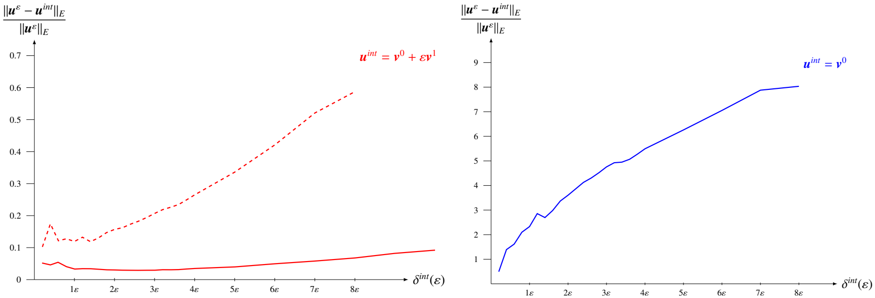


Figure: Holes