A time discretization for a nonstandard viscous Cahn–Hilliard system

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(joint work with P. Colli, G. Gilardi, P. Podio-Guidugli and P. Krejčí)
Dedicated to Gianni!
A modified \textsc{Cahn–Hilliard} system

We consider the modified Cahn–Hilliard system

\begin{align*}
(\varepsilon + 2g(\rho)) \mu_t + \mu g'(\rho) \rho_t - \Delta \mu &= 0 \quad \text{in} \quad Q := \Omega \times (0, T) \\
\rho_t - \Delta \rho + f'(\rho) &= \mu g'(\rho) \quad \text{in} \quad Q \\
\partial_{\nu} \mu = \partial_{\nu} \rho &= 0 \quad \text{on} \quad \Sigma := \partial \Omega \times (0, T) \\
\mu|_{t=0} = \mu^0, \quad \rho|_{t=0} = \rho^0 \quad \text{in} \quad \Omega
\end{align*}
We consider the modified Cahn–Hilliard system

\[(\varepsilon + 2g(\rho)) \mu_t + \mu g'(\rho) \rho_t - \Delta \mu = 0 \quad \text{in} \quad Q := \Omega \times (0, T)\]  
\[\rho_r - \Delta \rho + f'(\rho) = \mu g'(\rho) \quad \text{in} \quad Q\]  
\[\partial_\nu \mu = \partial_\nu \rho = 0 \quad \text{on} \quad \Sigma := \partial \Omega \times (0, T)\]  
\[\mu|_{t=0} = \mu^0, \quad \rho|_{t=0} = \rho^0 \quad \text{in} \quad \Omega\]

Model by P. Podio-Guidugli (2006) for phase segregation through atom rearrangement on a lattice
A modified \textbf{CAHN–HILLIARD} system

We consider the modified Cahn–Hilliard system

\[
(\varepsilon + 2g(\rho))\mu_t + \mu g'(\rho) \rho_t - \Delta \mu = 0 \quad \text{in} \quad Q := \Omega \times (0, T) \tag{1}
\]

\[
\rho_r - \Delta \rho + f'(\rho) = \mu g'(\rho) \quad \text{in} \quad Q \tag{2}
\]

\[
\partial_\nu \mu = \partial_\nu \rho = 0 \quad \text{on} \quad \Sigma := \partial \Omega \times (0, T) \tag{3}
\]

\[
\mu|_{t=0} = \mu^0, \quad \rho|_{t=0} = \rho^0 \quad \text{in} \quad \Omega \tag{4}
\]

\begin{itemize}
  \item Model by P. Podio-Guidugli (2006) for phase segregation through atom rearrangement on a lattice
  \item Studied for the special case \( g(\rho) = \rho \) in a series of papers by P. Colli, G. Gilardi, P. Podio-Guidugli and J. S. for the Allen–Cahn version (2010) and the Cahn–Hilliard version (2011 ff.) concerning well-posedness, optimal control and asymptotic behavior as \( t \to \infty \) and \( \varepsilon \searrow 0 \).
\end{itemize}
General assumptions

- $\Omega \subset \mathbb{R}^M$ is an open and bounded domain with smooth boundary $\partial \Omega$ and outward unit normal field $\nu$.

- $f = f_1 + f_2$, where $f_2 \in C^2[0, 1]$, and where $f_1 \in C^2(0, 1)$ is convex and satisfies
  \[ \lim_{r \searrow 0} f'_1(r) = -\infty \quad \text{and} \quad \lim_{r \nearrow 1} f'_1(r) = +\infty. \]

- $g \in W^{2,\infty}(0, 1)$ and $g(\rho) \geq 0 \quad \forall \rho \in [0, 1]$.

- $\mu^0 \in V \cap L^\infty(\Omega)$, and $\mu^0 \geq 0$ a.e. in $\Omega$.

- $\rho^0 \in W$, $0 < \rho^0 < 1$ in $\bar{\Omega}$, and $f'(\rho^0) \in H$.

Here, we set: $V := H^1(\Omega)$, $H := L^2(\Omega)$, $W := \{v \in H^2(\Omega); \partial_\nu v = 0 \quad \text{on} \ \partial \Omega\}$. 
Well-posedness (see Gianni’s talk)

In the recent CGPS paper “Global existence and uniqueness for a singular/degenerate Cahn–Hilliard system with viscosity” (submitted, see WIAS preprint No. 1713 (2012))

it was shown that (1)–(4) has a unique solution $(\mu, \rho)$ having the following properties:

- $\mu \in H^1(0, T; H) \cap L^2(0, T; W) \cap L^\infty(Q)$
- $\rho \in W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W)$
- $\mu \geq 0$ a.e. in $Q$
- There exist $\rho_*, \rho^* \in (0, 1)$ such that $\rho_* \leq \rho \leq \rho^*$ a.e. in $Q$.

A similar result can also be obtained under weaker conditions (see the above paper).
In this talk, we make the first step towards the numerical approximation of (1)–(4). We choose $N \in \mathbb{N}$, put $h = \frac{T}{N}$, and consider for $0 \leq n \leq N - 1$ the time-discretized problem (where we put $\varepsilon = 1$)

\[
(1 + 2g(\rho_n)) \frac{\mu_{n+1} - \mu_n}{h} + \frac{g(\rho_{n+1}) - g(\rho_n)}{h} \mu_{n+1} - \Delta \mu_{n+1} = 0 \quad \text{in } \Omega \tag{5}
\]

\[
\frac{\rho_{n+1} - \rho_n}{h} - \Delta \rho_{n+1} + f'(\rho_{n+1}) = \mu_n g'(\rho_n) \quad \text{in } \Omega \tag{6}
\]

\[
\partial_\nu \mu_{n+1} = \partial_\nu \rho_{n+1} = 0 \quad \text{on } \partial \Omega \tag{7}
\]

\[
\mu_0 = \mu^0, \quad \rho_0 = \rho^0 \quad \text{in } \Omega \tag{8}
\]

**Aim:** Well-posedness of the scheme, convergence of discrete solutions to $(\mu, \rho)$ as $N \to \infty$, error estimates
We argue by induction for $n \in \mathbb{N}$. Suppose that for some $0 \leq n < N - 1$ we have found $(\mu_n, \rho_n) \in W \times W$ such that $\mu_n \geq 0$ a.e. in $\Omega$, $f'(\rho_n) \in H$ and $0 < \rho_n < 1$ in $\overline{\Omega}$.

We rewrite (5), (6) in the form

$$
(1 + g(\rho_n) + g(\rho_{n+1})) \mu_{n+1} - h \Delta \mu_{n+1} = (1 + 2g(\rho_n)) \mu_n \quad \text{in } \Omega \quad (9)
$$

$$
\rho_{n+1} - h \Delta \rho_{n+1} + h f'(\rho_{n+1}) = \rho_n + h \mu_n g'(\rho_n) \quad \text{in } \Omega \quad (10)
$$

Now let $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$, where

- $\tilde{f}_2$ is any smooth extension of $f_2$ to $\mathbb{R}$,
- $\tilde{f}_1$ is the unique convex and l.s.c. extension of $f_1$ to $\mathbb{R}$ that satisfies $\tilde{f}(r) = +\infty$ if $r \not\in (0, 1)$.

Then the function $r \mapsto \frac{1}{2} r^2 + h \tilde{f}(r)$ is strictly convex provided that

$$
h \max_{0 \leq r \leq 1} |f''_2(r)| < 1 \quad (11)
$$

We will always assume this in the following.
Existence of the discrete solution II

For $h > 0$ satisfying (11), it follows from standard arguments that the strictly convex, coercive and l.s.c. functional $J : V \to (−\infty, +\infty]$, 

$$J(v) = \begin{cases} 
\frac{h}{2} \int_{\Omega} |\nabla v|^2 \, dx + \int_{\Omega} \left( \frac{v^2}{2} + h\tilde{f}(v) \right) \, dx - \int_{\Omega} (\rho_n + h\mu_n g'(\rho_n)) v \, dx, & \text{if } \tilde{f}(v) \in L^1(\Omega) \\
+\infty, & \text{otherwise}
\end{cases}$$

has a unique minimizer $\rho_{n+1}$ on $V$. Standard arguments (maximal monotonicity, Euler–Lagrange, elliptic regularity) then show that actually $\rho_{n+1} \in W$ solves (10) and that $0 < \rho_{n+1} < 1$ in $\overline{\Omega}$.

But then also $g(\rho_{n+1}) \geq 0$ in $\Omega$, and the elliptic boundary value problem (9), $\partial_{\nu} \mu_{n+1} = 0$ on $\partial\Omega$, has a unique solution $\mu_{n+1} \in W$. Testing (9) by $-\mu_{n+1}^- \leq 0$ yields immediately that $\mu_{n+1} \geq 0$ a.e. in $\Omega$. 

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Step 1: Test (5) by $h \mu_{n+1} \Rightarrow$

$$\frac{1}{2} \| \mu_{n+1} \|_H^2 + \frac{1}{2} \| \mu_{n+1} - \mu_n \|_H^2 - \frac{1}{2} \| \mu_n \|_H^2 + h \int_{\Omega} |\nabla \mu_{n+1}|^2 \, dx$$

$$+ \int_{\Omega} \left[ g(\rho_{n+1}) \mu_{n+1}^2 - g(\rho_n) \mu_n^2 + g(\rho_n)(\mu_{n+1} - \mu_n)^2 \right] \, dx = 0$$

Summation $\Rightarrow$
A priori estimates

Step 1: Test (5) by $h \mu_{n+1}$

$$\frac{1}{2} \| \mu_{n+1} \|^2_H + \frac{1}{2} \| \mu_{n+1} - \mu_n \|^2_H - \frac{1}{2} \| \mu_n \|^2_H + h \int_\Omega |\nabla \mu_{n+1}|^2 \, dx$$

$$+ \int_\Omega \left[ g(\rho_{n+1}) \mu_{n+1}^2 - g(\rho_n) \mu_n^2 + g(\rho_n) (\mu_{n+1} - \mu_n)^2 \right] \, dx = 0$$

Summation

$$\int_\Omega \left( \frac{1}{2} + g(\rho_n) \right) \mu_m^2 \, dx + \sum_{n=0}^{m-1} \int_\Omega \left( \frac{1}{2} + g(\rho_n) \right) |\mu_{n+1} - \mu_n|^2 \, dx$$

$$+ h \sum_{n=0}^{m-1} \int_\Omega |\nabla \mu_{n+1}|^2 \, dx \leq C, \quad 1 \leq m \leq N.$$  \hspace{1cm} (12)
Step 2: Test (6) by $\rho_{n+1} - \rho_n$

$$h \left\| \frac{\rho_{n+1} - \rho_n}{h} \right\|_H^2 + \frac{1}{2} \left( \left\| \nabla \rho_{n+1} \right\|_H^2 - \left\| \nabla \rho_n \right\|_H^2 + \left\| \nabla (\rho_{n+1} - \rho_n) \right\|_H^2 \right)$$

$$+ \int_{\Omega} \left( f_1(\rho_{n+1}) - f_1(\rho_n) \right) \, dx \leq \int_{\Omega} C \, h \left( 1 + |\mu_n| \right) \frac{\rho_{n+1} - \rho_n}{h} \, dx ,$$

using the boundedness of $g'$ and $f_2'$. Young’s inequality and summation over $n$ yield
Step 2: Test (6) by $ρ_{n+1} - ρ_n$ 

$$h \left\| \frac{ρ_{n+1} - ρ_n}{h} \right\|_H^2 + \frac{1}{2} \left( \left\| \nabla ρ_{n+1} \right\|_H^2 - \left\| \nabla ρ_n \right\|_H^2 + \left\| \nabla (ρ_{n+1} - ρ_n) \right\|_H^2 \right)$$

$$+ \int_Ω \left( f_1(ρ_{n+1}) - f_1(ρ_n) \right) dx \leq \int_Ω C' h \left( 1 + |μ_n| \right) \frac{ρ_{n+1} - ρ_n}{h} dx ,$$

using the boundedness of $g'$ and $f'_2$. Young's inequality and summation over $n$ yield

$$\sum_{n=0}^{m-1} h \left\| \frac{ρ_{n+1} - ρ_n}{h} \right\|_H^2 + \left\| \nabla ρ_m \right\|_H^2 + \sum_{n=0}^{m-1} \left\| \nabla (ρ_{n+1} - ρ_n) \right\|_H^2 + \int_Ω f_1(ρ_m) dx \leq C.$$

(13)
A priori estimates II

**Step 2:** Test (6) by $\rho_{n+1} - \rho_n \\n
$$h \left\| \frac{\rho_{n+1} - \rho_n}{h} \right\|_H^2 + \frac{1}{2} \left( \left\| \nabla \rho_{n+1} \right\|_H^2 - \left\| \nabla \rho_n \right\|_H^2 + \left\| \nabla (\rho_{n+1} - \rho_n) \right\|_H^2 \right)$$ \\

$$+ \int_\Omega \left( f_1(\rho_{n+1}) - f_1(\rho_n) \right) dx \leq \int_\Omega C' h \left( 1 + |\mu_n| \right) \frac{\rho_{n+1} - \rho_n}{h} \, dx,$$

using the boundedness of $g'$ and $f_2'$. Young's inequality and summation over $n$ yield

$$\sum_{n=0}^{m-1} h \left\| \frac{\rho_{n+1} - \rho_n}{h} \right\|_H^2 + \left\| \nabla \rho_m \right\|_H^2 + \sum_{n=0}^{m-1} \left\| \nabla (\rho_{n+1} - \rho_n) \right\|_H^2 + \int_\Omega f_1(\rho_m) \, dx \leq C.$$ (13)

**Step 3:** Test (6) by $-h \Delta \rho_{n+1}$ and by $h f_1'(\rho_{n+1}) \\n
$$\sum_{n=0}^{m-1} h \left\| \rho_n \right\|_W^2 + \sum_{n=0}^{m-1} h \left\| f_1'(\rho_m) \right\|_H^2 \leq C.$$ (14)
**Step 4:** Take the difference of (6), written for $n + 1$ and $n$, and test by $h^{-1}(\rho_{n+2} - \rho_{n+1})$. We obtain

$$
\frac{1}{2} \left[ \left\| \frac{\rho_{n+2} - \rho_{n+1}}{h} \right\|_H^2 - \left\| \frac{\rho_{n+1} - \rho_n}{h} \right\|_H^2 + \left\| \frac{\rho_{n+2} - \rho_{n+1}}{h} - \frac{\rho_{n+1} - \rho_n}{h} \right\|_H^2 \right]
$$

$$
+ h \int_\Omega \left| \nabla \frac{\rho_{n+2} - \rho_{n+1}}{h} \right|^2 \, dx + \frac{1}{h} \int_\Omega \left( f'_1(\rho_{n+2}) - f'_1(\rho_{n+1}) \right) (\rho_{n+2} - \rho_{n+1}) \, dx
$$

$$
\leq C h \left\| \frac{\rho_{n+2} - \rho_{n+1}}{h} \right\|_H^2 + C h \int_\Omega \mu_{n+1} \left\| \frac{\rho_{n+2} - \rho_{n+1}}{h} \right\|_H \left\| \frac{\rho_{n+1} - \rho_n}{h} \right\|_H \, dx
$$

$$
+ \int_\Omega (\mu_{n+1} - \mu_n) g'(\rho_n) \frac{\rho_{n+2} - \rho_{n+1}}{h} \, dx.
$$

One now substitutes for $\mu_{n+1} - \mu_n$ from (5). Young, Hölder, the embedding $V \hookrightarrow L^4(\Omega)$ and the discrete Gronwall lemma imply the estimate
\[
\left\| \frac{\rho_{m+1} - \rho_m}{h} \right\|_H^2 + \sum_{n=0}^{m-1} \left\| \frac{\rho_{n+2} - \rho_{n+1}}{h} - \frac{\rho_{n+1} - \rho_n}{h} \right\|_H^2 + \sum_{n=0}^{m-1} h \left\| \nabla \frac{\rho_{n+2} - \rho_{n+1}}{h} \right\|_H^2 \leq C. \tag{15}
\]

**Step 5:** Write (5) in the form

\[
\left( 1 + g(\rho_n) + g(\rho_{n+1}) \right) \frac{\mu_{n+1} - \mu_n}{h} + \frac{g(\rho_{n+1}) - g(\rho_n)}{h} \mu_n - \Delta \mu_{n+1} = 0
\]

and test by \(h^{-1}(\mu_{n+1} - \mu_n)\). Hölder, Young, embeddings, discrete Gronwall

\[
\sum_{n=0}^{m-1} h \left\| \frac{\mu_{n+1} - \mu_m}{h} \right\|_H^2 + \left\| \nabla \mu_m \right\|_H^2 + \sum_{n=0}^{m-1} \left\| \nabla (\mu_{n+1} - \mu_n) \right\|_H^2 \leq C. \tag{16}
\]

From this, we can deduce further estimates by comparison.
Discrete approximations

Let \( t_n = n h \), \( 0 \leq n \leq N \). We define for \( t_{n-1} < t \leq t_n \), \( 1 \leq n \leq N \):

\[
\begin{align*}
\mu^h(t) &= \mu_n, \quad \bar{\mu}^h = \mu_{n-1}, \quad \rho^h(t) = \rho_n, \quad \bar{\rho}^h(t) = \rho_{n-1}, \quad \xi^h(t) = f_1'(\rho_n), \\
\tilde{\mu}^h(t) &= \mu_{n-1} + \frac{1}{h}(t - t_{n-1})(\mu_n - \mu_{n-1}), \quad \tilde{\rho}^h(t) = \rho_{n-1} + \frac{1}{h}(t - t_{n-1})(\rho_n - \rho_{n-1}), \\
\tilde{\eta}^h(t) &= g(\rho_{n-1}) + \frac{1}{h}(t - t_{n-1})(g(\rho_n) - g(\rho_{n-1})).
\end{align*}
\]
Discrete approximations

Let $t_n = nh$, $0 \leq n \leq N$. We define for $t_{n-1} < t \leq t_n$, $1 \leq n \leq N$: 

\[
\mu^h(t) = \mu_n, \quad \bar{\mu}^h = \mu_{n-1}, \quad \rho^h(t) = \rho_n, \quad \bar{\rho}^h(t) = \rho_{n-1}, \quad \xi^h(t) = f'_1(\rho_n), \\
\tilde{\mu}^h(t) = \mu_{n-1} + \frac{1}{h}(t-t_{n-1})(\mu_n - \mu_{n-1}), \quad \tilde{\rho}^h(t) = \rho_{n-1} + \frac{1}{h}(t-t_{n-1})(\rho_n - \rho_{n-1}), \\
\tilde{\eta}^h(t) = g(\rho_{n-1}) + \frac{1}{h}(t-t_{n-1})(g(\rho_n) - g(\rho_{n-1})).
\]

Obviously, for $t_{n-1} < t \leq t_n$, $1 \leq n \leq N$, we have 

\[
\tilde{\mu}^h_t(t) = \frac{\mu_n - \mu_{n-1}}{h}, \quad \tilde{\rho}^h_t = \frac{\rho_n - \rho_{n-1}}{h}, \quad \tilde{\eta}^h_t = \frac{g(\rho_n) - g(\rho_{n-1})}{h}.
\] (17)
Discrete approximations

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\begin{align*}
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\tilde{\mu}^h(t) &= \mu_{n-1} + \frac{1}{h}(t-t_{n-1})(\mu_n-\mu_{n-1}), \quad \tilde{\rho}^h(t) = \rho_{n-1} + \frac{1}{h}(t-t_{n-1})(\rho_n-\rho_{n-1}), \\
\tilde{\eta}^h(t) &= g(\rho_n) + \frac{1}{h}(t-t_{n-1})(g(\rho_n)-g(\rho_{n-1})).
\end{align*}
\]

Obviously, for \( t_{n-1} < t \leq t_n \), \( 1 \leq n \leq N \), we have

\[
\begin{align*}
\tilde{\mu}^h_t(t) &= \frac{\mu_n-\mu_{n-1}}{h}, \quad \tilde{\rho}^h_t = \frac{\rho_n-\rho_{n-1}}{h}, \quad \tilde{\eta}^h_t = \frac{g(\rho_n)-g(\rho_{n-1})}{h}.
\end{align*}
\]  \hspace{1cm} (17)

Notice that we have

\[
\frac{1}{2} \| \tilde{\mu}^h - \bar{\mu}^h \|_{L^2(Q)}^2 = \sum_{n=1}^{N} h^{-2} \| \mu_n - \mu_{n-1} \|_H^2 \int_{t_{n-1}}^{t_n} (t-t_{n-1})^2 dt = \frac{h}{3} \sum_{n=1}^{N} \| \mu_n - \mu_{n-1} \|_H^2 \leq Ch. \]  \hspace{1cm} (18)

Due to regularity of the time derivative of \( \tilde{\mu}^h \), we even have

\[
\| \tilde{\mu}^h - \bar{\mu}^h \|_{L^2(Q)}^2 = \frac{h^2}{3} \sum_{n=1}^{N} h \left\| \frac{\mu_n - \mu_{n-1}}{h} \right\|_H^2 = \frac{h^2}{3} \| \tilde{\mu}^h_t \|_{L^2(Q)}^2 \leq C h^2. \]  \hspace{1cm} (19)
Similar estimates can be proved for
\[ \| \tilde{\mu}^h - \mu^h \|^2_{L^2(Q)} , \quad \| \mu^h - \bar{\mu}^h \|^2_{L^2(Q)} , \quad \| \bar{\rho}^h - \rho^h \|^2_{L^2(Q)} , \quad \| \rho^h - \bar{\rho}^h \|^2_{L^2(Q)} . \]
and even for stronger norms.
Similar estimates can be proved for
\[ \| \tilde{\mu}^h - \mu^h \|_{L^2(Q)}^2, \quad \| \mu^h - \bar{\mu}^h \|_{L^2(Q)}^2, \quad \| \tilde{\rho}^h - \rho^h \|_{L^2(Q)}^2, \quad \| \rho^h - \bar{\rho}^h \|_{L^2(Q)}^2. \]
and even for stronger norms.

Owing to the a priori estimates, we have for \( h \downarrow 0 \) (i.e., for \( N \to \infty \)):

\[ \tilde{\mu}^h \to \mu \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \]  \hspace{1cm} (20)
\[ \mu^h \to \mu \quad \text{weakly in } L^2(0, T; W) \]  \hspace{1cm} (21)
\[ \tilde{\mu}^h, \bar{\mu}^h, \mu^h \to \mu \quad \text{strongly in } L^\infty(0, T; H) \quad \text{(by compactness)} \]  \hspace{1cm} (22)
Similar estimates can be proved for
\[ \|\tilde{\mu}^h - \mu^h\|^2_{L^2(Q)}, \quad \|\mu^h - \bar{\mu}^h\|^2_{L^2(Q)}, \quad \|\tilde{\rho}^h - \rho^h\|^2_{L^2(Q)}, \quad \|\rho^h - \bar{\rho}^h\|^2_{L^2(Q)}. \]
and even for stronger norms.

Owing to the a priori estimates, we have for \( h \downarrow 0 \) (i.e., for \( N \to \infty \)):

\begin{align*}
\tilde{\mu}^h &\to \mu \quad \text{weakly star in } H^1(0, T; H) \cap L^\infty(0, T; V) \quad (20) \\
\mu^h &\to \mu \quad \text{weakly in } L^2(0, T; W) \quad (21) \\
\tilde{\mu}^h, \bar{\mu}^h, \mu^h &\to \mu \quad \text{strongly in } L^\infty(0, T; H) \quad \text{(by compactness)} \quad (22) \\
\tilde{\rho}^h &\to \rho \quad \text{weakly star in } W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; W) \quad \text{(by compactness)} \\
\rho^h, \bar{\rho}^h &\to \rho \quad \text{weakly star in } L^\infty(0, T; W) \\
\end{align*}
Similar estimates can be proved for
\[
\|\tilde{\mu}^h - \mu^h\|_{L^2(Q)}^2, \quad \|\mu^h - \bar{\mu}^h\|_{L^2(Q)}^2, \quad \|\tilde{\rho}^h - \rho^h\|_{L^2(Q)}^2, \quad \|\rho^h - \bar{\rho}^h\|_{L^2(Q)}^2.
\]
and even for stronger norms.

Owing to the a priori estimates, we have for \( h \downarrow 0 \) (i.e., for \( N \to \infty \)):

\[
\tilde{\mu}^h \to \mu \quad \text{weakly star in } H^1(0,T;H) \cap L^\infty(0,T;V) \tag{20}
\]

\[
\mu^h \to \mu \quad \text{weakly in } L^2(0,T;W) \tag{21}
\]

\[
\tilde{\mu}^h, \bar{\mu}^h, \mu^h \to \mu \quad \text{strongly in } L^\infty(0,T;H) \quad \text{(by compactness)} \tag{22}
\]

\[
\tilde{\rho}^h \to \rho \quad \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W) \text{ and strongly in } C^0([0,T]; H^{2-\varepsilon}(\Omega)) \hookrightarrow C^0(\bar{Q}) \tag{23}
\]

\[
\rho^h, \bar{\rho}^h \to \rho \quad \text{weakly star in } L^\infty(0,T;W) \text{ and strongly in } L^\infty(0,T;V) \quad \text{(by compactness)} \tag{24}
\]

\[
\xi^h \to \xi \quad \text{weakly star in } L^\infty(0,T;H) \tag{25}
\]

\[
\tilde{\eta}^h \to \eta \quad \text{weakly star in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \text{ and strongly in } C^0([0,T];H) \tag{26}
\]
Identification of the limits:

- Since $f'_1$ is maximal monotone, we easily deduce that $\xi = f'_1(\rho)$
Identification of the limits:

- Since $f_1'$ is maximal monotone, we easily deduce that $\xi = f_1'(\rho)$
- Using $V \hookrightarrow L^6(\Omega)$ and the Lipschitz continuity of $g, g', f_2'$, we find

  \[ g(\bar{\rho}_h) \to g(\rho), \quad g'(\bar{\rho}_h) \to g'(\rho), \quad f_2'(\rho_h) \to f_2'(\rho), \]

  all strongly in $L^\infty(0, T; L^6(\Omega))$. (27)
Identification of the limits:

- Since $f_1'$ is maximal monotone, we easily deduce that $\xi = f_1'(\rho)$

- Using $V \hookrightarrow L^6(\Omega)$ and the Lipschitz continuity of $g, g', f_2'$, we find

$$g(\bar{\rho}^h) \to g(\rho), \quad g'(\bar{\rho}^h) \to g'(\rho), \quad f_2'(\rho^h) \to f_2'(\rho),$$

all strongly in $L^\infty(0, T; L^6(\Omega))$. \hfill (27)

- Similarly, we find

$$\eta^h, \bar{\eta}^h \to g(\rho) \quad \text{strongly in } L^\infty(0, T; L^6(\Omega)) \quad \Rightarrow$$

$$\tilde{\eta}^h \to \eta \equiv g(\rho) \quad \text{strongly in } L^\infty(0, T; L^6(\Omega)) \quad \hfill (28)$$
Identification of the limits:

- Since \( f_1' \) is maximal monotone, we easily deduce that \( \xi = f_1'(\rho) \)

- Using \( V \hookrightarrow L^6(\Omega) \) and the Lipschitz continuity of \( g, g', f_2' \), we find

\[
g(\bar{\rho}^h) \to g(\rho), \quad g'(\bar{\rho}^h) \to g'(\rho), \quad f_2'(\rho^h) \to f_2'(\rho),
\]
all strongly in \( L^\infty(0,T; L^6(\Omega)) \).

(27)

- Similarly, we find

\[
\eta^h, \quad \bar{\eta}^h \to g(\rho) \quad \text{strongly in } L^\infty(0,T; L^6(\Omega)) \quad \Rightarrow \quad \tilde{\eta}^h \to \eta \equiv g(\rho) \quad \text{strongly in } L^\infty(0,T; L^6(\Omega)) \]

(28)

- Moreover,

\[
\bar{\mu}^h g'(\bar{\rho}^h) \to \mu g'(\rho) \quad \text{strongly in } L^\infty(0,T; L^{3/2}(\Omega)) \quad (29)
\]
\[
g(\bar{\rho}^h) \tilde{\mu}^h \to g(\rho) \mu_t \quad \text{weakly in } L^2(0,T; L^{3/2}(\Omega)) \quad (30)
\]
\[
\tilde{\eta}^h \mu^h \to (g(\rho))_t \mu \quad \text{weakly in } L^2(0,T; L^{3/2}(\Omega)) \quad (31)
\]
Now observe that the discrete equations can be written as

\[(1 + 2g(\bar{\rho}^h)) \tilde{\mu}^h_t + (\tilde{\eta}^h_t) \mu^h - \Delta \mu^h = 0 \quad \text{in } Q\] (32)

\[\tilde{\rho}^h_t - \Delta \rho^h + \xi^h = \bar{\mu}^h g'(\bar{\rho}^h) - f'_2(\rho^h) \quad \text{in } Q\] (33)

\[\partial_\nu \mu^h = \partial_\nu \rho^h = 0 \quad \text{on } \Sigma\] (34)

\[\tilde{\mu}^h(0) = \mu^0, \quad \tilde{\rho}^h(0) = \rho^0, \quad \text{in } \Omega\] (35)

It thus follows that the limit \((\mu, \rho)\) is the (unique) solution to (1)–(4)!
General strategy:

- Subtract (1) from (32) and test the difference by $(\tilde{\rho}^h - \rho)_t$.
- Subtract (2) from (33) and test the difference by $\tilde{\mu}^h - \mu$.
- Add the results.
- Estimate!

This strategy works in principle but requires lengthy estimates using similar techniques as in the derivation of the a priori estimates. However, one needs two preparatory results:
Further estimates I

Preparation 1: We have for $0 < \varepsilon < 1$, by interpolation,

$$\| (\rho^h - \rho)(t) \|_{H^{2-\varepsilon}(\Omega)} \leq C \| (\rho^h - \rho)(t) \|_V^\alpha \| (\rho^h - \rho)(t) \|_{W^{1-\alpha}}^1$$

for some $\alpha \in (0, 1)$ depending on $\varepsilon$, and thus (23), (24) imply

$$\rho^h \to \rho \quad \text{strongly in } L^\infty(0, T; H^{2-\varepsilon}(\Omega))$$

(36)

We infer that there are $\bar{N} \in \mathbb{N}$ and $\bar{r}, \hat{r}$ such that for any $N \geq \bar{N}$ it holds (where $h = h_N = \frac{T}{N}$)

$$0 < \bar{r} \leq \rho, \rho^h, \bar{\rho}^h, \hat{\rho}^h \leq \hat{r} < 1$$

(37)

Assuming $N \geq \bar{N}$ in the following, we can claim henceforth that $f'_1$ and $f' = f'_1 + f'_2$ are Lipschitz in the range of relevant arguments.
Preparation 2:

Under the further assumption $-\Delta \rho_0 + f'(\rho_0) - \mu_0 g'(\rho_0) \in V$, which is satisfied if $\rho_0 \in H^3(\Omega)$, we can show that

$$\|\tilde{\rho}_h^t\|_{L^\infty(0,T;V) \cap L^2(0,T;W)} \leq C.$$  \hfill (38)

and

$$\|\Delta (\tilde{\rho}_h^h - \rho_h^h)\|_{L^2(0,T;H)} \leq C h^2.$$  \hfill (39)

We then obtain the following error estimate:
Theorem:

Let the general assumptions hold, and let $\rho_0 \in H^3$. Suppose that $N \in \mathbb{N}$ is so large that for $h = \frac{T}{N}$ we have:

1. $\max_{0 \leq r \leq 1} |f''_2(r)| < \frac{1}{h}$.
2. $0 < \bar{r} \leq \rho, \rho^h, \bar{\rho}^h, \tilde{\rho}^h \leq \hat{r} < 1.$

Then it holds

$$\|\tilde{\rho}^h - \rho\|_{H^1(0,T;H)} + \|\tilde{\mu}^h - \mu\|_{L^\infty(0,T;H)} \leq C h^{1/2},$$

where $C > 0$ depends only on the data.

Remark: One obtains $h$ in place of $h^{1/2}$ provided one can show that

$$\|\nabla \tilde{\mu}_t^h\|_{L^2(Q)} \leq C.$$  (Ongoing work; requires $\mu_0 \in W$)
References:

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Ad multos annos, Gianni!