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The conserved phase–field system with memory

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Standard conserved phase–field system in $\Omega \times (0,T)$:

$$\partial_t(\vartheta + \ell\chi) - \Delta\vartheta = g \tag{1}$$

$$\partial_t\chi - \Delta w = 0 \tag{2}$$

$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta \tag{3}$$

I.C. for ϑ, χ; B.C. for ϑ, χ, w

Ω = domain in $\mathbb{R}^3$ (bounded, etc.)
ϑ = relative temperature
χ = order parameter
w = chemical potential
ℓ = latent heat
g = source term

Homogeneous Neumann condition for w

$\stackrel{(2)}{\Longrightarrow}$ conserved, i.e.

$$\partial_t \int_\Omega \chi \, dx = 0$$

The above first equation

$$\partial_t(\vartheta + \ell\chi) - \Delta\vartheta = g$$

comes from the energy balance

$$\partial_t(\vartheta + \ell\chi) + \operatorname{div}\mathbf{q} = g$$

and the **Fourier law** for the heat flux

$$\mathbf{q} = -k_0\nabla\vartheta \qquad (k_0 = 1)$$

Gurtin–Pipkin law:

$$\begin{aligned}\mathbf{q}(x,t) &= -\int_{-\infty}^{t} k(t-s)\nabla\vartheta(x,s)\,ds \\ &= -\int_{-\infty}^{0} k(t-s)\nabla\vartheta(x,s)\,ds \\ &\quad -\int_{0}^{t} k(t-s)\nabla\vartheta(x,s)\,ds \\ &= \mathbf{q}_0(x,t) - (k * \nabla\vartheta)(x,t).\end{aligned}$$

Hence the energy balance becomes

$$\partial_t(\vartheta + \ell\chi) - \Delta(k * \vartheta) = \text{new } g$$

accounting also for the **known** past history $\mathbf{q}_0$.

Typical assumptions on the **memory kernel**:

$k \in L^1(0,T)$ and

either smooth and $k(0) > 0$

or pos. type, i.e., $\displaystyle\int_0^t (k * v)(s)v(s)\,ds \geq 0 \quad \forall\, t, v.$

Prototype:

$$k_\varepsilon(t) = \frac{k_0}{\varepsilon}\exp(-t/\varepsilon)$$

where $k_0, \varepsilon > 0$. It comes from

$$\varepsilon\partial_t \mathbf{q} + \mathbf{q} = -k_0 \nabla\vartheta \qquad \text{Cattaneo–Maxwell law.}$$

For small ε

$$k_\varepsilon \sim k_0 \times \{\text{Dirac mass}\} \quad \text{and} \quad \text{C-M} \sim \text{Fourier}$$

So, we consider the system

$$
\begin{aligned}
&\partial_t(\vartheta + \ell\chi) - \Delta(k * \vartheta) = g && (1)\\
&\partial_t\chi - \Delta w = 0 && (2)\\
&w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta && (3)\\
&\text{Neumann B.C. for } k * \vartheta, \chi, w && (4)\\
&\text{I.C. for } \vartheta, \chi && (5)
\end{aligned}
$$

or some generalization of it.

Conserved case, without memory:

$$
\begin{aligned}
&\partial_t(\vartheta + \ell\chi) - \Delta\vartheta = g \\
&\partial_t\chi - \Delta w = 0 \\
&w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta
\end{aligned}
$$

I.C. and B.C.

or some generalization.

Caginalp
Alt, Palow
Kenmochi, Niezgódka
Brochet, Hilhorst, Novick-Cohen
. . . .
existence, uniqueness, attractors. . .

Wide literature on Cahn–Hilliard equation, e.g.,

Novick-Cohen (survey)
Kenmochi, Niezgódka, Pawlow (C-H + constraints)

Some references and plan 5

Non-conserved case, with memory (G-P):

$$\partial_t(\vartheta + \ell\chi) - \Delta(k * \vartheta) = g$$
$$\partial_t\chi + w = 0 \quad \text{(instead of} \quad \partial_t\chi - \Delta w = 0)$$
$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta$$

I.C. and B.C.

or some generalization.

Aizicovici, Barbu
Colli, G., Grasselli
Colli, Laurençot
. . . .
existence, uniqueness, longtime behavior, asymptotic analyses. . .

Other laws with memory for **q**
. . . .

Conserved case, with memory:

$$\partial_t(\vartheta + \ell\chi) - \Delta(k * \vartheta) = g \qquad (1)$$
$$\partial_t\chi - \Delta w = 0 \qquad (2)$$
$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta \qquad (3)$$
$$\text{B.C. for } k * \vartheta, \chi, w \qquad (4)$$
$$\text{I.C. for } \vartheta, \chi \qquad (5)$$

Novick-Cohen:
existence, regularizing effect, partial uniqueness

- Colli, G., Laurençot, Novick-Cohen:
  uniqueness, longtime behavior
- Colli, G., Grasselli, Schimperna:
  existence, uniqueness, regularity
  for a generalized system
- Felli, Rocca:
  asymptotic analyses

Felli: limit as $\varepsilon \to 0$ of

$$\partial_t(\vartheta + \ell\chi) - \Delta(k_\varepsilon * \vartheta) = g \tag{1}$$

$$\text{where} \qquad k_\varepsilon(t) \sim \frac{k_0}{\varepsilon}\exp(-t/\varepsilon)$$

$$\partial_t\chi - \Delta w = 0 \tag{2}$$

$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta \tag{3}$$

$$\text{N.B.C. for } k_\varepsilon * \vartheta, \chi, w. \tag{4}$$

$$\text{I.C. for } \vartheta, \chi \tag{5}$$

The limit problem is the conserved phase field system without memory.

Rocca: limit as $\mu \to 0$ of

$$\partial_t(\vartheta + \ell\chi) - \Delta(k * \vartheta) = g \tag{1}$$

$$\mu\partial_t\chi - \Delta w = 0 \tag{2}$$

$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta \tag{3}$$

$$\text{N.B.C. for } k * \vartheta, \chi, w. \tag{4}$$

$$\text{I.C. for } \vartheta, \chi \tag{5}$$

with some restriction on ℓ. The limit problem is obtained taking $\mu = 0$ (I.C. just for $\vartheta + \ell\chi$). In the limit the subsystem (2-3) can be simplified.

Recall (forgetting about I.C.)

$$\partial_t(\vartheta + \ell\chi) - \Delta(k * \vartheta) = g \tag{1}$$
$$\partial_t\chi - \Delta w = 0 \tag{2}$$
$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta \tag{3}$$
$$\text{homogeneous N.B.C. for } k * \vartheta, \chi, w \tag{4}$$

$$H := L^2(\Omega), \quad V := H^1(\Omega),$$
$$W := \{v \in H^2(\Omega) : \partial_n v = 0\}$$
$$\implies \quad W \subset V \subset H \subset V' \subset W'$$
$$\langle Au, v\rangle := \int_\Omega (-\Delta u)v = \int_\Omega \nabla u \cdot \nabla v = \int_\Omega u(-\Delta v)$$
$$(u, v \in \text{ appropriate spaces})$$
$$\implies \quad A : W \to H, \quad A : V \to V', \quad A : H \to W'.$$

Then (1–4) becomes

$$(\vartheta + \ell\chi)' + A(k * \vartheta) = g$$
$$\chi' + Aw = 0$$
$$w := A\chi + \chi^3 - \chi - \ell\vartheta$$

and I.C. for ϑ, χ must be added.

Theorem 1. *Assume*

$$k \in L^1(0,T) \quad \textit{and} \quad k \textit{ of positive type, i.e.,}$$
$$\int_0^t (k * v)(s) v(s)\, ds \geq 0 \quad \forall\, t,\ v$$
$$g \in L^1(0,T;H), \quad \vartheta_0 \in H, \quad \chi_0 \in V.$$

Then the problem

$$\begin{aligned} &(\vartheta + \ell\chi)' + A(k * \vartheta) = g \\ &\chi' + Aw = 0 \\ &w := A\chi + \chi^3 - \chi - \ell\vartheta \\ &\vartheta(0) = \vartheta_0, \quad \chi(0) = \chi_0 \end{aligned}$$

has a unique solution (ϑ, χ) *satisfying*

$$\begin{aligned} &\vartheta \in C^0([0,T];H) \cap H^1(0,T;W') \\ &\chi \in L^2(0,T;W) \cap L^\infty(0,T;V) \cap H^1(0,T;V') \\ &w \in L^2(0,T;V). \ \blacksquare \end{aligned}$$

Theorem 2. *HP's as above for any $T > 0$ and*

$$k \in L^1(0,\infty) \quad \text{and} \quad \int_0^\infty k(s)\,ds \neq 0$$
$$g \in L^1(0,\infty;H).$$

Let ω be the ω – limit in $V' \times H$ according to: $(\vartheta_\infty, \chi_\infty) \in \omega$ iff

$$\begin{aligned} &\text{there exists} \quad \{t_n\} \nearrow +\infty \quad \text{such that} \\ &\vartheta(t_n) \to \vartheta_\infty \quad \text{in } V' \\ &\chi(t_n) \to \chi_\infty \quad \text{in } H. \end{aligned}$$

Then $\omega \neq \emptyset$, compact, connected. Moreover, $(\vartheta_\infty, \chi_\infty) \in \omega$ implies:

- *ϑ_∞ is constant, namely*

$$\vartheta_\infty := \fint_\Omega \left(\vartheta_0 + \int_0^\infty g\,dt \right)$$

and the whole $\{\vartheta(t)\}$ converges to ϑ_∞;

- *χ_∞ belongs to V and solves*

$$A\chi_\infty + \chi_\infty^3 - \chi_\infty - \ell\vartheta_\infty = \text{const.}$$
$$\int_\Omega \chi_\infty = \int_\Omega \chi_0. \ \blacksquare$$

Rewrite two equations as

$$\vartheta' + \ell\chi' + A(k * \vartheta) = g$$
$$w := A\chi + \chi^3 - \chi - \ell\vartheta.$$

Generalize as follows

$$\vartheta' + \overset{\overset{\lambda'(\chi)}{\uparrow}}{\ell}\chi' + A(k * \vartheta) = g$$

$$w := A\chi + \underbrace{\chi^3 - \chi}_{\downarrow} - \overset{\overset{\lambda'(\chi)}{\uparrow}}{\ell}\vartheta \qquad \underset{\downarrow}{\boxed{\frac{1}{4}(\chi^2 - 1)^2}}$$

$$\overbrace{\beta(\chi) + \sigma'(\chi)} = \frac{\partial}{\partial \chi}\big(\,\overbrace{j(\chi) + \sigma(\chi)}\,\big)$$

and obtain

$$(\vartheta + \lambda(\chi))' + A(k * \vartheta) = g$$
$$w := A\chi + \beta(\chi) + \sigma'(\chi) - \lambda'(\chi)\,\vartheta$$

or, more generally,

$$w = A\chi + \xi + \sigma'(\chi) - \lambda'(\chi)\,\vartheta, \qquad \xi \in \beta(\chi)$$

Hence, we consider the system

$$\begin{aligned}
&(\vartheta + \lambda(\chi))' + A(k * \vartheta) = f && (1)\\
&\chi' + Aw = 0 && (2)\\
&w = A\chi + \xi + \sigma'(\chi) - \lambda'(\chi)\,\vartheta && (3)\\
&\chi \in D(\beta) \qquad \text{and} \qquad \xi \in \beta(\chi) && (4)\\
&\vartheta(0) = \vartheta_0, \quad \chi(0) = \chi_0 && (5)
\end{aligned}$$

where λ, β, σ are as general as possible.

Structure assumptions:

$$\begin{aligned}
&k \in W^{2,1}(0,T) \qquad \text{and} \qquad k(0) > 0\\
&j : \mathbb{R} \to [0, +\infty] \quad \text{convex, proper, l.s.c.}\\
&j(0) = 0, \quad \beta = \partial j \quad (\Longrightarrow \quad \beta(0) \ni 0)\\
&\lambda, \sigma, \in C^1(\mathbb{R}) \quad \text{and} \quad \lambda', \sigma' \quad \text{Lipschitz}\\
&\text{maybe: } \ \text{also } \lambda \text{ Lipschitz}
\end{aligned}$$

New unknown. Replace ϑ by

$$u := 1 * \vartheta \qquad \textit{(freezing index)}$$

Then the system becomes

$$(u' + \lambda(\chi))' + k(0)Au = f - A(k' * u) \tag{1}$$

$$\chi' + Aw = 0 \tag{2}$$

$$w = A\chi + \xi + \sigma'(\chi) - \lambda'(\chi)u' \tag{3}$$

$$\chi \in D(\beta) \quad \text{and} \quad \xi \in \beta(\chi) \tag{4}$$

$$u(0) = 0, \quad u'(0) = \vartheta_0, \quad \chi(0) = \chi_0 \tag{5}$$

and we look for solutions (u, χ, w, ξ) satisfying

$$u \in C^0([0,T];V) \cap C^1([0,T];H)$$

$$\chi \in L^2(0,T;W) \cap L^\infty(0,T;V) \cap H^1(0,T;V')$$

$$j(\chi) \in L^\infty(0,T;L^1(\Omega))$$

$$w \in L^2(0,T;V)$$

$$\xi \in L^2(0,T;H)$$

Theorem 1. *Let all the above structure assumptions hold and assume*

$$
\begin{aligned}
& f \in L^1(0,T;H) + W^{1,1}(0,T;V') \\
& \vartheta_0 \in H, \quad \chi_0 \in V, \quad j(\chi_0) \in L^1(\Omega) \\
& \fint_\Omega \chi_0 \in \operatorname{int} D(\beta).
\end{aligned}
$$

Then there exists a unique (u, χ) *and some* (w, ξ) *such that* (u, χ, w, ξ) *is a solution to (1–5) with the required regularity.* ■

Theorem 2. *HP's as in Thm 1 but the Lipschitz continuity of* λ.
Then the existence part still holds true. ■

Theorem 3. *HP's as in Thm 1. If moreover*

$$f \in W^{1,1}(0,T;H) + W^{2,1}(0,T;V'), \quad f(0) \in H$$
$$\vartheta_0 \in V$$

then

$$u + 1 * \lambda(\chi) \in W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V). \ \blacksquare$$

Theorem 4. *HP's as in Thm 1. If moreover*

$$f \in L^2(0,T;H) + W^{1,1}(0,T;V'), \quad \chi_0 \in D(\beta) \text{ a.e.}$$

and there exists ξ_0 *such that*

$$\xi_0 \in H, \qquad \xi_0 \in \beta(\chi_0) \text{ a.e.}$$
$$w_0 := A\chi_0 + \xi_0 + \sigma'(\chi_0) - \lambda'(\chi_0)\vartheta_0 \in V$$

then

$$\chi \in L^\infty(0,T;W) \cap H^1(0,T;V) \cap W^{1,\infty}(0,T;V'). \ \blacksquare$$