On the Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions

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Abstract

The Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions is considered and well-posedness results are proved.

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1 Introduction

The Cahn-Hilliard equation

$$\partial_t u = \kappa \Delta w \quad \text{and} \quad w = -\vartheta \Delta u + g'(u) \quad \text{in } \Omega \tag{1.1}$$

is central to materials science; it describes important qualitative features of two-phase systems, namely, the transport of atoms between unit cells. This phenomenon can be observed, e.g., when a binary alloy is cooled down sufficiently. One then observes a partial nucleation (i.e., the apparition of nucleides in the material) or a total nucleation, the so-called spinodal decomposition: the material quickly becomes inhomogeneous, forming a fine-grained structure in which each of the two components appears more or less alternatively. In a second stage, which is called coarsening, occurs at a slower time scale and is less understood, these microstructures coarsen. We refer the reader to, e.g., [5], [6], [22] and [32]; see also [25] and [26] for qualitative studies of the spinodal decomposition and [21] for studies on the coarsening. Here, Ω is the domain occupied by the material, u is the order parameter (it corresponds to a density of atoms), w is the chemical potential and g is a double-well potential whose wells correspond to the phases of the material; one usually considers polynomial potentials of degree 4, typically, $g(s) = \frac{1}{4}(s^2 - b^2)^2$, $b > 0$. Now, such potentials are approximations of the following thermodynamically relevant potential:

$$g(s) = -c_0 s^2 + c_1((1+s)\ln(1+s) + (1-s)\ln(1-s)), \quad s \in (-1,1), \quad c_0 > c_1 > 0. \tag{1.2}$$

Furthermore, κ is the mobility (we assume that it is a positive constant; more generally it should depend on the order parameter) and $\vartheta > 0$ is related to the surface tension at the interface.

Equation (1.1) has been extensively studied and is now essentially well understood as far as the existence, uniqueness and regularity of solutions and the asymptotic behavior of the solutions are concerned. We refer the reader, among a vast literature, to, e.g., [1], [4], [7], [8], [10], [11], [12], [14], [17], [19], [23], [27], [28], [29], [30], [31], [32], [33], [34], [35], [37], and [38].

In most works, the equations are endowed with Neumann boundary conditions for both u and w (which means that the interface is orthogonal to the boundary and that there is no mass flux at the boundary) or with periodic boundary conditions. Now, recently, physicists have introduced the so-called dynamic boundary conditions, in the sense that the kinetics, i.e., $\partial_t u$, appears explicitly in the boundary conditions, in order to account for the interaction of the components with the walls for a confined system (see [15] and [16]; see also [20] where numerical simulations are performed).

From a phenomenological point of view, such boundary conditions can be derived as follows. Consider, in addition to the usual Ginzburg-Landau free energy

$$\Psi_{GL}(u, \nabla u) = \int_\Omega \left(\frac{\vartheta}{2}|\nabla u|^2 + g(u)\right) dx, \quad \vartheta > 0$$

(the chemical potential w is defined as a variational derivative of Ψ_{GL} with respect to u), the following boundary free energy:

$$\Psi_{DC}(u, \nabla_\Gamma u) = \int_\Gamma \left(\frac{\nu}{2}|\nabla_\Gamma u|^2 + h(u)\right) d\sigma, \quad \nu > 0$$

(thus, $\Psi = \Psi_{GL} + \Psi_{DC}$ is the total free energy of the system), where Γ is the boundary of Ω and ∇_Γ is the surface gradient. Then, writing that the density on the boundary Γ relaxes towards equilibrium with a rate which is proportional to the variational derivative of Ψ with respect to u (the test function z below being taken in a suitable space),

$$\langle \frac{\delta \Psi(u)}{\delta u}, z \rangle = \int_\Omega \big(-\vartheta \Delta u + g'(u)\big) z \, dx + \int_\Gamma \big(-\nu \Delta_\Gamma u + \vartheta(\partial_n u)|_\Gamma + h'(u)\big) z \, d\sigma,$$

we obtain the dynamic boundary condition

$$\frac{1}{d}\partial_t u = \nu \Delta_\Gamma u - \vartheta(\partial_n u)|_\Gamma - h'(u) \quad \text{on } \Gamma, \quad d > 0. \tag{1.3}$$

Here, Δ_Γ is the Laplace-Beltrami operator and $(\partial_n u)|_\Gamma$ denotes the outer normal derivative of u on Γ. However, we can consider a boundary condition like

$$\frac{1}{d}\,\partial_t u = -\vartheta(\partial_n u)|_\Gamma - h'(u) \quad \text{on } \Gamma \tag{1.4}$$

as well. This is seen as a particular case of (1.3) if we allow the choice $\nu = 0$ there. Indeed, the operators Δ_Γ and ∇_Γ formally disappear in such a case and it is understood that the corresponding contributions have to be ignored. In particular, this is mandatory in one space dimension, since the above boundary operators are meaningless in this case.

The Cahn-Hilliard equation (1.1), endowed with the dynamic boundary condition (1.3), has been studied in [7], [17], [28], [33], [34], and [37]. In particular, one now has the existence and uniqueness of solutions and results on the asymptotic behavior of the solutions. We should note however that all these results have been obtained for regular potentials g, only (typically, g is a polynomial potential of degree 4 as above or is at least of class C^2).

In this paper, we are interested in the more general equations

$$\partial_t u - \Delta w = 0 \quad \text{in } \Omega \tag{1.5}$$

$$\partial_n w = 0 \quad \text{on } \Gamma \tag{1.6}$$

$$w = \tau\,\partial_t u - \Delta u + \beta(u) + \pi(u) - f \quad \text{in } \Omega \tag{1.7}$$

$$v = u|_\Gamma \quad \text{and} \quad \partial_t v + (\partial_n u)|_\Gamma - \nu\Delta_\Gamma v + \beta_\Gamma(v) + \pi_\Gamma(v) = f_\Gamma \quad \text{on } \Gamma \tag{1.8}$$

$$u(0) = u_0 \quad \text{in } \Omega \tag{1.9}$$

where the potentials have been generalized. Namely, we have split g' and h' as $g' = \beta + \pi$ and $h' = \beta_\Gamma + \pi_\Gamma$, respectively, where β and β_Γ are monotone and possibly non-smooth, while π and π_Γ are more regular perturbations, and we are interested in keeping β and β_Γ as general as possible (especially β). Moreover, we have introduced a nonnegative parameter τ in (1.7) and kept the coefficient ν in (1.8), by allowing the value $\nu = 0$, while we have normalized the other positive constants to 1, for simplicity. Furthermore, we have added the forcing terms f and f_Γ. Note that we can thus consider the dynamic boundary condition

$$\partial_t v + (\partial_n u)|_\Gamma + k_0(v - u_\Gamma) = 0, \quad k_0 > 0 \tag{1.10}$$

with a given u_Γ as a particular case of (1.8) with $\nu = 0$.

We remark that, for $\tau > 0$, we obtain the viscous Cahn-Hilliard equation introduced in [30] (see also [18] where similar models are derived and, e.g., [2], [13] and [27] for the mathematical analysis of the viscous Cahn-Hilliard equation with classical boundary conditions and [28] for dynamic boundary conditions and regular potentials). Furthermore, following [28], we view the dynamic boundary condition as a separate (parabolic if $\nu > 0$) equation on the boundary.

Compared with the previous results, we consider in this paper less regular nonlinear terms; in particular, π and π_Γ are only Lipschitz continuous, while β and β_Γ are subdifferentials of convex functions (the classical Cahn-Hilliard equation, with such potentials and with classical boundary conditions, is considered in [19]). In that case, equations (1.7) and (1.8) have to be read as differential inclusions. We are then able to obtain well-posedness results for problem (1.5)–(1.9) on any finite time interval $(0, T)$.

More precisely, we can show uniqueness in a very general framework and global existence under further assumptions on the nonlinearities, and the most interesting case is the following.

Roughly speaking, we allow β and β_Γ to be essentially arbitrary (in particular, they could have a bounded domain like β does in (1.2)), but we assume that β grows faster than β_Γ and that the other boundary contributions satisfy a sign condition. However, we can avoid compatibility conditions on the nonlinearities, provided that both β and β_Γ are everywhere defined and satisfy some growth conditions that depend on the dimension d of Ω.

2 Main results

In this section, we carefully describe the problem we are going to deal with and state our results. As in the Introduction, Ω is the body where the evolution is considered and $\Gamma := \partial\Omega$. Moreover, ∂_n still denotes the outward normal derivative on Γ. We assume $\Omega \subset \mathbb{R}^d$, with $1 \leq d \leq 3$, to be bounded, connected, and smooth, and write $|\Omega|$ for its Lebesgue measure. Similarly, $|\Gamma|$ denotes the $(d-1)$-dimensional measure of Γ. As far as the one-dimensional case is concerned, see the forthcoming Remark 2.1. Given a finite final time T, we set for convenience

$$Q_t := \Omega \times (0,t) \quad \text{and} \quad \Sigma_t := \Gamma \times (0,t) \quad \text{for every } t \in (0,T] \tag{2.1}$$

$$Q := Q_T\,, \quad \text{and} \quad \Sigma := \Sigma_T\,. \tag{2.2}$$

Now, we describe the main features of the structure of our system. Further assumptions will be made later on. We are given functions $\widehat{\beta}$, $\widehat{\beta}_\Gamma$, π, π_Γ and constants τ, ν satisfying the conditions listed below.

$$\widehat{\beta}, \widehat{\beta}_\Gamma : \mathbb{R} \to [0,+\infty] \text{ are convex, proper, and l.s.c., and } \widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0 \tag{2.3}$$

$$\pi \text{ and } \pi_\Gamma \text{ are Lipschitz continuous} \tag{2.4}$$

$$\tau, \nu \geq 0. \tag{2.5}$$

We define the graphs β and β_Γ in $\mathbb{R} \times \mathbb{R}$ by

$$\beta := \partial\widehat{\beta} \quad \text{and} \quad \beta_\Gamma := \partial\widehat{\beta}_\Gamma \tag{2.6}$$

and note that β and β_Γ are maximal monotone. Moreover, $\beta(0) \ni 0$ and $\beta_\Gamma(0) \ni 0$. Furthermore, we note that both β and β_Γ might have effective domains, denoted by $D(\beta)$ and $D(\beta_\Gamma)$, respectively, which might be different from the whole real line. In the sequel, for any maximal monotone graph $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$, we introduce the notation (see, e.g., [3, p. 28])

$$\gamma^\circ(r) \text{ is the element of } \gamma(r) \text{ having minimum modulus} \tag{2.7}$$

$$\gamma^Y_\varepsilon := \varepsilon^{-1}\big(I - (I + \varepsilon\gamma)^{-1}\big), \text{ the Yosida regularization of } \gamma, \text{ for } \varepsilon > 0 \tag{2.8}$$

and still use the symbol γ (and, e.g., γ^Y_ε as a particular case) for the maximal monotone operator induced by γ on any L^2-space.

Next, in order to state our concept of solution in a simple way, we set

$$\begin{aligned} &V := H^1(\Omega), \quad H := L^2(\Omega), \quad H_\Gamma := L^2(\Gamma) \\ &V_\Gamma := H^1(\Gamma) \quad \text{if } \nu > 0 \quad \text{and} \quad V_\Gamma := H^{1/2}(\Gamma) \quad \text{if } \nu = 0 \\ &\mathcal{V} := \{ v \in V : \ v|_\Gamma \in V_\Gamma \} \end{aligned} \tag{2.9}$$

the latter being endowed with the graph norm. Note that $\mathcal{V} = V$ if $\nu = 0$. As $H = H^0(\Omega)$, we denote the (standard) norms of H and V by $\|\cdot\|_{k,\Omega}$ with $k = 0, 1$, respectively. More generally, we use such a symbol for $H^k(\Omega)$ with real $k > 0$ and the analogous one, namely, $\|\cdot\|_{k,\Gamma}$, for $H^k(\Gamma)$ with real $k > 0$. In particular, $\|\cdot\|_{0,\Gamma}$ is the norm in H_Γ. On the contrary, we write $\|\cdot\|_{V_\Gamma}$ for the norm in V_Γ since the definition of such a space depends on ν. For the sake of simplicity, the same notation will be used for both a space and any power of it. We recall the optimal trace theorem for V, namely, the inequality

$$\|z|_\Gamma\|_{1/2,\Gamma} \le M_\Omega \|z\|_{1,\Omega} \quad \text{for every } z \in V \tag{2.10}$$

where M_Ω depends on Ω, only. Finally, the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the duality pairing between V^* and V. In the sequel, it is understood that H is embedded in V^* in the usual way, i.e., so that $\langle u, v\rangle = (u, v)$, the inner product of H, for every $u \in H$ and $v \in V$.

Remark 2.1. In the one-dimensional case $d = 1$, the open set Ω is a bounded interval (x_1, x_2). Hence, $\Gamma = \{x_1, x_2\}$ and $V_\Gamma = H_\Gamma = \mathbb{R}^2$, since the "surface measure" is the 0-dimensional Hausdorff measure (i.e., the counting measure) in this case, and the same holds for other spaces, e.g., $L^\infty(\Gamma)$. Moreover, $\mathcal{V} = V$ and $\partial_n v(x_i) = (-1)^i dv(x_i)/dx$ for any smooth v and $i = 1, 2$. Finally, we set

$$\nu = 0 \quad \text{if} \quad d = 1 \tag{2.11}$$

since the surface gradient ∇_Γ and the related Laplace-Beltrami operator Δ_Γ are meaningless and have to be ignored in that case, as said in the Introduction.

At this point, we can describe our problem, which consists in the variational formulation of system (1.5)–(1.9). Namely, we formally multiply the equations by test functions free on Γ, integrate by parts both in Ω and on Γ, and take the boundary conditions into account. However, as β and β_Γ might be multi-valued, we have to include selections ξ and ξ_Γ of $\beta(u)$ and of $\beta_\Gamma(v)$ in the definition of solution. Moreover, the regularity of all ingredients has to be made precise. So, just to start, we give the data f, f_Γ, and u_0 satisfying (further assumptions will be specified later on)

$$f \in L^2(0,T;H), \quad f_\Gamma \in L^2(0,T;H_\Gamma), \quad \text{and} \quad u_0 \in V \tag{2.12}$$

and look for a quadruplet (u, w, ξ, ξ_Γ) such that

$$u \in L^\infty(0,T;V) \cap H^1(0,T;V^*) \quad \text{and} \quad \tau\,\partial_t u \in L^2(0,T;H) \tag{2.13}$$

$$v := u|_\Gamma \in L^\infty(0,T;V_\Gamma) \cap H^1(0,T;H_\Gamma) \tag{2.14}$$

$$w \in L^2(0,T;V) \tag{2.15}$$

$$\xi \in L^2(0,T;H) \quad \text{and} \quad \xi \in \beta(u) \quad \text{a.e. in } Q \tag{2.16}$$

$$\xi_\Gamma \in L^2(0,T;H_\Gamma) \quad \text{and} \quad \xi_\Gamma \in \beta_\Gamma(v) \quad \text{a.e. on } \Sigma \tag{2.17}$$

$$u(0) = u_0 \tag{2.18}$$

and satisfying for a.a. $t \in (0,T)$

$$\langle \partial_t u(t), z\rangle + \int_\Omega \nabla w(t) \cdot \nabla z = 0 \tag{2.19}$$

$$\begin{aligned} \int_\Omega w(t) z = {} & \int_\Omega \tau\,\partial_t u(t)\, z + \int_\Gamma \partial_t v(t)\, z + \int_\Omega \nabla u(t) \cdot \nabla z + \int_\Gamma \nu \nabla_\Gamma v(t) \cdot \nabla_\Gamma z \\ & + \int_\Omega \bigl(\xi(t) + \pi(u(t)) - f(t)\bigr)\, z + \int_\Gamma \bigl(\xi_\Gamma(t) + \pi_\Gamma(v(t)) - f_\Gamma(t)\bigr)\, z \end{aligned} \tag{2.20}$$

for every $z \in V$ and every $z \in \mathcal{V}$, respectively.

Remark 2.2. We note that the definition of v given by (2.14) has to be read as $v(t) = u(t)|_\Gamma$ for a.a. $t \in (0,T)$ and it is understood that the symbol z in the boundary terms of (2.20) actually means $z|_\Gamma$. However, in order to simplify the notation, we use the same symbol for a function and its trace on the boundary, unless some misunderstanding arises. As far as this point is concerned, we note that $\partial_t(z|_\Gamma) = (\partial_t z)|_\Gamma$ whenever z is a smooth function, while the right-hand side of such a formula is meaningless in the opposite case. On the contrary, the left-hand side exists (at least in a generalized sense) whenever $z \in L^2(0,T;V)$. Therefore, the true meaning of $\partial_t z$ on the boundary is the latter. Moreover, we note that an equivalent formulation of (2.19)–(2.20) is given by

$$\int_0^T \langle \partial_t u(t), z(t) \rangle \, dt + \int_Q \nabla w \cdot \nabla z = 0 \tag{2.21}$$

$$\int_Q wz = \int_Q \tau \partial_t u \, z + \int_\Sigma \partial_t u \, z + \int_Q \nabla u \cdot \nabla z + \int_\Sigma \nu \nabla_\Gamma u \cdot \nabla_\Gamma z$$
$$+ \int_Q \bigl(\xi + \pi(u) - f \bigr) \, z + \int_\Sigma \bigl(\xi_\Gamma + \pi_\Gamma(u) - f_\Gamma \bigr) \, z \tag{2.22}$$

for every $z \in L^2(0,T;V)$ and every $z \in L^2(0,T;\mathcal{V})$, respectively, where we have simply written u instead of v in the boundary terms. Finally, we point out that the regularity requirements (2.13)–(2.17) are choosen just in order that the variational problem makes sense and a general uniqueness result holds (Theorem 2.4 below). Indeed, every solution satisfying (2.13)–(2.17) is automatically smoother and fulfils (1.7) and (1.8) a.e. in Q and a.e. on Σ, respectively, as we show in the forthcoming Remark 5.4. On the other hand, also some regularity in a different direction (like some boundedness with respect to time) holds for the solution we construct under the assumptions of our existence results (Theorems 2.8 and 2.9 below), as the a priori estimates we establish in the proof given in Section 5 clearly show.

Remark 2.3. Note that, by testing (2.19) by the constant $1/|\Omega|$, we obtain

$$\partial_t(u(t)_\Omega) = 0 \quad \text{for a.a. } t \in (0,T) \quad \text{and} \quad u(t)_\Omega = (u_0)_\Omega \quad \text{for every } t \in [0,T] \tag{2.23}$$

where, more generally, we set

$$v^*_\Omega := \frac{1}{|\Omega|} \, \langle v^*, 1 \rangle \quad \text{for } v^* \in V^*. \tag{2.24}$$

Clearly, (2.24) gives the usual mean value when applied to elements of H.

Now we state our results. The simplest one regards uniqueness. However, our conclusion is partial even though it holds in a very general case.

Theorem 2.4. *Assume* (2.3)–(2.6) *and* (2.12) *with the notation* (2.1)–(2.2) *and* (2.9)–(2.11). *Then, any two solutions to problem* (2.13)–(2.20) *have the same first component.*

Remark 2.5. A statement like Theorem 2.4 is typical for problems having some bad multi-valued nonlinearities and cannot be improved, unless further assumptions are made (see also the forthcoming Remark 3.1). In particular, if β is single-valued, the component ξ is uniquely determined as well. Then, a comparison in (2.20) with $z \in H^1_0(\Omega)$ shows that the same happens for the component w. Finally, writing (2.20) once more with such an information, we see that even the component ξ_Γ is uniquely determined and we have a full uniqueness result.

While the above uniqueness result is rather general and even continuous dependence can be proved under the same hypotheses (see the forthcoming Remark 3.2), we can ensure existence just under further assumptions. We recall that our aim is to keep the maximal monotone operators as general as we can, mainly, and we can do that under suitable conditions. As far as the data are concerned, we assume that

$$f \in H^1(0,T;H) \quad \text{and} \quad f_\Gamma \in H^1(0,T;H_\Gamma) \cap L^\infty(\Sigma) \tag{2.25}$$

$$u_0 \in H^2(\Omega) \quad \text{and} \quad \partial_n u_0 = 0 \tag{2.26}$$

$$\nu\, u_0|_\Gamma \in H^2(\Gamma) \tag{2.27}$$

$$\widehat{\beta}(u_0) \in L^1(\Omega), \quad \widehat{\beta}(u_0|_\Gamma) \in L^1(\Gamma), \quad \text{and} \quad \widehat{\beta}_\Gamma(u_0|_\Gamma) \in L^1(\Gamma) \tag{2.28}$$

$$\text{the mean value of } u_0 \text{ belongs to the interior of } D(\beta). \tag{2.29}$$

Moreover, a further assumption is needed, which is weaker or stronger depending on whether or not the viscosity constant τ is positive (see (2.8) and (2.7) for notation).

$$\beta^\circ(u_0) \in H \quad \text{and} \quad \beta^\circ_\Gamma(u_0|_\Gamma) \in H_\Gamma \tag{2.30}$$

$$-\Delta u_0 + \beta^Y_\varepsilon(u_0) - f(0) \quad \text{remains bounded in } V \text{ as } \varepsilon \to 0^+ \text{ if } \tau = 0. \tag{2.31}$$

Remark 2.6. If $\nu > 0$ and (2.26) is taken into account, (2.27) is equivalent to $u_0 \in H^{5/2}(\Omega)$, and to $\Delta_\Gamma u_0|_\Gamma \in H_\Gamma$. Let us comment (2.31), which looks involved. If $u_0 \in H^3(\Omega)$ and $f(0) \in V$, then it regards just boundedness for $\beta^Y_\varepsilon(u_0)$. In such a case, β cannot be too irregular on the range of u_0, and a sufficient condition for (2.31) is the following: the closure of the range of u_0 is included in some open interval where β is one-valued and Lipschitz continuous. For instance, if β comes from the logarithmic potential (1.2), this simply means that $\sup|u_0| < 1$. Finally, we note that (2.28) and (2.30) are not independent (the latter implies some of the former, indeed), and we have written all of them just for convenience.

As far as the structure of the system is concerned, we need some compatibility condition on the main nonlinearities and on the perturbation π_Γ on the boundary. Namely, we assume that

$$D(\beta_\Gamma) \supseteq D(\beta) \quad \text{and} \quad \beta_\Gamma(0) = \{0\} \tag{2.32}$$

and that real constants α, C_Γ, σ, L_Γ, M_Γ, and $r_\pm$ exist such that

$$\alpha > 0, \quad C_\Gamma \ge 0, \quad \sigma \in (0,1), \quad L_\Gamma > \sup|\pi'_\Gamma| \quad \text{and} \quad M_\Gamma > |\pi_\Gamma(0)| + \|f_\Gamma\|_{L^\infty(\Gamma)} \tag{2.33}$$

$$r_- \le 0 \le r_+\,, \text{ and } r_\pm \text{ belong to the interior of } D(\beta) \tag{2.34}$$

$$|\beta^\circ(r)| \ge \alpha|\beta^\circ_\Gamma(r)| - C_\Gamma \quad \text{for every } r \in D(\beta) \tag{2.35}$$

$$\sigma|\beta^\circ_\Gamma(r)| \ge L_\Gamma|r| + M_\Gamma \quad \text{for every } r \in D(\beta_\Gamma) \setminus (r_-, r_+). \tag{2.36}$$

Remark 2.7. The above assumptions merit some comment. The first of (2.32) is quite natural and the second one is not restrictive in the applications. Assumption (2.35) is the main compatibility condition. Clearly, it is satisfied whenever β is singular and β_Γ is not. Moreover, if $D(\beta_\Gamma) = D(\beta)$ and both β and β_Γ are regular, it becomes a growth condition on β_Γ with respect to β, and the same happens if both β and β_Γ are singular. Let us come to (2.36) and to the restrictions on the constants given by (2.33)–(2.34). If $D(\beta) = D(\beta_\Gamma) = \mathbb{R}$, then (2.36) is surely satisfied (with arbitrary L_Γ, M_Γ and suitably big $r_\pm$) if $\beta_\Gamma + \pi_\Gamma$ is stricly superlinear at infinity. Indeed, for any decomposition $\beta_\Gamma + \pi_\Gamma$ with a Lipschitz continuous π_Γ, exactly β_Γ

is superlinear in such a case. Finally, if $D(\beta)$ is bounded while $D(\beta_\Gamma)$ is not, (2.36) essentially requires that the nonlinear boundary term has a decomposition $\beta_\Gamma+\pi_\Gamma$ such that the monotone part is much bigger than the perturbation near the boundary of the body double-well potential. In particular, in the case the latter is the logarithmic potential (1.2) and the boundary condition has the form (1.10) for some given u_Γ, then (2.36) is fulfilled provided that $\sup|u_\Gamma|<1$. Indeed, we can take $\beta_\Gamma(r)=k_0 r$ and $\pi_\Gamma(r)=0$ as far as the decomposition is concerned. Then, it suffices to choose $\sigma, M_\Gamma\in(0,1)$ sufficiently close to 1 and L_Γ small enough.

Here is our main result.

Theorem 2.8. *Assume* (2.3)–(2.6) *and* (2.12) *with the notation* (2.1)–(2.2) *and* (2.9)–(2.11). *Moreover, assume* (2.25)–(2.36). *Then, there exists a quadruplet* (u,w,ξ,ξ_Γ) *satisfying* (2.13)–(2.18) *and solving problem* (2.19)–(2.20).

However, as said in the Introduction, we can prove a different existence result that requires growth conditions on β and β_Γ instead of compatibility and sign assumptions. On the contrary, less is required on the data. As far as the structure of the system is concerned, we ask that

$$D(\beta)=D(\beta_\Gamma)=\mathbb{R} \tag{2.37}$$

in any case, while the details of the further hypotheses depend on the dimension d of Ω and on whether or not the boundary differential operators actually appear in the equations (i.e., on ν). If $d=1$ that is all. If $d=2$, we require that

$$\begin{aligned}&\beta^\circ(r)=O(|r|^p)\quad \text{as } |r|\to+\infty, \quad \text{for some } p\ge 1 \quad \text{and}\\ &\text{either}\quad \nu>0 \quad \text{or}\quad \beta^\circ_\Gamma(r)=O(|r|^q)\quad \text{as } |r|\to+\infty, \quad \text{for some } q\ge 1.\end{aligned} \tag{2.38}$$

If $d=3$, we assume that

$$\begin{aligned}&\beta^\circ(r)=O(|r|^3)\quad \text{and}\quad \beta^\circ_\Gamma(r)=O(|r|^q)\quad \text{as } |r|\to+\infty, \quad \text{for some } q \text{ with}\\ &q\ge 1\quad \text{if } \nu>0\quad \text{and}\quad q\in[1,2]\quad \text{if } \nu=0.\end{aligned} \tag{2.39}$$

Theorem 2.9. *Assume* (2.3)–(2.6) *and* (2.12) *with the notation* (2.1)–(2.2) *and* (2.9)–(2.11). *Moreover, assume* (2.37) *and either i)* $d=1$, *or ii)* $d=2$ *and* (2.38), *or iii)* $d=3$ *and* (2.39). *Finally, assume* (2.26), (2.28), *and either* $\tau>0$ *or* $f\in H^1(0,T;H)$. *Then, there exists a quadruplet* (u,w,ξ,ξ_Γ) *satisfying* (2.13)–(2.18) *and solving problem* (2.19)–(2.20).

Remark 2.10. In connection with the definition of mean value given in Remark 2.3, we recall some facts. First of all, as Ω is bounded and smooth, the well-known Poincaré inequality holds true, namely

$$\|v\|_{1,\Omega}^2\le M_\Omega(\|\nabla v\|_{0,\Omega}^2+|v_\Omega|^2)\quad \text{for every } v\in V \tag{2.40}$$

where M_Ω depends on Ω, only. Next, we define

$$\operatorname{dom}\mathcal{N}:=\{v^*\in V^*:\ v^*_\Omega=0\}\quad \text{and}\quad \mathcal{N}:\operatorname{dom}\mathcal{N}\to\{v\in V:\ v_\Omega=0\} \tag{2.41}$$

by setting for $v^*\in\operatorname{dom}\mathcal{N}$

$$\mathcal{N}v^*\in V,\quad (\mathcal{N}v^*)_\Omega=0,\quad \text{and}\quad \int_\Omega \nabla\mathcal{N}v^*\cdot\nabla z=\langle v^*,z\rangle\quad \text{for every } z\in V \tag{2.42}$$

i.e., $\mathcal{N}v^*$ is the solution v to the generalized Neumann problem for $-\Delta$ with datum v^* that satisfies $v_\Omega = 0$. As Ω is bounded, smooth, and connected, it turns out that (2.42) yields a well-defined isomorphism, which satisfies

$$\langle u^*, \mathcal{N}v^* \rangle = \langle v^*, \mathcal{N}u^* \rangle = \int_\Omega (\nabla \mathcal{N}u^*) \cdot (\nabla \mathcal{N}v^*) \quad \text{for } u^*, v^* \in \operatorname{dom} \mathcal{N}. \tag{2.43}$$

Moreover, if we define $\|\cdot\|_* : V^* \to [0, +\infty)$ by the formula

$$\|v^*\|_*^2 := \|\nabla \mathcal{N}(v^* - (v^*)_\Omega)\|_{0,\Omega}^2 + |(v^*)_\Omega|^2 \quad \text{for } v^* \in V^* \tag{2.44}$$

it is straightforward to prove that $\|\cdot\|_*$ is a norm that makes V^* a Hilbert space. Therefore, if $\|\cdot\|_{V^*}$ stands for the dual norm to $\|\cdot\|_{1,\Omega}$, the following inequalities hold

$$\frac{1}{M_\Omega} \|v^*\|_{V^*} \le \|v^*\|_* \le M_\Omega \|v^*\|_{V^*} \quad \text{for } v^* \in V^* \tag{2.45}$$

where M_Ω depends on Ω, only. Indeed, as the latter holds thanks to (2.40), the former follows from the open mapping theorem, provided that we possibly replace M_Ω by a bigger constant. Note that

$$\langle v^*, \mathcal{N}v^* \rangle = \|v^*\|_*^2 \quad \text{for every } v^* \in \operatorname{dom} \mathcal{N} \tag{2.46}$$

by (2.43)–(2.44). Finally, owing to (2.43) once more, we see that

$$2\langle \partial_t v^*(t), \mathcal{N}v^*(t) \rangle = \frac{d}{dt} \int_\Omega |\nabla \mathcal{N}v^*(t)|^2 = \frac{d}{dt} \|v^*(t)\|_*^2 \quad \text{for a.a. } t \in (0,T) \tag{2.47}$$

for every $v^* \in H^1(0,T;V^*)$ satisfying $v^*_\Omega(t) = 0$ for every $t \in [0,T]$.

Throughout the whole paper, we widely use the notation and the properties introduced in the above remark, as well as the elementary inequality

$$ab \le \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for every } a, b \ge 0 \text{ and } \delta > 0. \tag{2.48}$$

Moreover, we account for the easy inequalities we derive at once. We have

$$\begin{aligned} \|z(t)\|_{0,\Omega}^2 &= \|z(0)\|_{0,\Omega}^2 + 2 \int_0^t \langle \partial_t z(s), z(s) \rangle \, ds \\ &\le \|z(0)\|_{0,\Omega}^2 + \delta \int_0^t \|\partial_t z(s)\|_{V^*}^2 \, ds + \frac{1}{\delta} \int_0^t \|z(s)\|_{1,\Omega}^2 \, ds \end{aligned}$$

for every $t \in [0,T]$, $z \in L^2(0,T;V) \cap H^1(0,T;V^*)$, and $\delta > 0$. Owing to (2.45), we conclude that

$$\|z(t)\|_{0,\Omega}^2 \le \|z(0)\|_{0,\Omega}^2 + \delta \int_0^t \|\partial_t z(s)\|_*^2 \, ds + c_\delta \int_0^t \|z(s)\|_{1,\Omega}^2 \, ds \tag{2.49}$$

where c_δ depends on Ω as well. Using a similar argument for boundary terms, we easily see that the inequality below also holds true

$$\|z(t)\|_{0,\Gamma}^2 \le \|z(0)\|_{0,\Gamma}^2 + \delta \int_0^t \|\partial_t z(s)\|_{0,\Gamma}^2 \, ds + c_\delta \int_0^t \|z(s)\|_{0,\Gamma}^2 \, ds. \tag{2.50}$$

Here, we have to assume that $z \in H^1(0,T;L^2(\Gamma))$. Finally, we recall the inequality

$$\|z\|_{0,\Omega}^2 \le \delta \|\nabla z\|_{0,\Omega}^2 + c_\delta \|z\|_*^2 \quad \text{for every } z \in V \tag{2.51}$$

which holds for every $\delta > 0$ and some constant c_δ depending on Ω as well.

We conclude this section by stating a general rule we use as far as constants are concerned, in order to avoid a boring notation. Throughout the paper, the symbol c stands for different constants which depend only on Ω, on the final time T, and on the constants and the norms of the functions involved in the assumptions of either our statements or our approximation. In particular, c is independent of the approximation parameter ε we introduce in the next section. A notation like c_δ (see, e.g., (2.51)) allows the constant to depend on the positive parameter δ, in addition. Hence, the meaning of c and c_δ might change from line to line and even in the same chain of inequalities. On the contrary, we use different symbols (see, e.g., (2.40)) to denote precise constants which we could refer to. By the way, all the constants we have termed M_Ω could be the same, since sharpness is not needed.

3 Uniqueness

In this section, we prove Theorem 2.4. We take two solutions and label their components with subscripts 1 and 2. First of all, we observe that u_1 and u_2 have the same mean value thanks to (2.23). Hence, we can write (2.19) for both solutions and test the difference by $\mathcal{N}u$ where $u := u_1 - u_2$. More precisely, we write such a difference at time $t = s$, choose $z = \mathcal{N}u(s)$ in it, and integrate what we get over $(0,t)$ with respect to s, where $t \in (0,T]$ is arbitrary. At the same time, we write (2.20) for both solutions, choose $z = -u(s)$ in the difference, and integrate. Finally, we add the obtained equalities to each other. If we set for convenience $w := w_1 - w_2$ and introduce an analogous notation for the other components, we have

$$\begin{aligned}
&\int_0^t \langle \partial_t u(s), \mathcal{N}u(s) \rangle \, ds + \int_{Q_t} \nabla w \cdot \nabla \mathcal{N}u - \int_{Q_t} wu \\
&\quad + \frac{\tau}{2} \int_\Omega |u(t)|^2 + \frac{1}{2} \int_\Gamma |u(t)|^2 + \int_{Q_t} |\nabla u|^2 + \nu \int_{\Sigma_t} |\nabla_\Gamma u|^2 + \int_{Q_t} \xi u + \int_{\Sigma_t} \xi_\Gamma u \\
&= \int_{Q_t} \bigl(\pi(u_2) - \pi(u_1)\bigr) \, u + \int_{\Sigma_t} \bigl(\pi_\Gamma(u_2) - \pi_\Gamma(u_1)\bigr) \, u .
\end{aligned}$$

Now, we use (2.47) for the first term on the left-hand side and cancel the next two integrals accounting for (2.42). Moreover, we observe that the last two integrals on the left-hand side are nonnegative since β and β_Γ are monotone. Finally, we owe to the Lipschitz continutity of π and π_Γ (see (2.4)). Hence, if we forget three nonnegative terms on the left-hand side, we obtain

$$\frac{1}{2} \|u(t)\|_*^2 + \frac{1}{2} \|u(t)\|_{0,\Gamma}^2 + \int_{Q_t} |\nabla u|^2 \le c \int_{Q_t} |u|^2 + \int_{\Sigma_t} |u|^2 .$$

At this point, we account for (2.51) and get

$$\int_{Q_t} |u|^2 \le \delta \int_{Q_t} |\nabla u|^2 + c_\delta \int_0^t \|u(s)\|_*^2 \, ds .$$

Therefore, it suffices to choose δ small enough and apply the Gronwall lemma to obtain $u = 0$. Hence $u_1 = u_2$, and the proof is complete.

Remark 3.1. In connection with Remark 2.5, we present a simple example of non-uniqueness in the case of a multi-valued β. Assume that π, $\widehat{\beta}_\Gamma$, π_Γ, f, and f_Γ vanish identically and let $r_0 \in \mathbb{R}$ be such that $\beta(r_0)$ is not a singleton. Now, pick any smooth w depending on time, only, such that $w(t) \in \beta(r_0)$ for every t and choose $u = r_0$, $\xi = w$, and $\xi_\Gamma = 0$. Then, (u, w, ξ, ξ_Γ) solves problem (2.13)–(2.20) with $u_0 = r_0$. Hence, there is a big family of solutions to the same problem.

Remark 3.2. The argument used in the above proof can be applied to obtain a continuous dependence result. Indeed, just the terms involving the data appear, in addition. If we consider the solutions corresponding to two sets of data, we have with a self-explaining notation

$$\begin{aligned}
&\|u_1 - u_2\|^2_{L^\infty(0,T;V^*)} + \tau\|u_1 - u_2\|^2_{L^\infty(0,T;H)} + \|u_1 - u_2\|^2_{L^\infty(0,T;H_\Gamma)} \\
&\quad + \|\nabla(u_1 - u_2)\|^2_{L^2(Q)} + \nu\|\nabla_\Gamma(u_1 - u_2)\|^2_{L^2(\Sigma)} \\
&\leq c\Big\{ \|u_{0,1} - u_{0,2}\|^2_* + \tau\|u_{0,1} - u_{0,2}\|^2_{0,\Omega} + \|u_{0,1} - u_{0,2}\|^2_{0,\Gamma} \\
&\qquad + \|f_1 - f_2\|^2_{L^2(0,T;H)} + \|f_{\Gamma,1} - f_{\Gamma,2}\|^2_{L^2(0,T;H_\Gamma)} \Big\}
\end{aligned}$$

provided that $u_{0,1}$ and $u_{0,2}$ have the same mean value.

4 Approximating problems

This section contains a preliminary work in the direction of proving Theorems 2.8 and 2.9. We consider an approximating problem, depending on the parameter $\varepsilon \in (0,1)$, obtained by smoothing the worst nonlinearities β and β_Γ of problem (2.19)–(2.20). Moreover, we replace the coefficient τ (which might vanish) by a positive value in order to make the solution more regular. So, we define the real number τ_ε and the functions $\beta_\varepsilon, \beta_{\Gamma,\varepsilon} : \mathbb{R} \to \mathbb{R}$ by the formulas

$$\tau_\varepsilon := \max\{\tau, \varepsilon\} \tag{4.1}$$

$$\beta_\varepsilon(r) := \beta^Y_\varepsilon(r) \quad \text{for } r \in \mathbb{R} \tag{4.2}$$

$$\begin{aligned}
\beta_{\Gamma,\varepsilon}(r) &:= \beta^Y_{\Gamma,\alpha\varepsilon}(r - \varepsilon C_\Gamma) \quad \text{if } r \leq -\varepsilon C_\Gamma \\
&:= \beta^Y_{\Gamma,\alpha\varepsilon}(r + \varepsilon C_\Gamma) \quad \text{if } r \geq \varepsilon C_\Gamma \\
&:= \frac{r}{\varepsilon C_\Gamma}\, \beta^Y_{\Gamma,\alpha\varepsilon}(-2\varepsilon C_\Gamma) \quad \text{if } -\varepsilon C_\Gamma < r < 0 \\
&:= \frac{r}{\varepsilon C_\Gamma}\, \beta^Y_{\Gamma,\alpha\varepsilon}(2\varepsilon C_\Gamma) \quad \text{if } 0 \leq r < \varepsilon C_\Gamma
\end{aligned} \tag{4.3}$$

where α and C_Γ are the same as in (2.35) and notation (2.8) for Yosida regularizations is used.

Remark 4.1. As far as the proof of Theorem 2.9 is concerned, we can simply take

$$\beta_{\Gamma,\varepsilon} := \beta^Y_{\Gamma,\varepsilon}\,. \tag{4.4}$$

The above complicated definition of $\beta_{\Gamma,\varepsilon}$ is justified by the forthcoming Lemma 5.1, which is needed just in the proof of Theorem 2.8.

Moreover, we define for convenience $\widehat{\beta}_\varepsilon, \widehat{\beta}_{\Gamma,\varepsilon} : \mathbb{R} \to \mathbb{R}$ by the formulas

$$\widehat{\beta}_\varepsilon(r) := \int_0^r \beta_\varepsilon(s)\, ds \quad \text{and} \quad \widehat{\beta}_{\Gamma,\varepsilon}(r) := \int_0^r \beta_{\Gamma,\varepsilon}(s)\, ds \quad \text{for } r \in \mathbb{R}. \tag{4.5}$$

As the Yosida regularization of a maximal monotone operator is monotone and Lipschitz continuous, such a property holds for both β_ε and $\beta_{\Gamma,\varepsilon}$. Moreover, such functions vanish at 0. It follows that $\widehat{\beta}_\varepsilon$ and $\widehat{\beta}_{\Gamma,\varepsilon}$ are nonnegative convex functions with (at most) a quadratic growth.

Then, the approximating problem consists in finding a pair $(u_\varepsilon, w_\varepsilon)$ satisfing the regularity properties and the Cauchy condition given below

$$u_\varepsilon \in L^\infty(0,T;V) \cap H^1(0,T;H) \tag{4.6}$$
$$u_\varepsilon|_\Gamma \in L^\infty(0,T;V_\Gamma) \cap H^1(0,T;H_\Gamma) \tag{4.7}$$
$$w_\varepsilon \in L^2(0,T;V) \tag{4.8}$$
$$u_\varepsilon(0) = u_0 \tag{4.9}$$

and solving, for a.a. $t \in (0,T)$, the variational equations

$$\int_\Omega \partial_t u_\varepsilon(t)\, z + \int_\Omega \nabla w_\varepsilon(t) \cdot \nabla z = 0 \tag{4.10}$$
$$\int_\Omega w_\varepsilon(t) z = \tau_\varepsilon \int_\Omega \partial_t u_\varepsilon(t)\, z + \int_\Gamma \partial_t u_\varepsilon(t)\, z + \int_\Omega \nabla u_\varepsilon(t) \cdot \nabla z + \nu \int_\Gamma \nabla_\Gamma u_\varepsilon(t) \cdot \nabla_\Gamma z$$
$$+ \int_\Omega \bigl(\beta_\varepsilon(u_\varepsilon(t)) + \pi(u_\varepsilon(t)) - f(t)\bigr)\, z + \int_\Gamma \bigl(\beta_{\Gamma,\varepsilon}(u_\varepsilon(t)) + \pi_\Gamma(u_\varepsilon(t)) - f_\Gamma(t)\bigr)\, z \tag{4.11}$$

for every $z \in V$ and for every $z \in \mathcal{V}$, respectively. An equivalent formulation of (4.10)–(4.11) is the following

$$\int_Q \partial_t u_\varepsilon\, z + \int_Q \nabla w_\varepsilon \cdot \nabla z = 0 \tag{4.12}$$
$$\int_Q w_\varepsilon z = \tau_\varepsilon \int_Q \partial_t u_\varepsilon\, z + \int_\Sigma \partial_t u_\varepsilon\, z + \int_Q \nabla u_\varepsilon \cdot \nabla z + \nu \int_\Sigma \nabla_\Gamma u_\varepsilon \cdot \nabla_\Gamma z$$
$$+ \int_Q \bigl(\beta_\varepsilon(u_\varepsilon) + \pi(u_\varepsilon) - f\bigr)\, z + \int_\Sigma \bigl(\beta_{\Gamma,\varepsilon}(u_\varepsilon) + \pi_\Gamma(u_\varepsilon) - f_\Gamma\bigr)\, z \tag{4.13}$$

for every $z \in L^2(0,T;V)$ and every $z \in L^2(0,T;\mathcal{V})$, respectively. Note that, as for (2.19)–(2.20), we have

$$\partial_t (u_\varepsilon(t)_\Omega) = 0 \quad \text{for a.a. } t \in (0,T) \quad \text{and} \quad u_\varepsilon(t)_\Omega = (u_0)_\Omega \quad \text{for every } t \in [0,T]. \tag{4.14}$$

We can prove a well-posedness result for the above problem as a particular case of the theorem stated below. Indeed, its proof does not require that the operators involved in the problem are exactly the previous ones. Just some more smoothness in addition to the regularity conditions required for β and β_Γ is needed, indeed.

Theorem 4.2. *Assume that $\beta_\varepsilon, \beta_{\Gamma,\varepsilon} : \mathbb{R} \to \mathbb{R}$ are monotone and Lipschitz continuous and that they vanish at* 0. *Moreover, assume $\tau_\varepsilon > 0$ and* (2.4). *Finally, assume* (2.12) *and* (2.26). *Then, there exists a unique pair $(u_\varepsilon, w_\varepsilon)$ satisfying* (4.6)–(4.9) *and solving* (4.10)–(4.11).

The uniquess part follows as a particular case of Theorem 2.4. Hence, the rest of the section is devoted to the proof of existence. For convenience, we refer to the precise notation introduced for the approximating problem, but it is clear from the proof we give that just the assumptions of the statement are used. Our argument relies on a Galerkin scheme and a compactness method based on suitable a priori estimates performed on the discrete solution.

The discretized problem. We consider the problem

$$\lambda \in \mathbb{R}, \quad e \in V \setminus \{0\}, \quad \text{and} \quad \int_\Omega \nabla e \cdot \nabla z = \lambda \int_\Omega e z \quad \text{for every } z \in V \tag{4.15}$$

which is the variational formulation of the eigenvalue problem $-\Delta e = \lambda e$ with homogeneous Neumann boundary conditions. It is well known that (4.15) has infinitely many eigenvalues. More precisely, there exist two sequences $\{\lambda_n\}_{n=1,2,\dots}$ and $\{e_n\}_{n=1,2,\dots}$ such that, for every $n \geq 1$, λ_n is an eigenvalue and e_n is a corresponding eigenfunction, the sequence $\{\lambda_n\}$ is nondecreasing, and the sequence $\{e_n\}$ is orthonormal and complete in $L^2(\Omega)$. We observe that

$$\int_\Omega \nabla e_j \cdot \nabla e_i = 0 \quad \text{for } i \neq j \quad \text{and} \quad \int_\Omega |\nabla e_i|^2 = \lambda_i > 0 \quad \text{for } i > 1. \tag{4.16}$$

Indeed, (4.15) clearly implies both equalities. Moreover the second one holds true for every i and implies that $\lambda_i \geq 0$ for every i. In order to verify that $\lambda_i > 0$ for $i > 1$, we notice that $\lambda = 0$ is an eigenvalue, whence $\lambda_1 = 0$, and that any non-zero constant is an eigenfunction. Furthermore, every eigenfunction is a constant since Ω is connected by assumption. We deduce that, for every $i > 1$, e_i cannot be a constant, whence $\lambda_i > 0$. Moreover, as e_1 is a constant and $\{e_n\}$ is orthonormal in H, from (4.15) and (2.41)–(2.42) we easily deduce that

$$e_i \in \operatorname{dom} \mathcal{N} \quad \text{and} \quad \mathcal{N} e_i = \frac{1}{\lambda_i}\, e_i \quad \text{for every } i > 1. \tag{4.17}$$

Now, we can introduce the discretized problem. We set

$$V_n := \operatorname{span}\{e_i : i = 1, \dots, n\} \quad \text{for every } n \geq 1 \tag{4.18}$$

$$u_0^n \text{ is the } L^2(\Omega)\text{-projection of } u_0 \text{ on } V_n \tag{4.19}$$

and note that $V_n \subset \mathcal{V}$ since $e_i \in H^2(\Omega)$ for every i. Then, we look for a pair $(u_\varepsilon^n, w_\varepsilon^n)$ satisfying

$$u_\varepsilon^n \in H^1(0,T;V_n) \quad \text{and} \quad w_\varepsilon^n \in L^2(0,T;V_n) \tag{4.20}$$

solving the following variational equations

$$\int_\Omega \partial_t u_\varepsilon^n(t)\, z + \int_\Omega \nabla w_\varepsilon^n(t) \cdot \nabla z = 0 \tag{4.21}$$

$$\begin{aligned} \int_\Omega w_\varepsilon^n(t) z &= \tau_\varepsilon \int_\Omega \partial_t u_\varepsilon^n(t)\, z + \int_\Gamma \partial_t u_\varepsilon^n(t)\, z \\ &\quad + \int_\Omega \nabla u_\varepsilon^n(t) \cdot \nabla z + \nu \int_\Gamma \nabla_\Gamma u_\varepsilon^n(t) \cdot \nabla_\Gamma z \\ &\quad + \int_\Omega \bigl(\beta_\varepsilon(u_\varepsilon^n(t)) + \pi(u_\varepsilon^n(t)) - f(t)\bigr)\, z \\ &\quad + \int_\Gamma \bigl(\beta_{\Gamma,\varepsilon}(u_\varepsilon^n(t)) + \pi_\Gamma(u_\varepsilon^n(t)) - f_\Gamma(t)\bigr)\, z \end{aligned} \tag{4.22}$$

for a.a. $t \in (0,T)$ and every $z \in V_n$, and fulfilling the Cauchy condition

$$u_\varepsilon^n(0) = u_0^n\,. \tag{4.23}$$

Before studying the discretized problem, we make a remark. As $e_1 \in V_n$ for every n, we can choose $z = e_1$ in (4.21). On the other hand, e_1 is a constant. Hence, we deduce that

$$\partial_t (u_\varepsilon^n(t))_\Omega = \frac{1}{|\Omega|} \int_\Omega \partial_t u_\varepsilon^n(t) = 0 \quad \text{for a.a. } t \in (0,T) \tag{4.24}$$

i.e., we have the same property as in the continous case (see Remark 2.3 and (4.14)).

Theorem 4.3. *The discretized problem* (4.21)–(4.23) *has a unique solution satisfying* (4.20).

Proof. Clearly, an equivalent formulation is obtained just taking $z = e_i$, $i = 1, \dots, n$, in equations (4.21)–(4.23), and this leads to a system of ordinary differential equations. Precisely, (the dependence on ε and n is not stressed to simplify the notation, here and later on) let $\mathbf{u}(t)$ and $\mathbf{w}(t)$ be the vectors of the coordinates of $u_\varepsilon^n(t)$ and $w_\varepsilon^n(t)$ with respect to the base of V_n we have chosen, i.e., the (column) n-vectors $\mathbf{u}(t) := (u_j(t))$ and $\mathbf{w}(t) := (w_j(t))$ satisfying

$$u_\varepsilon^n(t) = \sum_{j=1}^n u_j(t) e_j \quad \text{and} \quad w_\varepsilon^n(t) = \sum_{j=1}^n w_j(t) e_j$$

and consider the $n \times n$ matrices $A := (a_{ij})$, $B := (b_{ij})$, $A^\Gamma := (a_{ij}^\Gamma)$, and $B^\Gamma := (b_{ij}^\Gamma)$ defined by

$$a_{ij} := \int_\Omega e_j e_i \,, \quad b_{ij} := \int_\Omega \nabla e_j \cdot \nabla e_i \,, \quad a_{ij}^\Gamma := \int_\Gamma e_j e_i \,, \quad \text{and} \quad b_{ij}^\Gamma := \nu \int_\Gamma \nabla_\Gamma e_j \cdot \nabla_\Gamma e_i \,.$$

Then, equalities (4.21)–(4.22) take the form

$$\begin{aligned} & A\mathbf{u}'(t) + B\mathbf{w}(t) = 0 \\ & A\mathbf{w}(t) = \tau_\varepsilon A\mathbf{u}'(t) + A^\Gamma \mathbf{u}'(t) + B\mathbf{u}(t) + B^\Gamma \mathbf{u}(t) + \mathbf{F}(\mathbf{u}(t)) - \mathbf{f}(t) + \mathbf{G}(\mathbf{u}(t)) - \mathbf{g}(t) \end{aligned}$$

where the components of the functions $\mathbf{F}, \mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{f}, \mathbf{g} : (0,T) \to \mathbb{R}^n$ are given by

$$\begin{aligned} & F_i(y) := \int_\Omega (\beta_\varepsilon + \pi)\Big(\sum_{j=1}^n y_j e_j\Big)\, e_i \quad \text{for } y = (y_1, \dots, y_n) \in \mathbb{R}^n \\ & G_i(y) := \int_\Gamma (\beta_{\Gamma,\varepsilon} + \pi_\Gamma)\Big(\sum_{j=1}^n y_j e_j\Big)\, e_i \quad \text{for } y = (y_1, \dots, y_n) \in \mathbb{R}^n \\ & f_i(t) := \int_\Omega f(t)\, e_i \quad \text{and} \quad g_i(t) := \int_\Gamma f_\Gamma(t)\, e_i \quad \text{for a.a. } t \in (0,T) \end{aligned}$$

for $i = 1, \dots, n$. Moreover, (4.23) becomes

$$\mathbf{u}(0) = (u_{0,1}, \dots, u_{0,n}) \quad \text{where} \quad u_{0,i} := \int_\Omega u_0 e_i \quad \text{for } i = 1, \dots, n. \tag{4.25}$$

As $A = I_n$, the identity matrix, we can rewrite the above system as

$$\begin{aligned} & (I_n + \tau_\varepsilon B + BA^\Gamma)\mathbf{u}'(t) \\ & \quad = -B\big(B\mathbf{u}(t) + B^\Gamma \mathbf{u}(t) + \mathbf{F}(\mathbf{u}(t)) - \mathbf{f}(t) + \mathbf{G}(\mathbf{u}(t)) - \mathbf{g}(t)\big) \end{aligned} \tag{4.26}$$

$$\mathbf{w}(t) = \tau_\varepsilon \mathbf{u}'(t) + A^\Gamma \mathbf{u}'(t) + B\mathbf{u}(t) + B^\Gamma \mathbf{u}(t) + \mathbf{F}(\mathbf{u}(t)) - \mathbf{f}(t) + \mathbf{G}(\mathbf{u}(t)) - \mathbf{g}(t) \tag{4.27}$$

and we have to look for a pair $(\mathbf{u},\mathbf{w})$ satisfying $\mathbf{u}\in H^1(0,T;\mathbb{R}^n)$, $\mathbf{w}\in L^2(0,T;\mathbb{R}^n)$, equations (4.26)–(4.27), and the Cauchy condition (4.25). To discuss the latter version of the system, we recall that β_ε, $\beta_{\Gamma,\varepsilon}$, π, and π_Γ are Lipschitz continuous, whence $\mathbf{F}$ and $\mathbf{G}$ enjoy the same property. Furthermore, from (2.12), we infer that $\mathbf{f},\mathbf{g}\in L^2(0,T;\mathbb{R}^n)$. Therefore, the proof is complete once we show that we can solve (4.26).

To this end, we note that $b_{ij}=0$ if either $i=1$ or $j=1$ by (4.16). Hence, the first scalar equation of (4.26) becomes $u_1'(t)=0$, so that $u_1(t)$ is a known constant (see (4.25) with $i=1$). Hence, by assuming $n>1$, we eliminate u_1 from (4.26) and just look for the components u_j of the solution with $j>1$. If we set $\mathbf{v}:=(u_2,\dots,u_n)$ and remember that the first row and column of B vanish, we see that the remaining part of (4.26) is the $(n-1)$-dimensional system

$$\bigl(I_{n-1}+\tau_\varepsilon C+CD\bigr)\mathbf{v}'(t)=\mathbf{h}(t)-\mathbf{H}(\mathbf{v}(t)). \tag{4.28}$$

In (4.28), the matrices C and D are obtained by deleting both the first row and the first column of B and A^Γ, respectively, and the symbols I_{n-1}, $\mathbf{h}$, and $\mathbf{H}$ have an obvious meaning. We just note that $\mathbf{h}\in L^2(0,T;\mathbb{R}^{n-1})$ and that $\mathbf{H}:\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$ is Lipschitz continuous. Now, C is positive definite by (4.16). Therefore, (4.28) is equivalent to

$$\bigl(C^{-1}+\tau_\varepsilon I_{n-1}+D\bigr)\mathbf{v}'(t)=C^{-1}\bigl(\mathbf{h}(t)-\mathbf{H}(\mathbf{v}(t))\bigr)$$

and the matrix in front of $\mathbf{v}'(t)$ is positive definite. Indeed, $\tau_\varepsilon>0$ and D is at least positive semidefinite, since such a property holds for A^Γ, as one immediately sees just by owing to the definition. Hence, the above system can be solved for $\mathbf{v}'(t)$, thus for $\mathbf{v}(t)$ by the standard theory, and the proof is complete. □

Once we know that the discretized problem (4.21)–(4.23) has a solution, we would like to let n tend to infinity and conclude the proof of Theorem 4.2. However, before doing that, we prepare some preliminary density results. We note at once that the first part of the next lemma applies with $z=u_0$, due to (2.26). The last sentence holds for u_0 if $\nu>0$ and (2.27) is assumed as well (see also the first part of Remark 2.6).

Lemma 4.4. *Assume $z\in H^2(\Omega)$ and $\partial_n z|_\Gamma=0$ and set*

$$z_n \text{ is the } L^2(\Omega)\text{-projection of } z \text{ on } V_n. \tag{4.29}$$

Then, we have that

$$z_n\to z \quad \text{strongly in } \mathcal{V}. \tag{4.30}$$

Moreover, the sequence $\{z_n\}$ is bounded in $H^{5/2}(\Omega)$ whenever $z\in H^{5/2}(\Omega)$.

Proof. We represent z and z_n by means of their Fourier coefficients with respect to the system $\{e_i\}$ and apply (4.29). Thus, we have

$$z=\sum_{i=1}^\infty a_ie_i \quad\text{and}\quad z_n=\sum_{i=1}^n a_ie_i\,,\quad\text{whence}\quad z-z_n=\sum_{i=n+1}^\infty a_ie_i\,,$$

for some real sequence $\{a_i\}\in\ell^2$. Now, we observe that our assumptions on z imply that

$$-\Delta z=\sum_{i=1}^\infty \lambda_ia_ie_i\,,\quad -\Delta z_n=\sum_{i=1}^n\lambda_ia_ie_i\,,\quad\text{and}\quad \{\lambda_ia_i\}\in\ell^2. \tag{4.31}$$

Therefore, using the trace theorem for $H^2(\Omega)$, the regularity theory for the Neumann problem, and the Parseval identity, we obtain (in both cases $\nu > 0$ and $\nu = 0$)

$$\|z - z_n\|_{\mathcal{V}}^2 = \|z - z_n\|_{1,\Omega}^2 + \|z - z_n\|_{V_\Gamma}^2 \le c\|z - z_n\|_{H^2(\Omega)}^2$$
$$\le c\left(\|z - z_n\|_{L^2(\Omega)}^2 + \|-\Delta(z - z_n)\|_{L^2(\Omega)}^2\right) = c\sum_{i=n+1}^{\infty}\left(a_i^2 + \lambda_i^2 a_i^2\right)$$

and (4.30) follows. Finally we observe that the norms

$$\|z\|_{5/2,\Omega} \quad \text{and} \quad \left(\sum_{i=1}^{\infty}\left(a_i^2 + \lambda_i^{5/2}a_i^2\right)\right)^{1/2}$$

are equivalent on the space $\{z \in H^{5/2}(\Omega) : \ \partial_n z|_\Gamma = 0\}$. Therefore, the last sentence follows with the same argument as above. □

Lemma 4.5. *The set of functions $z \in H^2(\Omega)$ such that $\partial_n z|_\Gamma = 0$ is dense in $\mathcal{V}$.*

Proof. Let $\mathcal{V}_0$ be the set of functions $z \in H^2(\Omega)$ such that $\partial_n z|_\Gamma = 0$. We assume $u \in \mathcal{V}$ to be orthogonal to $\mathcal{V}_0$ in $\mathcal{V}$ and prove that $u = 0$. We recall that $\mathcal{V}$ is endowed with the graph norm. However, we can use the inner product defined by

$$(u, z)_{\mathcal{V}} := \int_\Omega \nabla u \cdot \nabla z + \int_\Gamma \left(uz + \nu\nabla_\Gamma u \cdot \nabla_\Gamma z\right)$$

since it induces an equivalent norm. Therefore, our assumption means that

$$\int_\Omega \nabla u \cdot \nabla z + \int_\Gamma \left(uz + \nu\nabla_\Gamma u \cdot \nabla_\Gamma z\right) = 0 \quad \text{for every } z \in \mathcal{V}_0. \tag{4.32}$$

By taking first $z \in C^\infty(\Omega)$ with compact support, we infer that $-\Delta u = 0$ in Ω in the sense of distributions (i.e., ∇u is divergence free), whence also (see [9, Thm 1, Ch. IX A, p. 240] for such a trace theorem and [24] for the general theory of Sobolev spaces with real index and for the notation used in the present paper)

$$\partial_n u|_\Gamma \in H^{-1/2}(\Gamma) \quad \text{and} \quad \int_\Omega \nabla u \cdot \nabla z = \langle \partial_n u|_\Gamma, z|_\Gamma \rangle_\Gamma \quad \text{for every } z \in V \tag{4.33}$$

where $\langle \cdot\,, \cdot \rangle_\Gamma$ stands for the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. It follows that

$$\langle \partial_n u|_\Gamma, z|_\Gamma \rangle_\Gamma + \int_\Gamma \left(uz + \nu\nabla_\Gamma u \cdot \nabla_\Gamma z\right) = 0 \quad \text{for every } z \in \mathcal{V}_0.$$

On the other hand, the map $z \mapsto (z|_\Gamma, \partial_n z|_\Gamma)$ maps $H^2(\Omega)$ onto $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ (see, e.g., [24, Thm. 8.3, p. 44]). In particular, every element of $H^{3/2}(\Gamma)$ is the trace of some $z \in \mathcal{V}_0$. Hence, the above conclusion becomes

$$\langle \partial_n u|_\Gamma, z_\Gamma \rangle_\Gamma + \int_\Gamma \left(uz_\Gamma + \nu\nabla_\Gamma u \cdot \nabla_\Gamma z_\Gamma\right) = 0 \quad \text{for every } z_\Gamma \in H^{3/2}(\Gamma).$$

Now, we observe that $H^{3/2}(\Gamma)$ is dense in both $H^1(\Gamma)$ and $H^{1/2}(\Gamma)$ (thus in V_Γ), as is well known, and that the left-hand side of the above equality defines a functional (z_Γ being the variable) which is linear and continuous with respect to the norm of $H^{1/2}(\Gamma)$, i.e., an element of the dual space V_Γ', in both cases $\nu > 0$ and $\nu = 0$. We conclude that such an equality actually holds for every $z_\Gamma \in V_\Gamma$. By combining with (4.33), we deduce that (4.32) holds for every $z \in \mathcal{V}$, in particular with $z = u$, and immediately derive that $u = 0$. □

By combining the first part of Lemma 4.4 and Lemma 4.5, it is straightforward to deduce the corollary stated below.

Corollary 4.6. *Let V_∞ be the union of the family $\{V_n : n \geq 1\}$. Then, the set of V_∞-valued step functions is dense in $L^2(0,T;\mathcal{V})$.*

Now, we perform a priori estimates on the discrete solution $(u_\varepsilon^n, w_\varepsilon^n)$ in order to solve the approximating problem by letting n tend to infinity. In the sequel, δ is a positive parameter.

First a priori estimate. By (4.24), we have $\partial_t u_\varepsilon^n(t) \in \operatorname{dom} \mathcal{N}$ for a.a. $t \in (0,T)$. Moreover, (4.17) implies that $\mathcal{N}\partial_t u_\varepsilon^n(t) \in V_n$ for a.a. $t \in (0,T)$. Hence, we can test (4.21) by $\mathcal{N}\partial_t u_\varepsilon^n$ and integrate over $(0,t)$, where $t \in (0,T]$ is arbitrary. More precisely, we write (4.21) at time $t = s$ and choose $z = \mathcal{N}\partial_t u_\varepsilon^n(s)$. Then we integrate over $(0,t)$ with respect to s. At the same time, we note that $\partial_t u_\varepsilon^n(t) \in V_n$ for a.a. $t \in (0,T)$ and test (4.22) by $-\partial_t u_\varepsilon^n$. Then, we add the equalities we have got to each other and add the same quantity to both sides for convenience. Accounting for (2.46), we obtain

$$
\begin{aligned}
&\int_0^t \|\partial_t u_\varepsilon^n(s)\|_*^2 \, ds + \int_{Q_t} \nabla w_\varepsilon^n \cdot \nabla \mathcal{N}\partial_t u_\varepsilon^n - \int_{Q_t} w_\varepsilon^n \, \partial_t u_\varepsilon^n + \tau_\varepsilon \int_{Q_t} |\partial_t u_\varepsilon^n|^2 + \int_{\Sigma_t} |\partial_t u_\varepsilon^n|^2 \\
&\quad + \frac{1}{2} \int_\Omega |\nabla u_\varepsilon^n(t)|^2 + \frac{\nu}{2} \int_\Gamma |\nabla_\Gamma u_\varepsilon^n(t)|^2 + \int_\Omega \widehat{\beta}_\varepsilon(u_\varepsilon^n(t)) + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(u_\varepsilon^n(t)) \\
&\quad + \|u_\varepsilon^n(t)\|_{0,\Omega}^2 + \|u_\varepsilon^n(t)\|_{0,\Gamma}^2 \\
&= - \int_\Omega \widehat{\pi}(u_\varepsilon^n(t)) - \int_\Gamma \widehat{\pi_\Gamma}(u_\varepsilon^n(t)) + \int_{Q_t} f \, \partial_t u_\varepsilon^n + \int_{\Sigma_t} f_\Gamma \, \partial_t u_\varepsilon^n \\
&\quad + \frac{1}{2} \int_\Omega |\nabla u_0^n|^2 + \frac{\nu}{2} \int_\Gamma |\nabla_\Gamma u_0^n|^2 \\
&\quad + \int_\Omega \widehat{\beta}_\varepsilon(u_0^n) + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(u_0^n) + \int_\Omega \widehat{\pi}(u_0^n) + \int_\Gamma \widehat{\pi_\Gamma}(u_0^n) \\
&\quad + \|u_\varepsilon^n(t)\|_{0,\Omega}^2 + \|u_\varepsilon^n(t)\|_{0,\Gamma}^2
\end{aligned}
\tag{4.34}
$$

where we have set for convenience

$$
\widehat{\pi}(r) := \int_0^r \pi(s) \, ds \quad \text{and} \quad \widehat{\pi_\Gamma}(r) := \int_0^r \pi_\Gamma(s) \, ds \quad \text{for } r \in \mathbb{R}. \tag{4.35}
$$

The second term on the left-hand side and the third one cancel out, due to the definition (2.42) of $\mathcal{N}$. As all the other integrals are nonnegative, we consider the right-hand side. Owing to (2.12) and recalling (4.1), we immediately have

$$
\int_{Q_t} f \, \partial_t u_\varepsilon^n + \int_{\Sigma_t} f_\Gamma \, \partial_t u_\varepsilon^n \leq \frac{\tau_\varepsilon}{2} \int_{Q_t} |\partial_t u_\varepsilon^n|^2 + \frac{1}{2} \int_{\Sigma_t} |\partial_t u_\varepsilon^n|^2 + c_\varepsilon.
$$

Now, we recall that all the nonlinearities have a quadratic growth. Hence, collecting the first two terms on the right-hand side of (4.34) and the last six ones, their sum is estimated by

$$
c \int_\Omega |u_\varepsilon^n(t)|^2 + c \int_\Gamma |u_\varepsilon^n(t)|^2 + c_\varepsilon \int_\Omega |u_0^n|^2 + c_\varepsilon \int_\Gamma |u_0^n|^2 + c_\varepsilon \, . \tag{4.36}
$$

On the other hand, owing to (2.49)–(2.50), we see that the sum of the first two integrals of (4.36) is bounded by the quantity

$$c\int_\Omega |u_0^n|^2 + c\int_\Gamma |u_0^n|^2 + \delta\int_0^t \big(\|\partial_t u_\varepsilon^n(s)\|_*^2 + \|\partial_t u_\varepsilon^n(s)\|_{0,\Gamma}^2\big)\,ds$$
$$+ c_\delta \int_0^t \big(\|u_\varepsilon^n(s)\|_{0,\Omega}^2 + \|\nabla u_\varepsilon^n(s)\|_{0,\Omega}^2 + \|u_\varepsilon^n(s)\|_{0,\Gamma}^2\big)\,ds.$$

Finally, we have to consider all the terms involving u_0^n that either come from the above estimates or have not yet been considered. To this aim, it suffices to apply Lemma 4.4. Therefore, we can choose δ small enough and apply the Gronwall lemma. We conclude that

$$\|u_\varepsilon^n\|_{L^\infty(0,T;V)\cap H^1(0,T;H)} + \|u_\varepsilon^n|_\Gamma\|_{L^\infty(0,T;V_\Gamma)\cap H^1(0,T;H_\Gamma)} \le c_\varepsilon \tag{4.37}$$

just forgetting some positive integrals.

Second a priori estimate. If we test (4.21) by w_ε^n and integrate with respect to time, owing to (4.37), we easily obtain

$$\int_0^T \|\nabla w_\varepsilon^n(t)\|_{0,\Omega}^2\,dt = -\int_0^T \langle \partial_t u_\varepsilon^n(t), w_\varepsilon^n(t)\rangle\,dt \le \delta\int_0^T \|w_\varepsilon^n(t)\|_{1,\Omega}^2\,dt + c_\delta\,. \tag{4.38}$$

On the other hand, by testing (4.22) by $1/|\Omega|$, recalling that all the nonlinearities are Lipschitz, squaring, integrating in time, and owing to (4.37) once more, we get

$$\begin{aligned}\int_0^T |(w_\varepsilon^n(t))_\Omega|^2\,dt &\le c\|\partial_t u_\varepsilon^n\|_{L^2(0,T;H)}^2 + c\|\partial_t u_\varepsilon^n\|_{L^2(0,T;H_\Gamma)}^2\\ &\quad + c\|f\|_{L^2(0,T;H)}^2 + c\|f_\Gamma\|_{L^2(0,T;H_\Gamma)}^2\\ &\quad + c_\varepsilon\big(1 + \|u_\varepsilon^n\|_{L^2(0,T;H)}^2 + \|u_\varepsilon^n\|_{L^2(0,T;H_\Gamma)}^2\big) \le c_\varepsilon\,.\end{aligned} \tag{4.39}$$

Adding (4.38) and (4.39) to each other and using the Poincaré inequality (2.40), we get

$$\int_0^T \|w_\varepsilon^n(t)\|_{1,\Omega}^2\,dt \le M_\Omega\int_0^T \big(\|\nabla w_\varepsilon^n(t)\|_{0,\Omega}^2 + |(w_\varepsilon^n(t))_\Omega|^2\big)\,dt$$
$$\le \delta M_\Omega \int_0^T \|w_\varepsilon^n(t)\|_{1,\Omega}^2\,dt + c_\delta + c_\varepsilon$$

whence immediately

$$\|w_\varepsilon^n\|_{L^2(0,T;V)} \le c_\varepsilon \tag{4.40}$$

by choosing δ small enough.

Conclusion. By well-known weak and weak star compactness results, we see that $(u_\varepsilon, w_\varepsilon)$ exists such that

$$u_\varepsilon^n \to u_\varepsilon \qquad \text{weakly star in } L^\infty(0,T;V)\cap H^1(0,T;H) \tag{4.41}$$
$$u_\varepsilon^n|_\Gamma \to u_\varepsilon|_\Gamma \qquad \text{weakly star in } L^\infty(0,T;V_\Gamma)\cap H^1(0,T;H_\Gamma) \tag{4.42}$$
$$w_\varepsilon^n \to w_\varepsilon \qquad \text{weakly in } L^2(0,T;V) \tag{4.43}$$

as n tends to infinity, at least for a subsequence. Moreover, owing to the compact embeddings $V \subset H$ and $V_\Gamma \subset H_\Gamma$, we can apply [36, Sect. 8, Cor. 4] and derive the strong convergence

$$u_\varepsilon^n \to u_\varepsilon \quad \text{strongly in } C^0([0,T];H) \quad \text{and} \quad u_\varepsilon^n|_\Gamma \to u_\varepsilon|_\Gamma \quad \text{strongly in } C^0([0,T];H_\Gamma).$$

In particular, we have that $u_\varepsilon(0) = u_0$ by (4.23) and (4.30) applied to u_0, and that

$$\begin{aligned} \beta_\varepsilon(u_\varepsilon^n) \to \beta_\varepsilon(u_\varepsilon) \quad &\text{and} \quad \pi(u_\varepsilon^n) \to \pi(u_\varepsilon) \qquad &&\text{strongly in } C^0([0,T];H) \\ \beta_{\Gamma,\varepsilon}(u_\varepsilon^n) \to \beta_{\Gamma,\varepsilon}(u_\varepsilon) \quad &\text{and} \quad \pi_\Gamma(u_\varepsilon^n) \to \pi_\Gamma(u_\varepsilon) \qquad &&\text{strongly in } C^0([0,T];H_\Gamma) \end{aligned}$$

just by Lipschitz continuity. Now, we fix $m \geq 1$ and note that $V_m \subset V_n$ for every $n \geq m$. Therefore, (4.21)–(4.22) imply that

$$\begin{aligned} &\int_Q \partial_t u_\varepsilon^n \, z + \int_Q \nabla w_\varepsilon^n \cdot \nabla z = 0 \\ &\int_Q w_\varepsilon^n z = \tau_\varepsilon \int_Q \partial_t u_\varepsilon^n \, z + \int_\Sigma \partial_t u_\varepsilon^n \, z + \int_Q \nabla u_\varepsilon^n \cdot \nabla z + \nu \int_\Sigma \nabla_\Gamma u_\varepsilon^n \cdot \nabla_\Gamma z \\ &\quad + \int_Q \bigl(\beta_\varepsilon(u_\varepsilon^n) + \pi(u_\varepsilon^n) - f\bigr) \, z + \int_\Sigma \bigl(\beta_{\Gamma,\varepsilon}(u_\varepsilon^n) + \pi_\Gamma(u_\varepsilon^n) - f_\Gamma\bigr) \, z \end{aligned}$$

for every $n \geq m$ and every V_m-valued step function z. By applying the above convergence, we see that the variational equations (4.12)–(4.13) are satisfied for the above test functions. As m is arbitrary, the same holds for every $z \in L^2(0,T;\mathcal{V})$ by Corollary 4.6. As (4.12)–(4.13) and (4.10)–(4.11) are equivalent to each other, the proof of Theorem 4.2 is complete.

Remark 4.7. In the a priori estimates of the above proof we have used the generic notation c_ε for brevity. However, it is clear that, for fixed ε (whence τ_ε is fixed too), if we let β_ε, $\beta_{\Gamma,\varepsilon}$, π, π_Γ, and the data u_0, f, f_Γ vary in some families (e.g., depending on some parameter ε' in addition) such that the assumptions of Theorem 4.2 are fulfilled and the inequalities

$$\begin{aligned} &\sup \beta_\varepsilon' + \sup \beta_{\Gamma,\varepsilon}' + \sup |\pi'| + \sup |\pi_\Gamma'| + |\pi(0)| + |\pi_\Gamma(0)| \\ &\quad + \|u_0\|_{2,\Omega} + \|f\|_{H^1(0,T;H)} + \|f_\Gamma\|_{L^2(0,T;H_\Gamma)} \leq M = M_\varepsilon \end{aligned} \tag{4.44}$$

hold true for some constant M and all the functions and the data of such families, then the corresponding solutions $(u_\varepsilon, w_\varepsilon)$ satisfy

$$\|u_\varepsilon\|_{L^\infty(0,T;V)\cap H^1(0,T;H)} + \|u_\varepsilon\|_{L^\infty(0,T;V_\Gamma)\cap H^1(0,T;H_\Gamma)} + \|w_\varepsilon\|_{L^2(0,T;V)} \leq M' = M_\varepsilon' \tag{4.45}$$

where the constant M' depends on ε and M, only. In particular, if we fix ε but perturb the given functions π, π_Γ, the approximating monotone functions β_ε and $\beta_{\Gamma,\varepsilon}$ given by (4.2) and (4.3), and the data by replacing them with smoother functions and data, depending on some parameter ε', and a bound like (4.44) holds uniformly with respect to ε', then the corresponding solutions satisfy the analogue of (4.45) uniformly with respect to ε'.

If the functions entering the structure of the system and the data are smoother, further regularity of the solution can be proved. We confine ourselves to show the result stated below, even though it is not sharp. However, it will be sufficient for our purpose in the sequel. It is understood that all previous assumptions (e.g., (2.26)–(2.27)) are satisfied.

Proposition 4.8. *Assume $f \in C^1([0,T];H)$, $f_\Gamma \in C^1([0,T];H_\Gamma)$, and let β, β_Γ, π, π_Γ be C^2-functions with bounded second derivatives. Then, we have*

$$\partial_t u_\varepsilon \in L^\infty(0,T;V) \cap H^1(0,T;H) \quad \text{and} \quad \partial_t u_\varepsilon|_\Gamma \in L^\infty(0,T;V_\Gamma) \cap H^1(0,T;H_\Gamma). \tag{4.46}$$

Proof. From the above argument, it is clear that (4.46) follows whenever we can perform the corresponding a priori estimate on the solution of the discretized problem. To this aim, we observe that the first component u_ε^n of such a discrete solution actually has a second time derivative since the nonlinear functions involved in the structure of the system of ordinary differential equations are of class C^1 under the above further assumption. Hence, we can differentiate (4.21) and (4.22) with respect to time and test the equalities we get by $\mathcal{N}\partial_t^2 u_\varepsilon^n$ and $-\partial_t^2 u_\varepsilon^n$, repectively. By doing that, summing, integrating over $(0,t)$, and adding the same integrals to both sides for convenience, we obtain

$$
\begin{aligned}
&\int_0^t \|\partial_t^2 u_\varepsilon^n(s)\|_*^2\,ds + \tau_\varepsilon \int_{Q_t} |\partial_t^2 u_\varepsilon^n|^2 + \int_{\Sigma_t} |\partial_t^2 u_\varepsilon^n|^2 \\
&\quad + \frac{1}{2}\,\|\partial_t u_\varepsilon^n(t)\|_{1,\Omega}^2 + \frac{\nu}{2}\int_\Gamma |\nabla_\Gamma \partial_t u_\varepsilon^n(t)|^2 + \frac{\nu}{2}\int_\Gamma |\partial_t u_\varepsilon^n(t)|^2 \\
&= -\frac{1}{2}\int_{Q_t} (\beta_\varepsilon + \pi)'(u_\varepsilon^n)\,\partial_t\bigl(|\partial_t u_\varepsilon^n|^2\bigr) - \frac{1}{2}\int_{\Sigma_t} (\beta_{\Gamma,\varepsilon} + \pi_\Gamma)'(u_\varepsilon^n)\,\partial_t\bigl(|\partial_t u_\varepsilon^n|^2\bigr) \\
&\quad + \int_{Q_t} \partial_t f\,\partial_t^2 u_\varepsilon^n + \int_{\Sigma_t} \partial_t f_\Gamma\,\partial_t^2 u_\varepsilon^n + \frac{1}{2}\int_\Omega |\nabla \partial_t u_\varepsilon^n(0)|^2 + \frac{\nu}{2}\int_\Gamma |\nabla_\Gamma \partial_t u_\varepsilon^n(0)|^2 \\
&\quad + \frac{1}{2}\int_\Omega |\partial_t u_\varepsilon^n(t)|^2 + \frac{\nu}{2}\int_\Gamma |\partial_t u_\varepsilon^n(t)|^2. \qquad (4.47)
\end{aligned}
$$

We deal with each term on the right-hand side, separately. We integrate the first term by parts and owe to boundedness of both β_ε' and $(\beta_\varepsilon + \pi)''$. We obtain

$$
-\frac{1}{2}\int_{Q_t} (\beta_\varepsilon + \pi)'(u_\varepsilon^n)\,\partial_t\bigl(|\partial_t u_\varepsilon^n|^2\bigr) \le c_\varepsilon \int_\Omega |\partial_t u_\varepsilon^n(t)|^2 + c_\varepsilon \int_\Omega |\partial_t u_\varepsilon^n(0)|^2 + c_\varepsilon \int_{Q_t} |\partial_t u_\varepsilon^n|^3.
$$

A term like the first one on the right-hand side is already present on the right-hand side of (4.47) and the last one can be treated owing to the Hölder inequality and to the continuous embedding $V \subset L^4(\Omega)$ as follows

$$
\int_{Q_t} |\partial_t u_\varepsilon^n|^3 \le \int_0^t \|\partial_t u_\varepsilon^n(s)\|_{L^2(\Omega)} \|\partial_t u_\varepsilon^n(s)\|_{L^4(\Omega)}^2\,ds \le c\int_0^t \|\partial_t u_\varepsilon^n(s)\|_{L^2(\Omega)} \|\partial_t u_\varepsilon^n(s)\|_{1,\Omega}^2\,ds.
$$

We observe at once that the first factor in the last integral has already been estimated in $L^2(0,T)$ by (4.37) (whence we are allowed to apply the Gronwall lemma below). We argue similarly for the second term of the right-hand side of (4.47) and just modify the last inequalities this way

$$
\begin{aligned}
&\int_{\Sigma_t} |\partial_t u_\varepsilon^n|^3 \le \int_0^t \|\partial_t u_\varepsilon^n(s)\|_{L^2(\Gamma)} \|\partial_t u_\varepsilon^n(s)\|_{L^4(\Gamma)}^2\,ds \\
&\le c\int_0^t \|\partial_t u_\varepsilon^n(s)\|_{L^2(\Gamma)} \|\partial_t u_\varepsilon^n(s)\|_{1/2,\Gamma}^2\,ds \le c\int_0^t \|\partial_t u_\varepsilon^n(s)\|_{L^2(\Gamma)} \|\partial_t u_\varepsilon^n(s)\|_{1,\Omega}^2\,ds
\end{aligned}
$$

by owing to the continuous embedding $H^{1/2}(\Gamma) \subset L^4(\Gamma)$ and the optimal trace inequality (2.10). Also in this case, we note that the first factor in the last integral has already been estimated in $L^2(0,T)$ by (4.37). As the subsequent two terms of (4.47) can be easily treated, let us come to the last two integrals. As the second one is similar if $\nu > 0$ and vanishes if $\nu = 0$, we consider the first one, only. We have

$$\int_\Omega |\partial_t u_\varepsilon^n(t)|^2 = \int_\Omega |\partial_t u_\varepsilon^n(0)|^2 + 2\int_{Q_t} \partial_t u_\varepsilon^n \, \partial_t^2 u_\varepsilon^n$$
$$\leq \int_\Omega |\partial_t u_\varepsilon^n(0)|^2 + \delta \int_{Q_t} |\partial_t^2 u_\varepsilon^n|^2 + c_\delta \int_{Q_t} |\partial_t u_\varepsilon^n|^2 \leq \int_\Omega |\partial_t u_\varepsilon^n(0)|^2 + \delta \int_{Q_t} |\partial_t^2 u_\varepsilon^n|^2 + c_\delta c_\varepsilon$$

by (4.37). So, it remains to find a bound for all the integrals involving $\partial_t u_\varepsilon^n(0)$. To this aim, we recall that $\partial_t u_\varepsilon^n$ is a continuous V_n-valued function. In particular, both equations (4.21)–(4.22) and (4.24) hold for every $t \in [0,T]$. Therefore, we can write the equations at $t = 0$ and test them by $\mathcal{N}\partial_t u_\varepsilon^n(0)$ and $-\partial_t u_\varepsilon^n(0)$, respectively, and take the sum. By recalling (2.26)–(2.27), integrating by parts in space, and applying Lemma 4.4, we get

$$\|\partial_t u_\varepsilon^n(0)\|_*^2 + \tau_\varepsilon \|\partial_t u_\varepsilon^n(0)\|_{0,\Omega}^2 + \|\partial_t u_\varepsilon^n(0)\|_{0,\Gamma}^2$$
$$= \int_\Omega \bigl(f(0) + \Delta u_\varepsilon^n(0) - (\beta_\varepsilon + \pi)(u_\varepsilon^n(0))\bigr)\partial_t u_\varepsilon^n(0)$$
$$+ \int_\Gamma \bigl(f_\Gamma(0) + \nu \Delta_\Gamma u_\varepsilon^n(0) - (\beta_{\Gamma,\varepsilon} + \pi_\Gamma)(u_\varepsilon^n(0))\bigr)\partial_t u_\varepsilon^n(0)$$
$$\leq \frac{\tau_\varepsilon}{2} \|\partial_t u_\varepsilon^n(0)\|_{0,\Omega}^2 + \frac{1}{2} \|\partial_t u_\varepsilon^n(0)\|_{0,\Gamma}^2 + c_\varepsilon$$

whence a bound for the initial values. Therefore, by collecting the previous inequalities, choosing δ small enough, and applying the Gronwall lemma, we obtain the desired estimate

$$\|\partial_t u_\varepsilon^n\|_{L^\infty(0,T;V)} + \|\partial_t u_\varepsilon^n\|_{L^\infty(0,T;V_\Gamma)} + \|\partial_t^2 u_\varepsilon^n\|_{L^2(0,T;H)} + \|\partial_t^2 u_\varepsilon^n\|_{L^2(0,T;H_\Gamma)} \leq c_\varepsilon$$

uniformly with respect to n. Hence, (4.46) is proved. □

5 Existence

In this section, we prove Theorems 2.8 and 2.9 at the same time. In principle, our argument is similar to the one used to prove Theorem 4.2. Here, the starting point is a solution $(u_\varepsilon, w_\varepsilon)$ to the approximating problem (4.9)–(4.11). However, as the nonlinearities β and β_Γ are not Lipschitz continuous, much more care is needed. Estimates for $\beta_\varepsilon(u_\varepsilon)$ and for $\beta_{\Gamma,\varepsilon}(u_\varepsilon)$ are crucial, indeed, and this is a difficulty. Moreover, as the solution to the approximating problem is less regular than the discrete one, some trouble might arise in justifying an analogous choice of the test functions. For instance, we would test (4.10) by $\mathcal{N}\partial_t u_\varepsilon$ and (4.11) by $-\partial_t u_\varepsilon$, and we are not allowed to do it. Indeed, while $\mathcal{N}\partial_t u_\varepsilon$ is well-defined and belongs to $L^2(0,T;V)$, there is no reason for $\partial_t u_\varepsilon$ to belong to $L^2(0,T;\mathcal{V})$ if $\nu > 0$, so that the desired choice of the test functions might not be admissible. Therefore, a more sophisticated procedure is needed, and we sketch a possibility here in connection with Remark 4.7 and Proposition 4.8. If we perturb the structure and the data according to the former remark (thus obtaining new nonlinearities and data depending, say, on some small parameter ε' in addition) so that the latter can be applied, then the desired choice of the above test functions is admissible and this produces an estimate. If such an estimate is uniform with respect to ε', it is conserved in the limit as $\varepsilon' \to 0$. On the other hand, it is easy to see that the solution $(u_{\varepsilon,\varepsilon'}, w_{\varepsilon,\varepsilon'})$ to the regularized problem converges to $(u_\varepsilon, w_\varepsilon)$ as $\varepsilon' \to 0$ in the appropriate topology (cf. (4.45)). Indeed, the argument we used at the end of Section 4 perfectly works. Hence, the estimate we conserve in the limit actually is an estimate for the approximated solution $(u_\varepsilon, w_\varepsilon)$.

Therefore, in order not to make the paper too heavy, we proceed formally in the sequel, e.g., by differentiating the equations and using some non-admissible test functions. For the same reason, we always owe to the assumptions of Theorem 2.8 and often do not take any advantage of possible positivity for τ. However, by going through the argument we present, it is clear that some assumptions are not used in the proof of Theorem 2.9 and that minor changes could be made. For instance, if $\tau > 0$, it suffices to suppose that $f \in L^2(0,T;H)$ in the next estimate, provided that we treat the f-term in the same way as the f_Γ-term.

Finally, in order to give unified proofs, we think of $\beta_{\Gamma,\varepsilon}$ defined by (4.3), in principle. However, when we conclude the proof of Theorem 2.9, we use (4.4), as said in Remark 4.1. Indeed, nothing more has to be proved regarding such a $\beta_{\Gamma,\varepsilon}$, since its properties are formally the same as those of β_ε.

However, before starting estimating, it is convenient to prepare some auxiliary material (needed for Theorem 2.8, mainly). By recalling (2.7)–(2.8), we note that the Yosida regularization γ_ε of every maximal monotone operator $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$ is monotone and Lipschitz continuous with constant $1/\varepsilon$. Moreover, we have $\gamma_\varepsilon(0) = 0$ whenever $\gamma(0) \ni 0$ and the inequality $|\gamma_\varepsilon(r)| \leq |\gamma^\circ(r)|$ holds true for every $r \in D(\gamma)$ and $\varepsilon > 0$ (see, e.g., [3, Prop. 2.6, p. 28]). By recalling (2.3) and (4.2), we see that the above inequality holds for β_ε and we derive the analogous one for $\beta_{\Gamma,\varepsilon}$. If $r \in D(\beta_\Gamma)$ and $r \geq \varepsilon C_\Gamma$, we have

$$0 \leq \beta_{\Gamma,\varepsilon}(r) = \beta^Y_{\Gamma,\alpha\varepsilon}(r + \varepsilon C_\Gamma) \leq \beta^Y_{\Gamma,\alpha\varepsilon}(r) + \frac{C_\Gamma}{\alpha} \leq \beta^\circ_\Gamma(r) + \frac{C_\Gamma}{\alpha}\,.$$

Assume now $0 \leq r < \varepsilon C_\Gamma$. Then

$$0 \leq \beta_{\Gamma,\varepsilon}(r) \leq \beta^Y_{\Gamma,\alpha\varepsilon}(2\varepsilon C_\Gamma) \leq \frac{2C_\Gamma}{\alpha}\,.$$

By arguing analogously for $r < 0$, we obtain a similar inequality. By recalling (4.3) as well, we summarize the properties of the approximating nonlinearities as follows

$$\begin{aligned} &|\beta_\varepsilon(r)| \leq |\beta^\circ(r)| \quad \text{and} \quad |\beta_{\Gamma,\varepsilon}(r)| \leq |\beta^\circ_\Gamma(r)| + \frac{2C_\Gamma}{\alpha} \\ &|\widehat{\beta}_\varepsilon(r)| \leq |\widehat{\beta}(r)| \quad \text{and} \quad |\widehat{\beta}_{\Gamma,\varepsilon}(r)| \leq |\widehat{\beta}_\Gamma(r)| + \frac{2C_\Gamma}{\alpha}\,|r| \end{aligned} \tag{5.1}$$

for $r \in D(\beta)$ or $r \in D(\beta_\Gamma)$, accordingly.

Lemma 5.1. *There holds*

$$|\beta_\varepsilon(r)| \geq \alpha|\beta_{\Gamma,\varepsilon}(r)| - 2C_\Gamma \quad \textit{for every } r \in \mathbb{R}. \tag{5.2}$$

Proof. Let us observe that (2.3) and the first of (2.32) imply that r, $\beta^\circ(r)$, and $\beta^\circ_\Gamma(r)$ have the same sign for every $r \in D(\beta)$. Therefore, by starting from (2.35), we derive that

$$\big(I + \varepsilon(\beta + C_\Gamma)^\circ\big)(r) \geq \big(I + \alpha\varepsilon\beta^\circ_\Gamma\big)(r) \quad \text{for every nonnegative } r \in D(\beta)$$

whence easily (even, e.g., in the case of a bounded domain $D(\beta)$)

$$\big(I + \varepsilon(\beta + C_\Gamma)\big)^{-1}(r) \leq \big(I + \alpha\varepsilon\beta_\Gamma\big)^{-1}(r) \quad \text{for every } r \geq \varepsilon C_\Gamma.$$

This means that

$$\beta_\varepsilon^Y(r-\varepsilon C_\Gamma)+C_\Gamma=(\beta+C_\Gamma)_\varepsilon^Y(r)\geq\alpha\beta_{\Gamma,\alpha\varepsilon}^Y(r)\quad\text{for every } r\geq\varepsilon C_\Gamma$$

and we deduce that $\beta_\varepsilon^Y(r)+C_\Gamma\geq\alpha\beta_{\Gamma,\alpha\varepsilon}^Y(r+\varepsilon C_\Gamma)$ for every $r\geq 0$, whence in particular

$$\beta_\varepsilon(r)\geq\alpha\beta_{\Gamma,\varepsilon}(r)-C_\Gamma\geq\alpha\beta_{\Gamma,\varepsilon}(r)-2C_\Gamma\quad\text{for every } r\geq\varepsilon C_\Gamma.$$

Assume now $0\leq r<\varepsilon C_\Gamma$. Then, we have

$$\beta_\varepsilon(r)-\alpha\beta_{\Gamma,\varepsilon}(r)\geq-\alpha\beta_{\Gamma,\varepsilon}(r)\geq-\alpha\beta_{\Gamma,\alpha\varepsilon}^Y(2\varepsilon C_\Gamma)\geq-\alpha\cdot 2\varepsilon C_\Gamma\,\frac{1}{\alpha\varepsilon}=-2C_\Gamma.$$

By arguing similarly for $r\leq 0$, we deduce (5.2). □

Lemma 5.2. *There exist $\varepsilon_0>0$ and points $r_\pm^*\in D(\beta)$ such that*

$$\sigma|\beta_{\Gamma,\varepsilon}(r)|\geq(\sup|\pi_\Gamma'|)\,|r|+|\pi_\Gamma(0)|+\|f_\Gamma\|_{L^\infty(\Gamma)}\tag{5.3}$$

for every $r\in\mathbb{R}\setminus(r_-^*,r_+^*)$ and $\varepsilon\in(0,\varepsilon_0)$.

Proof. We just consider the construction of r_+^* since the other one is similar. We set for convenience

$$\gamma(r):=\alpha\beta_\Gamma(r)-C_\Gamma\quad\text{and}\quad\lambda(r):=\frac{\alpha L_\Gamma}{\sigma}\,r+\frac{\alpha M_\Gamma}{\sigma}-C_\Gamma\quad\text{for } r\in\mathbb{R}$$

so that (2.36) becomes

$$\gamma^\circ(r)\geq\lambda(r)\quad\text{for every } r\in D(\beta)\text{ satisfying } r\geq r_+.$$

As in the previous proof, we derive the corresponding inequality for the Yosida regularizations, namely

$$\gamma_\varepsilon^Y(r)\geq\lambda_\varepsilon^Y(r)\quad\text{for every } r\geq r_+^\varepsilon\,,\quad\text{where}\quad r_+^\varepsilon:=(I+\varepsilon\gamma)^{-1}(r_+).$$

By computation, we see that the above inequality reads

$$\alpha\beta_{\Gamma,\alpha\varepsilon}^Y(r+\varepsilon C_\Gamma)-C_\Gamma\geq\frac{1}{1+(\alpha\varepsilon L_\Gamma/\sigma)}\Big(\frac{\alpha L_\Gamma}{\sigma}\,r+\frac{\alpha M_\Gamma}{\sigma}-C_\Gamma\Big)\quad\text{for every } r\geq r_+^\varepsilon.$$

By recalling (4.3), we deduce that

$$\alpha\beta_{\Gamma,\varepsilon}(r)-C_\Gamma\geq\frac{1}{1+(\alpha\varepsilon L_\Gamma/\sigma)}\Big(\frac{\alpha L_\Gamma}{\sigma}\,r+\frac{\alpha M_\Gamma}{\sigma}-C_\Gamma\Big)\quad\text{for every } r\geq\max\{r_+^\varepsilon,\varepsilon C_\Gamma\}.$$

As r_+ belongs to the interior of $D(\beta)$ by (2.34), we can fix $r_+^*\in D(\beta)$ with $r_+^*>r_+$. On the other hand, the point r_+^ε converges to r_+ as $\varepsilon\to 0$. Therefore, we can choose $\varepsilon_1\in(0,1)$ such that $\max\{r_+^\varepsilon,\varepsilon C_\Gamma\}\leq r_+^*$ for $\varepsilon\in(0,\varepsilon_1)$ and deduce that the above inequality holds for every $r\geq r_+^*$. By rearranging, we obtain for $r\geq r_+^*$

$$\sigma\beta_{\Gamma,\varepsilon}(r)\geq\frac{1}{1+(\alpha\varepsilon L_\Gamma/\sigma)}\Big(L_\Gamma\,r+M_\Gamma-\frac{\sigma C_\Gamma}{\alpha}\Big)+\frac{\sigma C_\Gamma}{\alpha}=\frac{L_\Gamma}{1+(\alpha\varepsilon L_\Gamma/\sigma)}\,r+\frac{M_\Gamma+\varepsilon C_\Gamma L_\Gamma}{1+(\alpha\varepsilon L_\Gamma/\sigma)}\,.$$

On the other hand, the last two fractions converge to L_Γ and M_Γ, respectively, as $\varepsilon\to 0$ and the inequalities (2.33) hold. Therefore, (5.3) holds true as well with some $\varepsilon_0\in(0,\varepsilon_1)$. □

Lemma 5.3. *There holds* $\|\partial_t u_\varepsilon(0)\|_*^2 + \tau_\varepsilon \|\partial_t u_\varepsilon(0)\|_{0,\Omega}^2 + \|\partial_t u_\varepsilon(0)\|_{0,\Gamma}^2 \le c.$

Proof. We write equations (4.10)–(4.11) at $t = 0$ and test them by $\mathcal{N}\partial_t u_\varepsilon(0)$ and $-\partial_t u_\varepsilon(0)$, respectively, owing to (4.14). Then, we sum the equalities we get to each other and use the properties of $\mathcal{N}$ in order to cancel two terms, as usual. Accounting for the regularity of u_0 (see (2.26)–(2.27)) and rearranging, we obtain

$$\begin{aligned} &\|\partial_t u_\varepsilon(0)\|_*^2 + \tau_\varepsilon \|\partial_t u_\varepsilon(0)\|_{0,\Omega}^2 + \|\partial_t u_\varepsilon(0)\|_{0,\Gamma}^2 \\ &= - \int_\Omega \bigl(-\Delta u_0 + \beta_\varepsilon(u_0) + \pi(u_0) - f(0) \bigr) \, \partial_t u_\varepsilon(0) \\ &\quad - \int_\Gamma \bigl(-\nu \Delta_\Gamma u_0 + \beta_{\Gamma,\varepsilon}(u_0) + \pi_\Gamma(u_0) - f_\Gamma(0) \bigr) \, \partial_t u_\varepsilon(0). \end{aligned} \tag{5.4}$$

Now, we estimate the last term owing to (5.1) and (2.30) as follows

$$\begin{aligned} &- \int_\Gamma \bigl(-\nu \Delta_\Gamma u_0 + \beta_{\Gamma,\varepsilon}(u_0) + \pi_\Gamma(u_0) - f_\Gamma(0) \bigr) \, \partial_t u_\varepsilon(0) \\ &\le \frac{1}{2} \|\partial_t u_\varepsilon(0)\|_{0,\Gamma}^2 + \frac{1}{2} \|-\nu \Delta_\Gamma u_0 + \beta_{\Gamma,\varepsilon}(u_0) + \pi_\Gamma(u_0) - f_\Gamma(0)\|_{0,\Gamma}^2 \, . \\ &\le \frac{1}{2} \|\partial_t u_\varepsilon(0)\|_{0,\Gamma}^2 + c \bigl(\|\nu \Delta_\Gamma u_0|_\Gamma\|_{0,\Gamma}^2 + \|\beta_\Gamma^\circ(u_0)\|_{0,\Gamma}^2 + \|u_0\|_{0,\Gamma}^2 + \|f_\Gamma(0)\|_{0,\Gamma}^2 \bigr) \\ &\le \frac{1}{2} \|\partial_t u_\varepsilon(0)\|_{0,\Gamma}^2 + c. \end{aligned}$$

As far as the second to last term of (5.4) is concerned, we distinguish the cases $\tau > 0$ and $\tau = 0$. In the first one, we behave as for the above boundary term and easily obtain

$$\begin{aligned} &- \int_\Omega \bigl(-\Delta u_0 + \beta_\varepsilon(u_0) + \pi(u_0) - f(0) \bigr) \, \partial_t u_\varepsilon(0) \\ &\le \frac{\tau_\varepsilon}{2} \|\partial_t u_\varepsilon(0)\|_{0,\Omega}^2 + \frac{1}{2\tau} \|-\Delta u_0 + \beta_\varepsilon(u_0) + \pi(u_0) - f(0)\|_{0,\Omega}^2 \le \frac{\tau_\varepsilon}{2} \|\partial_t u_\varepsilon(0)\|_{0,\Omega}^2 + c. \end{aligned}$$

If instead $\tau = 0$, we account for (2.31) and estimate the same term this way

$$\begin{aligned} &- \int_\Omega \bigl(-\Delta u_0 + \beta_\varepsilon(u_0) + \pi(u_0) - f(0) \bigr) \, \partial_t u_\varepsilon(0) \\ &\le \frac{1}{2} \|\partial_t u_\varepsilon(0)\|_*^2 + c \, \|-\Delta u_0 + \beta_\varepsilon(u_0) + \pi(u_0) - f(0)\|_{1,\Omega}^2 \\ &\le \frac{1}{2} \|\partial_t u_\varepsilon(0)\|_*^2 + c \, \|-\Delta u_0 + \beta_\varepsilon(u_0) - f(0)\|_{1,\Omega}^2 + c \le \frac{1}{2} \|\partial_t u_\varepsilon(0)\|_*^2 + c. \end{aligned}$$

Therefore, the desired inequality obviously follows in any case. □

At this point, we can start estimating. We assume $\varepsilon \in (0,1)$, in principle, but we remark that some of the properties below may require $\varepsilon < \varepsilon_0$ for some $\varepsilon_0 \in (0,1)$ (e.g., according to Lemma 5.2). Moreover, δ is a positive parameter, as in the previous section.

First a priori estimate. Noting that $\partial_t u_\varepsilon$ has zero mean value by (4.14), we test (4.10) by $\mathcal{N}\partial_t u_\varepsilon$ and (4.11) by $-\partial_t u_\varepsilon$. By doing that, we account for (2.42) in order to cancel two terms in the sum at once. Then, by integrating over $(0,t)$, owing to (2.46), and adding the same quantity for convenience, we obtain (see also (4.35))

$$\begin{aligned}
&\int_0^t \|\partial_t u_\varepsilon(s)\|_*^2\,ds + \tau_\varepsilon \int_{Q_t} |\partial_t u_\varepsilon|^2 + \int_{\Sigma_t} |\partial_t u_\varepsilon|^2 \\
&\quad + \frac{1}{2}\int_\Omega |\nabla u_\varepsilon(t)|^2 + \frac{\nu}{2}\int_\Gamma |\nabla_\Gamma u_\varepsilon(t)|^2 + \int_\Omega \widehat{\beta}_\varepsilon(u_\varepsilon(t)) + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(u_\varepsilon(t)) \\
&\quad + \|u_\varepsilon(t)\|_{0,\Omega}^2 + \|u_\varepsilon(t)\|_{0,\Gamma}^2 \\
&= -\int_\Omega \widehat{\pi}(u_\varepsilon(t)) - \int_\Gamma \widehat{\pi_\Gamma}(u_\varepsilon(t)) + \int_{Q_t} f\,\partial_t u_\varepsilon + \int_{\Sigma_t} f_\Gamma\,\partial_t u_\varepsilon \\
&\quad + \frac{\nu}{2}\int_\Omega |\nabla u_0|^2 + \frac{1}{2}\int_\Gamma |\nabla_\Gamma u_0|^2 \\
&\quad + \int_\Omega \widehat{\beta}_\varepsilon(u_0) + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(u_0) + \int_\Omega \widehat{\pi}(u_0) + \int_\Gamma \widehat{\pi_\Gamma}(u_0) \\
&\quad + \|u_\varepsilon(t)\|_{0,\Omega}^2 + \|u_\varepsilon(t)\|_{0,\Gamma}^2 .
\end{aligned} \tag{5.5}$$

We can deal with the terms involving π, π_Γ, and f_Γ as we did in the previous section in order to estimate the corresponding ones by (4.36). On the contrary, we have to use a different argument for the integral containing f, since $\tau_\varepsilon = \varepsilon$ if $\tau = 0$. Accounting for (2.25), we integrate by parts and proceed as follows

$$\begin{aligned}
&\int_{Q_t} f\,\partial_t u_\varepsilon = \int_\Omega f(t)u_\varepsilon(t) - \int_\Omega f(0)u_0 - \int_{Q_t} \partial_t f\ u_\varepsilon \\
&\le \int_\Omega |u_\varepsilon(t)|^2 + \int_{Q_t} |u_\varepsilon|^2 + c.
\end{aligned} \tag{5.6}$$

For the same reason, no trouble arises from all the integrals involving u_0 but the ones related to $\widehat{\beta}_\varepsilon$ and $\widehat{\beta}_{\Gamma,\varepsilon}$. However, for such terms, we can apply (5.1) and (2.28) and derive that

$$\int_\Omega \widehat{\beta}_\varepsilon(u_0) + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(u_0) \le \int_\Omega \widehat{\beta}(u_0) + \int_\Gamma \bigl(\widehat{\beta}_\Gamma(u_0) + (2C_\Gamma/\alpha)|u_0|\bigr) = c.$$

Finally, noting that the last integral of (5.6) will be controlled by the second to last term on the left-hand side of (5.5) via Gronwall's lemma, we just need to estimate the two last norms of (5.5). To this aim, we apply (2.49)–(2.50) and have

$$\begin{aligned}
&\|u_\varepsilon(t)\|_{0,\Omega}^2 + \|u_\varepsilon(t)\|_{0,\Gamma}^2 \\
&\le \|u_0\|_{0,\Omega}^2 + \delta\int_0^t \|\partial_t u_\varepsilon(s)\|_*^2\,ds + c_\delta \int_0^t \|u_\varepsilon(s)\|_{1,\Omega}^2\,ds \\
&\quad + \|u_0\|_{0,\Gamma}^2 + \delta\int_0^t \|\partial_t u_\varepsilon(s)\|_{0,\Gamma}^2\,ds + c_\delta \int_0^t \|u_\varepsilon(s)\|_{0,\Gamma}^2\,ds.
\end{aligned}$$

At this point, we choose δ small enough, apply the Gronwall lemma, and conclude that

$$\begin{aligned}
&\|u_\varepsilon\|_{L^\infty(0,T;V)\cap H^1(0,T;V^*)} + \|u_\varepsilon\|_{L^\infty(0,T;V_\Gamma)\cap H^1(0,T;H_\Gamma)} + \tau_\varepsilon^{1/2}\|\partial_t u_\varepsilon\|_{L^2(0,T;H)} \\
&\quad + \|\widehat{\beta}_\varepsilon(u_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} + \|\widehat{\beta}_{\Gamma,\varepsilon}(u_\varepsilon)\|_{L^\infty(0,T;L^1(\Gamma))} \le c.
\end{aligned} \tag{5.7}$$

Second a priori estimate (partial). We have to find a bound for w_ε in $L^2(0,T;V)$. We set $w_{\varepsilon,\Omega} := (w_\varepsilon)_\Omega$ and test (4.10) by $w_\varepsilon(t) - w_{\varepsilon,\Omega}(t)$. We obtain

$$\begin{aligned}&\|\nabla(w_\varepsilon(t) - w_{\varepsilon,\Omega}(t))\|_{0,\Omega}^2 = -\langle \partial_t u_\varepsilon(t), w_\varepsilon(t) - w_{\varepsilon,\Omega}(t)\rangle \\ &\leq \delta \|w_\varepsilon(t) - w_{\varepsilon,\Omega}(t)\|_{1,\Omega}^2 + c_\delta \, \|\partial_t u_\varepsilon(t)\|_* \quad \text{for a.a. } t \in (0,T).\end{aligned}$$

Therefore, accounting for the Poincaré inequality (2.40) and choosing δ small enough, we conclude that

$$\|w_\varepsilon(t) - w_{\varepsilon,\Omega}(t)\|_{1,\Omega} \leq c\|\partial_t u_\varepsilon(t)\|_* \quad \text{for a.a. } t \in (0,T) \tag{5.8}$$

and deduce that

$$\|w_\varepsilon - w_{\varepsilon,\Omega}\|_{L^2(0,T;V)} \leq c \tag{5.9}$$

by accounting for (5.7). As our aim is to get rid of $w_{\varepsilon,\Omega}$, we prepare a relationship (to be used later on) between some mean values. We set for convenience

$$\xi_\varepsilon := \beta_\varepsilon(u_\varepsilon), \quad \xi_{\Gamma,\varepsilon} := \beta_{\Gamma,\varepsilon}(u_\varepsilon), \quad \xi_{\varepsilon,\Omega} := (\xi_\varepsilon)_\Omega, \quad \text{and} \quad \xi_{\Gamma,\varepsilon,\Gamma} := (\xi_{\Gamma,\varepsilon})_\Gamma \tag{5.10}$$

where the notation $z_\Gamma := |\Gamma|^{-1} \int_\Gamma z$ is used. We test (4.11) by the constant $1/|\Omega|$ and recall that $(\partial_t u_\varepsilon)_\Omega = 0$. Hence, we obtain

$$w_{\varepsilon,\Omega}(t) = \xi_{\varepsilon,\Omega}(t) + \frac{|\Gamma|}{|\Omega|} \, \xi_{\Gamma,\varepsilon,\Gamma}(t) + F_\varepsilon(t) \quad \text{for a.a. } t \in (0,T) \tag{5.11}$$

where we have set

$$F_\varepsilon(t) = \frac{1}{|\Omega|} \int_\Omega \bigl(\pi(u_\varepsilon(t)) - f(t)\bigr) + \frac{1}{|\Omega|} \int_\Gamma \bigl(\partial_t u_\varepsilon(t) + \pi_\Gamma(u_\varepsilon(t)) - f_\Gamma(t)\bigr). \tag{5.12}$$

Consequences under the assumptions of Theorem 2.9. First of all, it is trivial to deduce estimates for $\pi(u_\varepsilon)$ and $\pi_\Gamma(u_\varepsilon)$ from (5.7), just by Lipschitz continuity. Moreover, from the growth conditions of the assumptions, we easily derive estimates even for the main nonlinearities. Here, we think of $\beta_{\Gamma,\varepsilon}$ defined by (4.4) (see Remark 4.1). If $d = 1$, then u_ε is bounded in $L^\infty(Q)$ by some constant M since $V \subset L^\infty(\Omega)$ and (5.7) holds. By (2.37), we derive that

$$|\beta_\varepsilon(u_\varepsilon)| \leq |\beta^\circ(u_\varepsilon)| \leq \sup_{|r| \leq M} |\beta^\circ(r)| = c.$$

Moreover, a (trivial) similar argument holds for $\beta_{\Gamma,\varepsilon}(u_\varepsilon)$. As far as the case $d > 1$ is concerned, note that both (2.38) and (2.39) imply corresponding global inequalities since β° and β_Γ° are monotone functions. Now, if $d = 2$, then $V \subset L^p(\Omega)$ for every $p < +\infty$, the embedding being continuous. Moreover, V_Γ is continuously embedded either in $L^\infty(\Gamma)$ or in $L^q(\Gamma)$ for every $q < +\infty$ according to whether $\nu > 0$ or $\nu = 0$. Therefore, (2.38) and (5.7) imply that

$$\|\beta_\varepsilon(u_\varepsilon)\|_{L^\infty(0,T;H)} + \|\beta_{\Gamma,\varepsilon}(u_\varepsilon)\|_{L^\infty(0,T;H_\Gamma)} \leq c. \tag{5.13}$$

Finally, if $d = 3$, V is continuously embedded in $L^6(\Omega)$ and V_Γ is continuously embedded either in $L^q(\Gamma)$ for every $q < +\infty$ or in $L^4(\Gamma)$ according to whether $\nu > 0$ or $\nu = 0$. Thus, (5.13) holds also in this case, by (2.39). Next, we observe that (5.13) obviously implies that the mean values $\xi_{\varepsilon,\Omega}$ and $\xi_{\Gamma,\varepsilon,\Gamma}$ are bounded in $L^\infty(0,T)$. As the function F_ε defined by (5.12) is bounded in $L^2(0,T)$ thanks to (5.7), we deduce that the same holds for $w_{\varepsilon,\Omega}$ in view of (5.11). Therefore, we conclude that

$$\|w_\varepsilon\|_{L^2(0,T;V)} \leq c \tag{5.14}$$

by accounting for (5.9).

Conclusion for Theorem 2.9. We still assume $\beta_{\Gamma,\varepsilon}$ to be defined by (4.4), so that both $\beta_\varepsilon(u_\varepsilon)$ and $\beta_{\Gamma,\varepsilon}(u_\varepsilon)$ actually are estimated by (5.13). However, in order to infer existence for problem (2.19)–(2.20), estimates in $L^2(0,T;H)$ and in $L^2(0,T;H_\Gamma)$, respectively, are sufficient, and we just owe to such a weaker information. By using standard compactness results, we see that limit functions exist such that

$$u_\varepsilon \to u \quad \text{weakly star in } L^\infty(0,T;V)\cap H^1(0,T;V^*) \tag{5.15}$$
$$u_\varepsilon|_\Gamma \to u|_\Gamma \quad \text{weakly star in } L^\infty(0,T;V_\Gamma)\cap H^1(0,T;H_\Gamma) \tag{5.16}$$
$$\tau_\varepsilon\partial_t u_\varepsilon \to \tau\partial_t u \quad \text{weakly in } L^2(0,T;H) \tag{5.17}$$
$$w_\varepsilon \to w \quad \text{weakly in } L^2(0,T;V) \tag{5.18}$$
$$\beta_\varepsilon(u_\varepsilon) \to \xi \quad \text{weakly in } L^2(0,T;H) \tag{5.19}$$
$$\beta_{\Gamma,\varepsilon}(u_\varepsilon) \to \xi_\Gamma \quad \text{weakly in } L^2(0,T;H_\Gamma) \tag{5.20}$$
$$\pi(u_\varepsilon) \to \zeta \quad \text{weakly star in } L^\infty(0,T;H) \tag{5.21}$$
$$\pi_\Gamma(u_\varepsilon|_\Gamma) \to \zeta_\Gamma \quad \text{weakly star in } L^\infty(0,T;H_\Gamma) \tag{5.22}$$

at least for a subsequence. Now, we prove that (u,w,ξ,ξ_Γ) is a solution to our problem. By (5.15)–(5.20), we see that the regularity requirements contained in (2.13)–(2.17) are fulfilled and that the following variational equations

$$\int_0^T \langle \partial_t u(t), z(t)\rangle\, dt + \int_Q \nabla w\cdot\nabla z = 0$$
$$\int_Q wz = \int_Q \tau\partial_t u\, z + \int_\Sigma \partial_t u\, z + \int_Q \nabla u\cdot\nabla z + \int_\Sigma \nabla_\Gamma u\cdot\nabla_\Gamma z$$
$$+ \int_Q \bigl(\xi+\zeta-f\bigr)\, z + \int_\Sigma \bigl(\xi_\Gamma+\zeta_\Gamma-f_\Gamma\bigr)\, z$$

hold for every $z\in L^2(0,T;V)$ and every $z\in L^2(0,T;\mathcal{V})$, respectively. Moreover, we recall that the embeddings $V\subset H$ and $V_\Gamma\subset H_\Gamma$ are compact. Hence, we can apply [36, Sect. 8, Cor. 4] and derive that

$$u_\varepsilon \to u \quad \text{strongly in } C^0([0,T];H) \quad \text{and} \quad u_\varepsilon|_\Gamma \to u|_\Gamma \quad \text{strongly in } C^0([0,T];H_\Gamma) \tag{5.23}$$

In particular, (2.18) holds as well and $\pi(u_\varepsilon)$ and $\pi_\Gamma(u_\varepsilon|_\Gamma)$ converge to $\pi(u)$ and to $\pi_\Gamma(u|_\Gamma)$ strongly in $C^0([0,T];H)$ and in $C^0([0,T];H_\Gamma)$, respectively, just by Lipschitz continuity, whence $\zeta=\pi(u)$ and $\zeta_\Gamma=\pi_\Gamma(u|_\Gamma)$. Finally, the convergence (5.19)–(5.20) and (5.23) and the maximal monotonicity of β and β_Γ allow us to conclude that $\xi\in\beta(u)$ a.e. in Q and that $\xi_\Gamma\in\beta_\Gamma(u|_\Gamma)$ a.e. on Σ (see, e.g., [3, Prop. 2.5, p. 27] for a similar result). Therefore, we see that (u,w,ξ,ξ_Γ) satisfies both the remaining conditions (2.16)–(2.17) and (2.21)–(2.22). As the latter are equivalent to (2.19)–(2.20), the proof is complete.

So, we continue the proof of Theorem 2.8.

Improvement of the first a priori estimate. Recalling Proposition 4.8, we can differentiate equations (4.10) and (4.11) with respect to time and test the equalities we obtain by $\mathcal{N}\partial_t u_\varepsilon$ and $-\partial_t u_\varepsilon$, respectively. Then, we integrate over $(0,t)$ and take the sum. As before, we use the properties of $\mathcal{N}$. We get

$$\frac{1}{2}\|\partial_t u_\varepsilon(t)\|_*^2 + \frac{\tau_\varepsilon}{2}\int_\Omega |\partial_t u_\varepsilon(t)|^2 + \frac{1}{2}\int_\Gamma |\partial_t u_\varepsilon(t)|^2$$
$$+ \int_{Q_t} |\partial_t \nabla u_\varepsilon|^2 + \nu \int_{\Sigma_t} |\partial_t \nabla_\Gamma u_\varepsilon|^2 + \int_{Q_t} \beta'_\varepsilon(u_\varepsilon)\,|\partial_t u_\varepsilon|^2 + \int_{\Sigma_t} \beta'_{\Gamma,\varepsilon}(u_\varepsilon)\,|\partial_t u_\varepsilon|^2$$
$$= \frac{1}{2}\|\partial_t u_\varepsilon(0)\|_*^2 + \frac{\tau_\varepsilon}{2}\int_\Omega |\partial_t u_\varepsilon(0)|^2 + \frac{1}{2}\int_\Gamma |\partial_t u_\varepsilon(0)|^2$$
$$- \int_{Q_t} \pi'(u_\varepsilon)\,|\partial_t u_\varepsilon|^2 - \int_{\Sigma_t} \pi'_\Gamma(u_\varepsilon)\,|\partial_t u_\varepsilon|^2 + \int_{Q_t} \partial_t f\,\partial_t u_\varepsilon + \int_{\Sigma_t} \partial_t f_\Gamma\,\partial_t u_\varepsilon\,.$$

All the terms on the left-hand side are nonnegative. As far as those on the right-hand side are concerned, the terms involving $\partial_t u_\varepsilon(0)$ are estimated by Lemma 5.3 and the integrals over Σ_t are estimated by (5.7) since π'_Γ is bounded and $\partial_t f_\Gamma \in L^2(0,T;H_\Gamma)$. Hence, we just have to deal with the integrals over Q_t. We estimate the first one by recalling that π' is bounded and applying inequality (2.51) as follows

$$-\int_{Q_t} \pi'(u_\varepsilon)\,|\partial_t u_\varepsilon|^2 \le \delta \int_{Q_t} |\partial_t \nabla u_\varepsilon|^2 + c_\delta \int_0^t \|\partial_t u_\varepsilon(s)\|_*^2\,ds.$$

Finally, we treat the second one by recalling that $(\partial_t u_\varepsilon)_\Omega = 0$ and thus using the Poincaré inequality (2.40) this way

$$\int_{Q_t} \partial_t f\,\partial_t u_\varepsilon \le \delta \int_{Q_t} |\partial_t u_\varepsilon|^2 + c_\delta \le \delta M_\Omega \int_{Q_t} |\partial_t \nabla u_\varepsilon|^2 + c_\delta.$$

Therefore, by choosing δ small enough and applying the Gronwall lemma, we conclude that

$$\|\partial_t u_\varepsilon\|_{L^\infty(0,T;V^*)\cap L^2(0,T;V)} + \|\partial_t u_\varepsilon\|_{L^\infty(0,T;H_\Gamma)\cap L^2(0,T;V_\Gamma)} + \tau_\varepsilon^{1/2}\|\partial_t u_\varepsilon\|_{L^\infty(0,T;H)} \le c \qquad (5.24)$$

where the trace inequality (2.10) is used in the case $\nu = 0$.

Third a priori estimate. We owe to an argument devised in [27, Appendix, Prop. A.1] and based on the easy inequalities

$$\beta_\varepsilon(r)\,(r - m_0) \ge \delta_0|\beta_\varepsilon(r)| - c \quad \text{and} \quad \beta_{\Gamma,\varepsilon}(r)\,(r - m_0) \ge \delta_0|\beta_{\Gamma,\varepsilon}(r)| - c \qquad (5.25)$$

where $m_0 := (u_0)_\Omega$ and δ_0 and $c > 0$ are some positive constants. Inequalities (5.25) hold for every $r \in \mathbb{R}$ and $\varepsilon > 0$, and we prove them by accounting for (2.3) and (2.29). We choose $m_\pm \in D(\beta)$ such that $m_- \le 0 \le m_+$ and $m_- < m_0 < m_+$ and define the positive number $\delta_0 := \min\{m_0 - m_-, m_+ - m_0\}$. Assume now $r \ge m_+$. Then, we have $\beta_\varepsilon(r) \ge 0$ and $r - m_0 \ge \delta_0$, whence $\beta_\varepsilon(r)\,(r - m_0) \ge \delta_0\beta_\varepsilon(r)$ and the first of (5.25) follows with any $c \ge 0$. The argument for $r \le m_-$ is similar. Next, by assuming $m_- \le r \le m_+$, we obtain

$$\delta_0|\beta_\varepsilon(r)| - \beta_\varepsilon(r)\,(r - m_0) \le (\delta_0 + m_+ - m_-)|\beta_\varepsilon(r)| \le c \sup_{m_- \le s \le m_+} |\beta^\circ(s)| = c$$

thanks to the first of (5.1). The second of (5.25) can be verified in the same way owing to the second of (5.1). Once (5.25) are established, we prove a bound for the mean values $\xi_{\varepsilon,\Omega}$ and $\xi_{\Gamma,\varepsilon,\Gamma}$ (see (5.10)). We recall that $(u_\varepsilon)_\Omega = m_0$ for all times. Therefore, we can test (4.10) and (4.11) by $\mathcal{N}(u_\varepsilon(t)-m_0)$ and $-(u_\varepsilon(t)-m_0)$, respectively. After summing the equalities we get to each other, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla u_\varepsilon(t)|^2 + \nu \int_\Gamma |\nabla_\Gamma u_\varepsilon(t)|^2 + \int_\Omega \xi_\varepsilon(t)\big(u_\varepsilon(t)-m_0\big) + \int_\Gamma \xi_{\Gamma,\varepsilon}(t)\big(u_\varepsilon(t)-m_0\big) \\
&= \int_\Omega \big(f(t)-\pi(u_\varepsilon(t))\big)\big(u_\varepsilon(t)-m_0)\big) + \int_\Gamma \big(f_\Gamma(t)-\pi_\Gamma(u_\varepsilon(t))\big)\big(u_\varepsilon(t)-m_0)\big) \\
&\quad - \langle \partial_t u_\varepsilon(t), \mathcal{N}(u_\varepsilon(t)-m_0)\rangle - \tau_\varepsilon \int_\Omega \partial_t u_\varepsilon(t)\,\big(u_\varepsilon(t)-m_0\big) - \int_{\Sigma_t} \partial_t u_\varepsilon(t)\,\big(u_\varepsilon(t)-m_0\big).
\end{aligned}$$

The term involving ξ_ε is estimated from below by using (5.25) as follows

$$\int_\Omega \xi_\varepsilon(t)\big(u_\varepsilon(t)-m_0\big) \geq \delta_0 \int_\Omega |\xi_\varepsilon(t)| - c$$

and the next one is treated in the same way. Finally, the whole right-hand side is bounded by a constant, due to (5.7) and (5.24). In particular, we deduce that

$$\|\xi_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} + \|\xi_{\Gamma,\varepsilon}\|_{L^\infty(0,T;L^1(\Gamma))} \leq c$$

whence immediately

$$\|\xi_{\varepsilon,\Omega}\|_{L^\infty(0,T)} + \|\xi_{\Gamma,\varepsilon,\Gamma}\|_{L^\infty(0,T)} \leq c. \tag{5.26}$$

Improvement of the second a priori estimate. We just recall the inequality (5.8), the relationship (5.11), and the definition (5.12) of F_ε, and observe that (5.24) implies that

$$\|w_\varepsilon - w_{\varepsilon,\Omega}\|_{L^\infty(0,T;V)} \leq c \quad \text{and} \quad \|F_\varepsilon\|_{L^\infty(0,T)} \leq c. \tag{5.27}$$

On the other hand, (5.26) has been established. We deduce that $w_{\varepsilon,\Omega}$ is bounded in $L^\infty(0,T)$ as well. Therefore, we see that the first (5.27) implies that

$$\|w_\varepsilon\|_{L^\infty(0,T;V)} \leq c \tag{5.28}$$

which improves (5.14).

Fourth a priori estimate. We simply test (4.11) by ξ_ε and integrate over $(0,T)$. Note that we are allowed to do that since $u_\varepsilon \in L^2(0,T;\mathcal{V})$ and β_ε is Lipschitz continuous, whence $\beta_\varepsilon(u_\varepsilon) \in L^2(0,T;\mathcal{V})$. After adding the same integral to both sides for convenience and rearranging, we obtain

$$\begin{aligned}
&\tau_\varepsilon \int_\Omega \widehat{\beta}_\varepsilon(u_\varepsilon(T)) + \int_\Gamma \widehat{\beta}_\varepsilon(u_\varepsilon(T)) + \int_Q \beta_\varepsilon'(u_\varepsilon)|\nabla u_\varepsilon|^2 + \nu \int_\Sigma {\beta_\varepsilon}'(u_\varepsilon)|\nabla_\Gamma u_\varepsilon|^2 \\
&\quad + \int_Q |\xi_\varepsilon|^2 + \delta \int_\Sigma |\xi_{\Gamma,\varepsilon}|^2 \\
&= \tau_\varepsilon \int_\Omega \widehat{\beta}_\varepsilon(u_0) + \int_\Gamma \widehat{\beta}_\varepsilon(u_0) + \int_Q \big(f - \pi(u_\varepsilon) + w_\varepsilon\big)\,\xi_\varepsilon \\
&\quad + \int_\Sigma \Big(\delta\,|\xi_{\Gamma,\varepsilon}|^2 + (\sigma-1)\xi_\varepsilon\,\xi_{\Gamma,\varepsilon}\Big) + \int_\Sigma \big(-\sigma\,\xi_{\Gamma,\varepsilon} - \pi_\Gamma(u_\varepsilon) + f_\Gamma\big)\,\xi_\varepsilon\,.
\end{aligned}$$

All the integrals on the left-hand side are nonnegative and the first three terms on the right-hand side are easily treated owing to (5.1), (2.28), and the previous estimates. Now, we show that the second to last integral is uniformly bounded by using Lemma 5.1 provided that δ is small enough. By recalling that u_ε, ξ_ε, and $\xi_{\Gamma,\varepsilon}$ have the same sign and that $\sigma < 1$, we easily have

$$\begin{aligned}
&\delta\,|\xi_{\Gamma,\varepsilon}|^2 + (\sigma-1)\xi_\varepsilon\,\xi_{\Gamma,\varepsilon} = \delta\,|\xi_{\Gamma,\varepsilon}|^2 - (1-\sigma)|\xi_\varepsilon||\xi_{\Gamma,\varepsilon}| \\
&\leq \bigl(\delta - \alpha(1-\sigma)\bigr)|\xi_{\Gamma,\varepsilon}|^2 + 2C_\Gamma(1-\sigma)|\xi_{\Gamma,\varepsilon}| \\
&\leq \bigl(\delta - (\alpha/2)(1-\sigma)\bigr)|\xi_{\Gamma,\varepsilon}|^2 + 2(1-\sigma)\,C_\Gamma^2/\alpha \leq 2(1-\sigma)\,C_\Gamma^2/\alpha \quad \text{a.e. on } \Sigma
\end{aligned}$$

whenever $\delta < (\alpha/2)(1-\sigma)$. Finally, the last integral is bounded too, as we show at once by accounting for Lemma 5.2 and the obvious inequality

$$|\pi_\Gamma(u_\varepsilon) - f_\Gamma| \leq (\sup|\pi_\Gamma'|)\,|u_\varepsilon| + |\pi_\Gamma(0)| + \|f_\Gamma\|_{L^\infty(\Gamma)} \quad \text{a.e. on } \Sigma.$$

Consider first the subset σ_ε^+ of Σ where $u_\varepsilon \geq r_+^*$. Then, $\xi_\varepsilon \geq 0$ and $\xi_{\Gamma,\varepsilon} \geq 0$ there. Moreover, (5.3) holds, whence $-\sigma\,\xi_{\Gamma,\varepsilon} + \pi_\Gamma(u_\varepsilon) - f_\Gamma \leq 0$ a.e. on Σ_ε^+. Hence, the corresponding contribution to the integral is nonpositive. Analogously, the same holds for the subset where $u_\varepsilon \leq r_-^*$. Therefore, if Σ_ε^* denotes the subset where $r_-^* \leq u_\varepsilon \leq r_+^*$, we have

$$\begin{aligned}
&\int_\Sigma \bigl(-\sigma\,\xi_{\Gamma,\varepsilon} + \pi_\Gamma(u_\varepsilon) - f_\Gamma\bigr)\,\xi_\varepsilon \leq \int_{\Sigma_\varepsilon^*} |-\sigma\,\xi_{\Gamma,\varepsilon} + \pi_\Gamma(u_\varepsilon) - f_\Gamma|\,|\xi_\varepsilon| \\
&\leq \Bigl(\sup_{r_-^* \leq r \leq r_+^*} |\beta_{\Gamma,\varepsilon}(r)| + c\Bigr) \sup_{r_-^* \leq r \leq r_+^*} |\beta_\varepsilon(r)| \leq c
\end{aligned}$$

by (5.1), since $r_\pm^* \in D(\beta) \subseteq D(\beta_\Gamma)$. Therefore, we deduce the basic estimate

$$\|\xi_\varepsilon\|_{L^2(0,T;H)} + \|\xi_{\Gamma,\varepsilon}\|_{L^2(0,T;H_\Gamma)} \leq c. \tag{5.29}$$

Conclusion for Theorem 2.8. Thanks to the estimates we have proved, we can easily infer existence for problem (2.19)–(2.20) also in this case. Indeed, using standard compactness results, we see that limit functions exist such that (5.15)–(5.22) hold at least for a subsequence. Actually, some convergence is related to some stronger topology. In order to prove that (u, w, ξ, ξ_Γ) is a solution, it suffices to argue as in the conclusion of the proof of Theorem 2.9, the only difference being the identification of ξ_Γ, since $\beta_{\Gamma,\varepsilon}$ differs from the Yosida regularization of β_Γ in the present case. It is clear that we can conclude provided that we find $u_\varepsilon^* \in L^2(0,T;V)$ such that the following convergence holds

$$u_\varepsilon^*|_\Gamma \to u|_\Gamma \qquad \text{strongly in } L^2(0,T;H_\Gamma) \tag{5.30}$$

$$\xi_{\Gamma,\varepsilon}^* := \beta_{\Gamma,\alpha\varepsilon}^Y(u_\varepsilon^*|_\Gamma) \to \xi_\Gamma \qquad \text{weakly in } L^2(0,T;H_\Gamma). \tag{5.31}$$

To this aim, we define

$$u_\varepsilon^* := u_\varepsilon + \min\{\varepsilon C_\Gamma, \max\{u_\varepsilon, -\varepsilon C_\Gamma\}\}.$$

Then, (5.30)–(5.31) immediately follow if we prove that

$$u_\varepsilon^* - u_\varepsilon \to 0 \quad \text{and} \quad \xi_{\Gamma,\varepsilon}^* - \xi_{\Gamma,\varepsilon} \to 0 \quad \text{uniformly in } Q \text{ and on } \Sigma, \text{ respectively} \tag{5.32}$$

since (5.20) and (5.23) hold also in the present case. Now, we have $|u_\varepsilon^* - u_\varepsilon| \leq \varepsilon C_\Gamma$ in Q, whence the first of (5.32), trivially. On the other hand, the difference $\xi_{\Gamma,\varepsilon}^* - \xi_{\Gamma,\varepsilon}$ vanishes at points of Σ where $|u_\varepsilon| > \varepsilon C_\Gamma$, by definition of $\beta_{\Gamma,\varepsilon}$ (see (4.3)), since $u_\varepsilon^* = u_\varepsilon + \varepsilon C_\Gamma \operatorname{sign} u_\varepsilon$ there. Next, assume u_ε to be evaluated at points of Σ where $0 \le u_\varepsilon \le \varepsilon C_\Gamma$. Then, both $\xi^*_{\Gamma,\varepsilon}$ and $\xi_{\Gamma,\varepsilon}$ are nonnegative and satisfy

$$\xi^*_{\Gamma,\varepsilon} \le \beta^Y_{\Gamma,\alpha\varepsilon}(2\varepsilon C_\Gamma) \le \beta^\circ_\Gamma(2\varepsilon C_\Gamma) \quad \text{and} \quad \xi_{\Gamma,\varepsilon} \le \beta_{\Gamma,\varepsilon}(2\varepsilon C_\Gamma) \le \beta^Y_{\Gamma,\alpha\varepsilon}(3\varepsilon C_\Gamma) \le \beta^\circ_\Gamma(3\varepsilon C_\Gamma).$$

Hence, by arguing similarly at points where $-\varepsilon C_\Gamma \le u_\varepsilon \le 0$, we conclude that

$$\sup_\Sigma |\xi^*_{\Gamma,\varepsilon} - \xi_{\Gamma,\varepsilon}| \le 2\max\{\beta^\circ_\Gamma(3\varepsilon C_\Gamma), |\beta^\circ_\Gamma(-3\varepsilon C_\Gamma)|\}.$$

Therefore, the second of (5.32) follows from the second of (2.32), which implies continuity for β°_Γ at 0 indeed, and the proof of Theorem 2.8 is complete.

Remark 5.4. Of course, the above proof shows more regularity for the solution, according to the estimates we have derived. For instance, under the assumptions of Theorem 2.8, we have $u \in W^{1,\infty}(0,T;V^*) \cap H^1(0,T;V)$ in the non-viscous case and something better in the viscous one by (5.24). Moreover, something more than required in (2.13)–(2.17) also holds under the assumptions of Theorem 2.9. Here, we point out that u automatically enjoys further regularity properties that follow directly from (2.13)–(2.17) and the variational formulation of problem (2.19)–(2.20). From (2.22) we derive both (1.7) and (1.8). Precisely, such equalities hold a.e. in Q and a.e. on Σ, respectively (i.e., all of their terms are functions rather than functionals). First of all, (1.7) clearly holds in the sense of distributions and a comparison in it immediately yields that $\Delta u \in L^2(Q)$. In particular, (1.7) holds a.e. in Q. Moreover, by using $u \in L^2(0,T;V)$ as well, we deduce that $(\partial_n u)|_\Gamma$ makes sense and belongs to $L^2(0,T;H^{-1/2}(\Gamma))$ and that (1.8) holds in a generalized sense (see (4.33) for a similar situation). Therefore, if $\nu = 0$, just by comparison in (1.8), we infer that $(\partial_n u)|_\Gamma \in L^2(\Sigma)$ and that (1.8) itself holds a.e. on Σ. Assume now $\nu > 0$ (whence $d = 2,3$). We read both (1.7) and (1.8) as nice elliptic equations, namely, $-\Delta u = g$ and $-\Delta_\Gamma u|_\Gamma + u|_\Gamma = g_\Gamma$ (with an obvious choice of g and g_Γ, where t is just seen as a parameter), and use a bootstrap argument. First of all, we have $u|_\Gamma \in L^2(0,T;H^1(\Gamma))$ and $\Delta u \in L^2(Q)$. Thus, we deduce that $u \in L^2(0,T;H^{3/2}(\Omega))$ (by the elliptic theory in Ω) and that $(\partial_n u)|_\Gamma$ belongs to $L^2(0,T;H^{-1/4}(\Gamma))$ (actually, to $L^2(0,T;H^s(\Gamma))$ for every $s < 0$). Hence, $u|_\Gamma$ is the variational solution to the above equation on Γ with $g_\Gamma \in L^2(0,T;H^{-1/4}(\Gamma))$. By applying the boundary version of [24, Thm. 7.5, p. 204], we derive that $u|_\Gamma \in L^2(0,T;H^{2-1/4}(\Gamma)) \subset L^2(0,T;H^{3/2}(\Gamma))$, whence also $u \in L^2(0,T;H^2(\Omega))$ by the elliptic theory in Ω once more. In particular, $u \in L^2(0,T;L^\infty(\Omega))$ since $d \le 3$. Furthermore, we deduce that $(\partial_n u)|_\Gamma \in L^2(0,T;H^{1/2}(\Gamma)) \subset L^2(\Sigma)$. Therefore, we have that $g_\Gamma \in L^2(\Sigma)$ as well, whence $\Delta_\Gamma u|_\Gamma \in L^2(\Sigma)$ by comparison, and conclude that (1.8) holds a.e. on Σ.

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