EXISTENCE FOR A DOUBLY NONLINEAR VOLTERRA EQUATION

GIANNI GILARDI* AND ULISSE STEFANELLI†

Abstract. We consider a doubly nonlinear Volterra equation involving a non-smooth kernel and two possibly degenerate monotone operators. By exploiting an implicit time-discretization procedure, we obtain the existence of a global strong solution and extend to the non-local in time situation some former results by Colli [12].

Key words. doubly nonlinear, Volterra equation, discretization, existence

AMS subject classifications. 45K05, 35K55

1. Introduction. Let V and H be reflexive Banach spaces with $V \subset H$ densely and compactly. The present analysis is concerned with the doubly nonlinear initial value problem

$$A(u') + B(u) + k * B(u) \ni f \quad \text{and} \quad u(0) = u^0. \tag{1.1}$$

Here, $A : H \to H^*$ (dual) and $B : V \to V^*$ are maximal monotone (possibly multivalued) operators, and the non-smooth kernel $k \in BV(0,T)$, and the data $u^0 \in V$ and $f : [0,T] \to V^*$ are given.

Existence results for (1.1) with $k = 0$ as well as some motivation of the applicative interest of doubly nonlinear relations of the form of (1.1) have been discussed by Colli & Visintin [15] in Hilbert spaces. Later on, these results have been extended to the reflexive Banach setting by Colli [12]. In particular, among the various different sets of assumptions considered in [12], suitable solutions to (1.1) for $k = 0$ are proved to exist if either

i) A is non-degenerate and bounded
and B is cyclically monotone and coercive [12, Thm. 1] or

ii) A is cyclically monotone and bounded
and B is Lipschitz continuous and strongly monotone [12, Thm. 2],

(see below for the definitions and details).

The aim of the present paper is to extend the latter existence results to the more general non-local in time case $k \neq 0$. In particular, we address situation i) in Theorem 2.1 and ii) in Theorem 2.3 below. This paper brings, to our knowledge, the first contribution in the direction of an existence theory for (1.1).

Our existence argument relies on an implicit time-discretization procedure. Letting $\tau := T/N$ ($N \in \mathbb{N}$) denote the time-step and $\{k_i\}_{i=1}^N \in \mathbb{R}^N$, and $\{B(u_i)\}_{i=1}^N \in V^*$ be approximations of k and $B(u)$, respectively, we replace $k * B(u)$ by the quantities

$$\tau \sum_{j=1}^{i} k_{i-j+1} B(u_j) \qquad i = 1, \ldots, N.$$

This choice has been firstly discussed by the second author in [26] and turns out to be especially well-suited for the aim of studying Volterra equations of convolution

*Dipartimento di Matematica "F. Casorati", Università di Pavia, via Ferrata 1, I-27100 Pavia, Italy (gianni.gilardi@unipv.it)

†Istituto di Matematica Applicata e Tecnologie Informatiche – CNR, via Ferrata 1, I-27100 Pavia, Italy (ulisse.stefanelli@imati.cnr.it)

type (see also [18]). In particular, it entails a useful discrete Young inequality (see Proposition 3.2 below) and the conditional stability of the time-discretization scheme. Moreover, the latter discrete convolution converges to its continuous counterpart as the time-step goes to 0 (Proposition 3.3) and a discrete resolvent theory is available (Proposition 3.5).

A remarkable fact is that our time-discretization technique allows the treatment of non-smooth convolution kernels. In particular, we ask k to be of bounded variation and even allow $k \in L^\infty(0,T)$ in case i) (see Remark 4.4). We shall mention that memory kernels are generally assumed to be non-increasing and non-negative. Hence, the latter turn out to be fairly natural regularity requirements.

We shall mention that existence results for different doubly nonlinear Volterra equations have already been obtained. Let us remark in particular that equation

$$(A(u))' + B(u) + k * B(u) \ni f \tag{1.2}$$

has recently attracted a good deal of attention. Of course the local-in-time case $k = 0$ has been deeply studied and we shall refer to Grange & Mignot [19], Barbu [9], DiBenedetto & Showalter [16], Alt & Luckhaus [6], and Bernis [11], Hokkanen [20, 21, 22], Aizicovici & Hokkanen [4, 5], Maitre & Witomski [24], and Gajewski & Skrypnik [17], among many others. The non-local case $k \neq 0$ and has been considered under various simplifications (linearized operators, smooth kernels, etc.) by Aizicovici, Colli, & Grasselli [2, 3], Barbu, Colli, Gilardi, & Grasselli [10], Colli & Grasselli [13, 14], Stefanelli [26, 27, 28], and Hokkanen [21]. Finally, Gilardi & Stefanelli [18] investigated (1.2) in great generality by means of the same discretization tools here exploited.

1.1. An integro-partial differential equation. Let us present here an example of a nonlinear integro-partial differential problem whose variational formulation leads to (1.1). To this aim, we consider the initial and boundary value problem

$$a(\partial_t u) - \operatorname{div}\big(b(\nabla u) - k * b(\nabla u)\big) = h \quad \text{in } \Omega \times (0,T), \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ is a suitably smooth and bounded open set. Here, the maximal monotone maps $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ and the datum $h : \Omega \times (0,T) \to \mathbb{R}$ are given.

We complement the latter equation by prescribing initial and mixed Dirichlet-Neumann boundary conditions (other choices are of course admissible, see below). In particular, we split $\partial\Omega$ into two parts, Γ_D and Γ_N, and ask for

$$u(0) = u^0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma_D, \quad \text{and} \quad b(\nabla u)\cdot\nu = g \quad \text{on } \Gamma_N \times (0,T), \tag{1.4}$$

almost everywhere in the respective domains, where $u^0 : \Omega \to \mathbb{R}$ is the initial datum, ν denotes the outward unit normal to $\partial\Omega$, and $g : \partial\Omega \times (0,T) \to \mathbb{R}$ is the Neumann datum.

Let now $p, q \in (1, +\infty)$ be such that $p < q^*$ (Sobolev exponent, $q^* = +\infty$ if $q > n$) and let

$$H := L^p(\Omega) \quad \text{and} \quad V := \{v \in W^{1,q}(\Omega) : \ v = 0 \ \text{ on } \ \Gamma_D\}. \tag{1.5}$$

The reader is referred to [1] for definitions and properties of Sobolev spaces.

We shall assume that

$$|a(r)| \le C(1 + |r|^{p-1}) \quad \text{and} \quad |b(\eta)| \le C(1 + |\eta|^{q-1}) \quad \forall r \in \mathbb{R}, \ \eta \in \mathbb{R}^n, \tag{1.6}$$

for some constant $C > 0$, and that, say, $u^0 \in V$, $h \in L^1(0,T;L^{p'}(\Omega))$, and $g \in L^1(0,T;L^{q'}(\Gamma_N))$ where $1/p + 1/p' = 1/q + 1/q' = 1$ (these assumptions will be refined and complemented below). We define the operators $A : H \to H^*$, and $B : V \to V^*$, and the datum $f : (0,T) \to V^*$ as

$$A(u)(x) := a(u(x)) \quad \text{for a.e. } x \in \Omega, \quad \forall u \in V, \tag{1.7}$$

$$\langle B(u), v\rangle := \int_\Omega b(\nabla u)\cdot\nabla v \quad \forall u,\, v \in V, \tag{1.8}$$

$$\langle f(t), v\rangle := \int_\Omega h(\cdot,t)\, v + \int_{\Gamma_N} (g + k * g)(\cdot,t)\, v \quad \forall v \in V, \quad \text{a.e. in } (0,T) \tag{1.9}$$

where the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the duality pairing between V^* and V (note that the above growth assumptions on a and b entail that these definitions make sense). Finally, along with these choices, Problem (1.1) arises as the variational formulation of Problem (1.3)-(1.4).

1.2. Plan of the paper. We shall collect the assumptions and state our main results (Theorems 2.1 and 2.3) in Section 2. Both results are obtained from the convergence of the same time-discrete scheme. Hence, we prepare in Section 3 some preliminary material on the discrete convolution, present the scheme, and prove its solvability. Finally, in Sections 4 and 5 we prove Theorems 2.1 and 2.3, respectively.

2. Main Results. As said in the Introduction, we can prove two different results that are related to different sets of assumptions. However, some common framework is used. Namely, we assume in the whole paper that the conditions listed below are fulfilled.

$$V \text{ and } H \text{ are separable reflexive real Banach spaces} \tag{2.1}$$

$$V \subset H \text{ with dense and compact embedding} \tag{2.2}$$

$$A : H \to 2^{H^*} \quad \text{and} \quad B : V \to 2^{V^*} \quad \text{are maximal monotone} \tag{2.3}$$

$$k \in BV(0,T) \tag{2.4}$$

where $T \in (0,+\infty)$ is a fixed final time.

The reader is referred to [7] for an extensive discussion on functions of bounded variation. Here, we confine ourselves to note that any $v \in BV(0,T)$ has a unique right-continuous representative (whose total variation in the elementary sense coincides with the total variation of v, see [7, Thm. 3.28, p. 136]), so that $v(t)$ has a precise meaning for any fixed $t \in [0,T)$. Moreover, such a representative has a limit as $t \nearrow T$ which we call $v(T)$.

We term V^* and H^* the dual spaces of V and H, respectively, and observe that $H^* \subset V^*$ with compact embedding. Moreover, the norms in the four spaces V, V^*, H, H^* are denoted by $\|\cdot\|$, $\|\cdot\|_*$, $|\cdot|$, $|\cdot|_*$, respectively. Finally, the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the duality pairing both between V^* and V and between H^* and H. In view of (2.3), we note that A and B induce maximal monotone operators in L^p-type spaces by a standard procedure. In the following, we do not distinguish between such operators and the original ones in the notation.

Now, we make the meaning of the Cauchy problem (1.1) a little more precise. Due to the fact that A and B are possibly multivalued, a solution of such a problem is actually a triple (u, ζ, w) of vector-valued functions on $(0,T)$ such that

$$\zeta(t) + w(t) + (k * w)(t) = f(t) \qquad \text{for a.a. } t \in (0,T), \tag{2.5}$$
$$\zeta(t) \in A(u'(t)) \quad \text{and} \quad w(t) \in B(u(t)) \quad \text{for a.a. } t \in (0,T), \tag{2.6}$$
$$u(0) = u^0. \tag{2.7}$$

Equation (2.5) has to be understood in V^* and some minimal regularity is needed in order that all the above conditions make sense, e.g., $u \in L^1(0,T;V) \cap W^{1,1}(0,T;H)$, $\zeta \in L^1(0,T;H^*)$, $w \in L^1(0,T;V^*)$, and $f \in L^1(0,T;V^*)$. Note that this implies $u \in C^0([0,T];H)$, so that the Cauchy condition (2.7) is meaningful. Such a regularity is surely given by our existence results that we state at once. In both theorems, the symbols C and α stand for given strictly positive constants.

Theorem 2.1. *Assume* (2.1)–(2.4) *and* $p, p' \in (1, +\infty)$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$. *Moreover, assume*

$$|z|_*^{p'} \le C(1 + |v|^p) \quad \textit{and} \quad \langle z, v\rangle \ge \alpha |v|^p - C \quad \textit{for every } v \in H \textit{ and } z \in Av, \tag{2.8}$$
$$B = \partial\psi \quad \textit{where} \quad \psi : V \to (-\infty, +\infty] \quad \textit{is convex, proper, and l.s.c.,} \tag{2.9}$$
$$\lim_{\|u\| \to \infty} \frac{\psi(u)}{\|u\|} = +\infty. \tag{2.10}$$

Finally, assume that

$$f \in L^{p'}(0,T;H^*) + W^{1,\infty}(0,T;V^*) \quad \textit{and} \quad u^0 \in D(\psi). \tag{2.11}$$

Then, there exists (u, ζ, w) *satisfying*

$$u \in L^\infty(0,T;V) \cap W^{1,p}(0,T;H), \quad \zeta \in L^{p'}(0,T;H^*), \quad w \in L^{p'}(0,T;V^*) \tag{2.12}$$

and solving the Cauchy problem (2.5)–(2.7).

Before moving on, let us briefly comment on how Theorem 2.1 can be applied to the concrete situation of equation (1.3). First of all, we are allowed to generalize the frame of Subsection 1.1 by letting a and b be possibly multivalued and asking for the bound (1.6) on a, only. On the other hand, a shall be asked to fulfill $a(r)r \ge \alpha'|r|^p - C'$ for all $r \in \mathbb{R}$ and some $\alpha' > 0$, $C' \ge 0$ (see (2.8)). We let the functional ψ be defined, for all $u \in V$ (where V is defined in (1.5)), as

$$\psi(u) := \begin{cases} \displaystyle\int_\Omega j(\nabla u) & \text{if } j(\nabla u) \in L^1(\Omega) \\ +\infty & \text{if } j(\nabla u) \notin L^1(\Omega) \end{cases} \tag{2.13}$$

where $j : \mathbb{R}^n \to (-\infty, +\infty]$ is convex, proper, l.s.c. and satisfies $\partial j = b$ and $\lim j(r)/|r| = +\infty$ as $|r| \to +\infty$. Hence, we formally have $Bu = -\operatorname{div}(b(\nabla u))$ (with Neumann boundary conditions on Γ_N due to the choice of V) and the simplest choice ensuing (2.10) for V as in (1.5) is $j(\cdot) = (1/q)|\cdot|^q$. As far as the datum f is concerned, we simply ask for $h \in L^{p'}(\Omega \times (0,T)) + W^{1,\infty}(0,T;L^{(q^*)'}(\Omega))$ and $g \in W^{1,\infty}(0,T;L^{q'}(\Gamma_N))$ in (1.9), so that (2.11) holds.

Remark 2.2. In fact Theorem 2.1 may be extended to the case of a bounded kernel k. However, for the sake of simplicity, we will prove our result as it has been stated, i.e., assuming (2.4), and confine ourselves to give an outline of the proof of its extension in the forthcoming Remark 4.4.

THEOREM 2.3. *Assume* (2.1)–(2.4) *and*

$$A = \partial\varphi \quad \textit{where} \quad \varphi : H \to \mathbb{R} \quad \textit{is convex and continuous} \tag{2.14}$$

$$|z|_* \le C(|v| + 1) \quad \textit{for every } v \in H \textit{ and } z \in A(v) \tag{2.15}$$

$$B \textit{ is single-valued and Lipschitz continuous} \tag{2.16}$$

$$\langle w_1 - w_2, u_1 - u_2 \rangle \ge \alpha \| w_1 - w_2 \|^2$$

$$\textit{for every } w_i \in V \textit{ and } w_i \in B(u_i),\ i = 1, 2. \tag{2.17}$$

Moreover, assume

$$f \in H^1(0,T;V^*), \quad u^0 \in V, \quad \textit{and} \quad f(0) - B(u^0) \in D(\varphi^*). \tag{2.18}$$

Then, there exists (u, ζ, w) *satisfying*

$$u \in H^1(0,T;V), \quad \zeta \in L^\infty(0,T;H^*) \cap H^1(0,T;V^*), \quad w \in H^1(0,T;V^*) \tag{2.19}$$

and solving the Cauchy problem (2.5)–(2.7).

In (2.18), $\varphi^* : H^* \to (-\infty, +\infty]$ is the conjugate function of φ, of course. As above, let us now reconsider the situation of Subsection 1.1 from the point of view of Theorem 2.3. To this end, we shall reinforce the bound on a in (1.6) by asking a to be linearly bounded. On the other hand, a multivalued graph a is admissible and, denoting by $\gamma : \mathbb{R} \to (-\infty, +\infty]$ a primitive, i.e. $\partial\gamma = a$, we define

$$\varphi(u) := \int_\Omega \gamma(u) \quad \forall u \in H = L^p(\Omega).$$

We impose both strong monotonicity and Lipschitz continuity on the function b and define $B = \partial\psi$ from (2.13). Finally, in order the first (2.18) to hold, we may ask for $h \in H^1(0,T;L^{(q^*)'}(\Omega))$ and $g \in H^1(0,T;L^{q'}(\Gamma_N))$ in (1.9), while the last (2.18) means that some $\zeta_0 \in D(\varphi^*)$ exists such that

$$\int_\Omega h(0)\, v + \int_{\Gamma_N} g(0)\, v - \int_\Omega b(\nabla u^0) \cdot \nabla v = \int_\Omega \zeta_0\, v \quad \forall v \in V$$

i.e., that the data and the distribution $\zeta_0 := h(0) + \operatorname{div} b(\nabla u^0)$ satisfy

$$\zeta_0 \in L^{p'}(\Omega), \qquad \int_\Omega \gamma^*(\zeta_0) < +\infty, \quad \text{and} \quad b(\nabla u^0) \cdot \nu = g(0) \quad \text{on } \Gamma_N$$

where γ^* is the conjugate of γ. Note that the last condition on Γ_N is meaningful in a generalized sense due to $\operatorname{div} b(\nabla u^0) = \zeta_0 - h(0) \in L^{p'}(\Omega) + L^{(q^*)'}(\Omega) = L^{(q^*)'}(\Omega)$.

REMARK 2.4. We note that assumption (2.15) implies that

$$\varphi^*(z) \ge \alpha |z|_*^2 - C \quad \text{for any } z \in H^* \tag{2.20}$$

where α and C are some strictly positive constants (as we do not need sharpness, α could have the same value as in (2.17)).

REMARK 2.5. We note that the regularity conditions (2.12) of Theorem 2.1 imply that u is a weakly continuous V-valued function. Moreover, (2.6) yields $u(t) \in D(B)$ a.e. Hence, the assumption $u^0 \in D(\psi)$ (see (2.11)) is quite natural. Moreover, we note that the regularity of the solution given by Theorem 2.3 is rather high. This is due, in particular, to (2.18), which looks like a compatibility condition, besides regularity. Finally, we observe that (2.14)–(2.15) allow (an even strong) degeneracy.

3. Time-discretization. In order to prove our existence results, we consider a fully implicit time-discretization of problem (2.5)–(2.7). Such a procedure is based on a discrete convolution, for which we directly refer to [18], [26], and to the references quoted there.

Let us start by fixing a uniform partition of the time interval $[0,T]$ by choosing a constant time-step $\tau = T/N$, $N \in \mathbb{N}$. Then, we consider both N-vectors $z \in E^N$ and $(N+1)$-vectors $z \in E^{N+1}$, where E is a Banach space and label their elements by z_i, where $i = 1, \ldots, N$ and $i = 0, \ldots, N$, respectively. First, we recall the basic definitions and properties.

Definition 3.1. *Let $a \in \mathbb{R}^N$ and $b \in E^N$. Then, we define $a *_\tau b \in E^{N+1}$ by*

$$(a *_\tau b)_0 := 0 \quad \textit{and} \quad (a *_\tau b)_i := \tau \sum_{j=1}^{i} a_{i-j+1} b_j \quad \textit{for } i = 1, \ldots, N. \tag{3.1}$$

Such a discrete convolution enjoys nice properties. Beside the most elementary ones, we mention the derivative formula, the discrete Young theorem, and a basic inequality which is useful when letting τ tend to zero. To this aim, we introduce a notation. If $z \in E^{N+1}$, we define the piecewise linear interpolant $\widehat{z}_\tau$ and the backward piecewise constant interpolant $\overline{z}_\tau$ of z as follows

$$\widehat{z}_\tau(0) := z_0 \quad \text{and} \quad \widehat{z}_\tau(t) := \gamma_i(t) z_i + \big(1 - \gamma_i(t)\big) z_{i-1} \quad \text{for } t \in I_i^\tau \text{ and } i = 1, \ldots, N$$
$$\overline{z}_\tau(0) := z_0 \quad \text{and} \quad \overline{z}_\tau(t) := z_i \quad \text{for } t \in I_i^\tau \text{ and } i = 1, \ldots, N$$

where

$$\gamma_i(t) := (t - (i-1)\tau)/\tau \quad \text{and} \quad I_i^\tau := ((i-1)\tau, i\tau].$$

The definition of $\overline{z}_\tau$ is extended to vectors $z \in E^N$ simply avoiding the definition of $\overline{z}_\tau(0)$. Moreover, we define $\delta z \in E^N$ this way

$$(\delta z)_i := \frac{z_i - z_{i-1}}{\tau} \quad \text{for } i = 1, \ldots, N \tag{3.2}$$

and simply write δz_i in place of $(\delta z)_i$ in the sequel. By the way, we notice that

$$\|\widehat{z}_\tau - \overline{z}_\tau\|_{L^\infty(0,T;E)}^r = \max_{1 \le i \le N} \|z_i - z_{i-1}\|_E^r \le \sum_{i=1}^{N} \|z_i - z_{i-1}\|_E^r = \tau^{r-1} \cdot \tau \sum_{i=1}^{N} \|\delta z_i\|^r$$

whence

$$\|\widehat{z}_\tau - \overline{z}_\tau\|_{L^\infty(0,T;E)} \le \tau^{1-(1/r)} \|\widehat{z}_\tau'\|_{L^r(0,T;E)} \quad \text{for } z \in E^{N+1} \text{ and } 1 \le r < +\infty. \tag{3.3}$$

As far as the convolution is concerned, we have

$$\widehat{(a *_\tau b)}_\tau = \overline{a}_\tau * \overline{b}_\tau. \tag{3.4}$$

Moreover, for $a \in \mathbb{R}^{N+1}$ and $b \in E^N$, the discrete derivative formula holds, namely

$$\delta(a *_\tau b) = a_0 b + (\delta a) *_\tau b \tag{3.5}$$

which is the discrete counterpart of

$$(a * b)' = a(0)b + a' * b. \tag{3.6}$$

Next, we recall the discrete Young theorem (where $1/\infty = 0$) [18, Lemma 3.2] .

Proposition 3.2. *Let $a \in \mathbb{R}^N$, $b \in E^N$, and $p, q, r \in [1, \infty]$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then*

$$\|\overline{(a *_\tau b)}_\tau\|_{L^r(0,T;E)} \le \|\overline{a}_\tau\|_{L^p(0,T)} \|\overline{b}_\tau\|_{L^q(0,T;E)}.$$

Finally, we recall the following useful tools (see [18, Prop. 3.3 and Cor. 3.4]).

Proposition 3.3. *Let $a \in \mathbb{R}^{N+1}$, $b \in E^N$, and $r \in [1, \infty]$. Then, we have*

$$\|\overline{(a *_\tau b)}_\tau - \overline{a}_\tau * \overline{b}_\tau\|_{L^r(0,T;E)} \le \tau \left(|a_0| + \operatorname{var} a\right) \|\overline{b}_\tau\|_{L^r(0,T;E)}$$

where $\operatorname{var} a := \sum_{i=1}^N |a_i - a_{i-1}|$.

Corollary 3.4. *Let E be a real reflexive Banach space, $r \in [1, \infty]$, and let the sequence $\{\overline{a}_\tau\}$ be bounded in $BV(0,T)$. If $\overline{a}_\tau \to a$ strongly in $L^1(0,T)$ and $\overline{b}_\tau \to b$ weakly star (strongly) in $L^r(0,T;E)$ as $\tau \to 0$, then $\overline{(a *_\tau b)}_\tau \to a * b$ weakly star (strongly, respectively) in $L^r(0,T;E)$.*

The next step consists in approximating the kernel k and its resolvent ρ. Let us recall that the resolvent of k is the unique function $\rho \in L^1(0,T)$ satisfying $\rho + k * \rho = k$. By (2.4), we think of a right-continuous representative of k and define the discrete kernel k^τ as follows

$$k^\tau = (k_i)_{i=0}^N \in \mathbb{R}^{N+1} \quad \text{where} \quad k_i = k(i\tau) \quad \text{for } i = 0, \dots, N. \tag{3.7}$$

Next, we define the discrete resolvent $\rho^\tau = (\rho_i)_{i=0}^N \in \mathbb{R}^{N+1}$ by the condition

$$\rho^\tau + k^\tau *_\tau \rho^\tau = k^\tau \quad \text{provided that } \tau\|k\|_{L^\infty(0,T)} < 1. \tag{3.8}$$

Indeed, the discrete resolvent is well defined if $\tau|k_1| < 1$, as shown in [18]. Moreover, for the sake of simplicity, we avoid the superscript τ in the notation when we consider the interpolants of k^τ and ρ^τ (as we did for the components k_i and ρ_i), i.e., we simply write, e.g., $\widehat{k}_\tau$ and $\overline{k}_\tau$.

We recall the basic property of the resolvent ρ and of the discrete resolvent ρ^τ. At the same time, we summarize the boundedness and convergence properties.

Proposition 3.5. *Assume* (2.4) *and $\tau\|k\|_{L^\infty(0,T)} < 1$. Then, for every $u, v \in L^1(0,T;E)$ and $a, b \in E^N$, we have*

$$u + k * u = v \quad \textit{if and only if} \quad u = v - \rho * v \tag{3.9}$$

$$a + k^\tau *_\tau a = b \quad \textit{if and only if} \quad a = b - \rho^\tau *_\tau b \tag{3.10}$$

respectively. The sequences $\{\widehat{k}_\tau\}$ and $\{\overline{k}_\tau\}$ are bounded in $BV(0,T)$ and converge to the given kernel k strongly in $L^1(0,T)$. The sequences $\{\widehat{\rho}_\tau\}$ and $\{\overline{\rho}_\tau\}$ are bounded in $BV(0,T)$ and converge to the resolvent ρ of k strongly in $L^1(0,T)$. In particular, $\rho \in BV(0,T)$.

Remark 3.6. Even though we have defined the discrete resolvent just in the case $k \in BV(0,T)$, an L^∞-type norm of ρ^τ can be estimated in terms of the L^∞-norm of k. Indeed, assuming $\tau\|k\|_{L^\infty(0,T)} \le 1/2$, we see that (3.8) implies

$$\frac{1}{2}|\rho_i| \le (1 + \tau k_1)|\rho_i| \le \|k\|_{L^\infty(0,T)} + \tau\|k\|_{L^\infty(0,T)} \sum_{j=1}^{i-1} |\rho_j| \quad \text{for } i = 1, \dots, N$$

and the discrete Gronwall lemma (see (3.23)) yields the (non sharp) estimate

$$|\rho_i| \le 2\|k\|_{L^\infty(0,T)} e^{2T\|k\|_{L^\infty(0,T)}} \quad \text{for } i = 1, \dots, N.$$

Similarly, the standard Gronwall lemma yields $\rho \in L^\infty(0,T)$ whenever $k \in L^\infty(0,T)$ and that

$$\|\rho\|_{L^\infty(0,T)} \le \|k\|_{L^\infty(0,T)} e^{T\|k\|_{L^\infty(0,T)}}.$$

These facts are used in the sequel in order to extend Theorem 2.1 to the case $k \in L^\infty(0,T)$ (see the forthcoming Remark 4.4).

Now, we are ready to introduce the discrete problem. Let $f^\tau = (f_i)_{i=1}^N \in (V^*)^N$ approximate the right hand side f of equation (2.5). A precise choice of f^τ will be made later on.

Definition 3.7. *Given $u^0 \in V$ and $f^\tau \in (V^*)^N$, a solution to the discrete problem is a triple $(u^\tau, \zeta^\tau, w^\tau)$ satisfying the following conditions*

$$u^\tau \in V^{N+1} \quad \textit{and} \quad u_0 = u^0, \quad \zeta^\tau \in (H^*)^N, \quad w^\tau \in (V^*)^N \tag{3.11}$$

$$\zeta^\tau + w^\tau + k^\tau *_\tau w^\tau = f^\tau \tag{3.12}$$

$$\zeta_i \in A(\delta u_i) \quad \textit{and} \quad w_i \in B(u_i) \quad \textit{for } i = 1, \dots, N \tag{3.13}$$

where u_i, ζ_i, and w_i are the components of u^τ, ζ^τ, and w^τ, respectively.

Now, we prove an existence result for the discrete problem. The assumption we need here are weaker than the assumptions of both Theorems 2.1 and 2.3.

Theorem 3.8. *Assume* (2.1)–(2.4) *and $\tau\|k\|_{L^\infty(0,T)} < 1$. Moreover, assume that A is bounded and B is coercive. Then, for any $u^0 \in V$ and $f^\tau \in (V^*)^N$, the discrete problem has a solution.*

Proof. Using (3.10), we rewrite (3.12) in the form

$$\zeta^\tau + w^\tau = \widetilde{f}^\tau + \rho^\tau *_\tau \zeta^\tau \quad \text{where} \quad \widetilde{f}^\tau = (\widetilde{f}_i)_{i=1}^N = f^\tau - \rho^\tau *_\tau f^\tau.$$

Hence, after setting $u_0 = u^0$, we just have to solve inductively the equation

$$(1 - \tau\rho_1)\zeta_i + w_i = \widetilde{f}_i + \tau \sum_{j=1}^{i-1} \rho_{i-j+1}\zeta_j \,, \quad \zeta_i \in A(\delta u_i), \quad \text{and} \quad w_i \in B(u_i) \tag{3.14}$$

for $i = 1, \dots, N$ with the convention that the empty sum is 0. At each step, u_{i-1} and the right hand side are known. On the other hand, a simple computation shows that the coefficient of ζ_i on the left hand side is $1/(1+\tau k_1) > 1/2$ if $\tau|k_1| < 1$, and this is the case if $\tau\|k\|_{L^\infty(0,T)} < 1$. Hence, (3.14) has the form

$$\sigma\zeta_i + w_i = f^*, \quad \zeta_i \in A\left(\frac{u_i - u^*}{\tau}\right), \quad \text{and} \quad w_i \in B(u_i)$$

where $\sigma > 0$, $u^* \in V$, and $f^* \in V^*$ are given. In other words, we have to solve

$$\sigma\zeta_i + w_i = f^*, \quad \zeta_i \in A_*(u_i), \quad \text{and} \quad w_i \in B(u_i), \quad \text{where} \quad A_*(u) := A\left(\frac{u - u^*}{\tau}\right).$$

As $A_* : H \to 2^{H^*}$ enjoys the same properties of A, namely, it is maximal monotone and bounded, and the same holds for σA_*, we avoid all the subscripts and superscripts and the factor σ, i.e., we look for $u \in V$ such that

$$Au + Bu \ni f \tag{3.15}$$

where $f \in V^*$ is given.

In order to solve (3.15), we introduce an approximating problem depending on the parameter $\varepsilon \in (0,1)$. We term A_ε the Yosida regularization of A and briefly recall its properties. We refer, e.g., to [8, Prop. 1.1, Lem. 1.3, and Thm. 1.3]. As H is reflexive, we can assume that both $|\cdot|$ and $|\cdot|_*$ are strictly convex norms. As A is maximal monotone and bounded, it turns out that

$$A_\varepsilon : H \to H^* \text{ is single-valued, maximal monotone, and demicontinuous} \tag{3.16}$$

for any bounded subset $S \subset H$, we have

$$\sup\{|A_\varepsilon v|_* : \ v \in S, \ \varepsilon \in (0,1)\} < +\infty \tag{3.17}$$

$$u_\varepsilon \rightharpoonup u \text{ in } H, \ A_\varepsilon u_\varepsilon \rightharpoonup \zeta \text{ in } H^*, \text{ and } \langle A_\varepsilon u_\varepsilon, u_\varepsilon\rangle \to \langle \zeta, u\rangle \text{ imply } \zeta \in Au. \tag{3.18}$$

Then, we first solve the approximating problem of finding $u_\varepsilon \in V$ such that

$$A_\varepsilon u_\varepsilon + Bu_\varepsilon \ni f. \tag{3.19}$$

As A_ε is everywhere defined, monotone, and demicontinuous, the same holds for $A_\varepsilon|_V : V \to V^*$. Hence, $A_\varepsilon|V + B : V \to 2^{V^*}$ is maximal monotone. Moreover, it is coercive since B is coercive. Therefore, (3.19) has a solution, namely, there exists $(u_\varepsilon, \zeta_\varepsilon, w_\varepsilon) \in V \times H^* \times V^*$ such that

$$\zeta_\varepsilon + w_\varepsilon = f, \quad \zeta_\varepsilon = A_\varepsilon u_\varepsilon\,, \quad \text{and} \quad w_\varepsilon \in Bu_\varepsilon. \tag{3.20}$$

Now, we perform an priori estimate. Setting $\zeta_{0,\varepsilon} := A_\varepsilon 0$ for convenience, we have

$$\langle w_\varepsilon, u_\varepsilon\rangle \le \langle \zeta_\varepsilon - \zeta_{0,\varepsilon} + w_\varepsilon, u_\varepsilon\rangle = \langle f - \zeta_{0,\varepsilon}, u_\varepsilon\rangle \le \|f\|_* \|u_\varepsilon\| + |\zeta_{0,\varepsilon}|_* |u_\varepsilon| \le M\|u_\varepsilon\|$$

for some constant M, since $\{\zeta_{0,\varepsilon}\}$ is bounded in H^* by (3.17) and the embedding $V \subset H$ is continuous. Owing to the coerciveness assumption on B, we derive that $\{u_\varepsilon\}$ is bounded in V, whence in H as well. Then, (3.17) implies that $\{\zeta_\varepsilon\}$ is bounded in H^*, and solving (3.20) for w_ε yields that $\{w_\varepsilon\}$ is bounded in V^*. Hence, for a subsequence, we have

$$u_\varepsilon \rightharpoonup u \text{ in } V, \quad \zeta_\varepsilon \rightharpoonup \zeta \text{ in } H^*, \quad \text{and} \quad w_\varepsilon \rightharpoonup w \text{ in } V^*.$$

Clearly, $\zeta + w = f$. Moreover, due to the compact embedding $V \subset H$ (see (2.2)), we derive the strong convergence $u_\varepsilon \to u$ in H, whence $\langle \zeta_\varepsilon, u_\varepsilon\rangle \to \langle \zeta, u\rangle$ and $\zeta \in Au$ by (3.18). Finally

$$\lim_{\varepsilon\to 0} \langle w_\varepsilon, u_\varepsilon\rangle = \lim_{\varepsilon\to 0} \langle f - \zeta_\varepsilon, u_\varepsilon\rangle = \langle f - \zeta, u\rangle = \langle w, u\rangle$$

and we conclude that $w \in Bu$ as well by [8, Lem. 1.3, p. 42]. ☐

In the next two sections, we prove our existence results. In each case, we first make the choice of f^τ precise. Then, we perform some a priori estimates. Finally, we let τ tend to zero owing to compactness and monotonicity arguments. Besides its discrete version given by Proposition 3.2, we widely use the Young theorem and the elementary Young inequality

$$\|a * b\|_{L^r(0,T;E)} \le \|a\|_{L^p(0,T)} \|b\|_{L^q(0,T;E)}, \ \ p,q,r \in [1,+\infty], \ \ 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \tag{3.21}$$

$$xy \le \sigma x^p + c_{p,\sigma} y^{p'}, \quad x,y \ge 0, \ \sigma > 0, \ p,p' \in (1,+\infty), \ \frac{1}{p} + \frac{1}{p'} = 1. \tag{3.22}$$

In (3.22), $c_{p,\sigma}$ is some constant that depends on p and σ, only. However, in the sequel the same symbol c will stand for different constants that depend only on the functions and quantities related to the assumptions of the theorem we want to prove (e.g., on the operators and on the norms of the data in the spaces we have specified). Hence, the meaning of c might change from line to line and even in the same chain of inequalities. Moreover, a symbol like c_σ allows the constant c to depend on the parameter σ, in addition. Finally, we use the following discrete Gronwall lemma: if $\{a_n\}$ and $\{b_n\}$ are non negative real sequences, $c_0 \geq 0$, and $N \in \mathbb{N}$, then

$$a_n \leq c_0 + \sum_{i=1}^{n-1} b_i a_i \quad \text{for } n = 1, \ldots, N \quad \text{implies that}$$
$$a_n \leq c_0 \prod_{i=1}^{n-1} (1 + b_i) \leq c_0 \exp\left(\sum_{i=1}^{n-1} b_i\right) \quad \text{for } n = 1, \ldots, N. \tag{3.23}$$

This can be easily proved by induction (see also, e.g., [23, Prop. 2.2.1]).

4. Proof of Theorem 2.1. Using assumption (2.11), we split f as

$$f = f_1 + f_2 \quad \text{with} \quad f_1 \in L^{p'}(0,T;H^*) \quad \text{and} \quad f_2 \in W^{1,\infty}(0,T;V^*) \tag{4.1}$$

and define f^τ by choosing suitable discretizations $f_1^\tau = (f_{1,i})$ and $f_2^\tau = (f_{2,i})$ of f_1 and f_2, respectively. We set

$$f_{1,i} := \frac{1}{\tau} \int_{(i-1)\tau}^{i\tau} f_1(t)\, dt, \quad i = 1, \ldots, N, \quad \text{and} \quad f_{2,i} := f_2(i\tau), \quad i = 0, \ldots, N. \tag{4.2}$$

Moreover, we introduce the transforms $\widetilde{f}_1$ and $\widetilde{f}_2$ and the discrete transforms $\widetilde{f}_1^\tau$ and $\widetilde{f}_2^\tau$ by

$$\widetilde{f}_j := f_j - \rho * f_j \quad \text{and} \quad \widetilde{f}_j^\tau := f_j^\tau - \rho^\tau *_\tau f_j^\tau \quad \text{for } j = 1, 2 \tag{4.3}$$

and avoid the superscript τ in the notation for the interpolants of the discretized data (as we did for their i-th components).

Remark 4.1. Owing to (3.21) and to Propositions 3.2 and 3.5, we note at once that the following estimates hold

$$\|\widetilde{f}_1\|_{L^{p'}(0,T;H^*)} \leq \big(1 + \|\rho\|_{L^1(0,T)}\big) \|f_1\|_{L^{p'}(0,T;H^*)} \tag{4.4}$$
$$\|\widetilde{f}_2\|_{L^\infty(0,T;V^*)} \leq \big(1 + \|\rho\|_{L^1(0,T)}\big) \|f_2\|_{L^\infty(0,T;V^*)} \tag{4.5}$$
$$\|\widetilde{f}_2'\|_{L^\infty(0,T;V^*)} \leq \big(1 + \|\rho\|_{L^1(0,T)}\big) \|f_2'\|_{L^\infty(0,T;V^*)} + \|\rho\|_{L^\infty(0,T)} \|f_2\|_{L^\infty(0,T;V^*)} \tag{4.6}$$
$$\|\overline{\widetilde{f}}_{1,\tau}\|_{L^{p'}(0,T;H^*)} \leq \big(1 + \|\overline{\rho}_\tau\|_{L^1(0,T)}\big) \|\overline{f}_{1,\tau}\|_{L^{p'}(0,T;H^*)} \tag{4.7}$$
$$\|\overline{f}_{1,\tau}\|_{L^{p'}(0,T;H^*)} \leq \|f_1\|_{L^{p'}(0,T;H^*)}. \tag{4.8}$$

In particular, such quantities are estimated by a known constant. For (4.6), we have used $\widetilde{f}_2' = f_2' - \rho * f_2' - \rho\, f_2(0)$, owing to the analogous of (3.6) obtained by interchanging a and b. Moreover, the following convergences hold

$$\overline{f}_{1,\tau} \to f_1 \quad \text{and} \quad \overline{\widetilde{f}}_{1,\tau} \to \widetilde{f}_1 \quad \text{strongly in } L^{p'}(0,T;H^*) \tag{4.9}$$
$$\overline{f}_{2,\tau} \to f_2 \quad \text{and} \quad \overline{\widetilde{f}}_{2,\tau} \to \widetilde{f}_2 \quad \text{strongly in } L^\infty(0,T;V^*) \tag{4.10}$$
$$\widehat{f}_{2,\tau} \to f_2 \quad \text{and} \quad \widehat{\widetilde{f}}_{2,\tau} \to \widetilde{f}_2 \quad \text{strongly in } W^{1,q}(0,T;V^*) \text{ for every } q < +\infty. \tag{4.11}$$

The discrete problem is now

$$\zeta_i + w_i + (k^\tau *_\tau w^\tau)_i = f_{1,i} + f_{2,i}, \quad \zeta_i \in A(\delta u_i), \quad \text{and} \quad w_i \in B(u_i). \tag{4.12}$$

for $i = 1, \dots, N$. Note that (3.10) yields an equivalent version of the equation in (4.12), namely

$$\zeta_i + w_i = \widetilde{f}_{1,i} + \widetilde{f}_{2,i} + (\rho^\tau *_\tau \zeta^\tau)_i \quad \text{for } i = 1, \dots, N. \tag{4.13}$$

Now, we test (4.13) by $\tau\delta u_i$ and sum over $i = 1, \dots, n$ with any $n \le N$. We obtain

$$\begin{aligned}
&\tau\sum_{i=1}^n \langle \zeta_i, \delta u_i\rangle + \tau\sum_{i=1}^n \langle w_i, \delta u_i\rangle \\
&= \tau\sum_{i=1}^n \langle \widetilde{f}_{1,i}, \delta u_i\rangle + \tau\sum_{i=1}^n \langle \widetilde{f}_{2,i}, \delta u_i\rangle + \tau\sum_{i=1}^n \langle (\rho^\tau *_\tau \zeta^\tau)_i, \delta u_i\rangle
\end{aligned} \tag{4.14}$$

and we now estimate each term of (4.14), separately.

Remark 4.2. Despite of the above-stated notational convention for constants (see the end of Section 3), we stress that the bounds below are going to depend just on the L^∞ norm of k, rather than on its BV norm, (see, in particular, Remark 3.6), even though assumption (2.4) is listed in the statement of the theorem and it is actually used in the proof. That is why we can deal even with the case $k \in L^\infty(0,T)$ (see Remarks 2.2 and 4.4).

Coming back to the treatment of the terms of (4.14), we immediately have

$$\tau\sum_{i=1}^n \langle \zeta_i, \delta u_i\rangle \ge \alpha\tau\sum_{i=1}^n |\delta u_i|^p - Cn\tau \ge \alpha\tau\sum_{i=1}^n |\delta u_i|^p - c \tag{4.15}$$

due to the second of (2.8). Next, the definition of $w_i \in Bu_i = \partial\psi(u_i)$ yields

$$\begin{aligned}
&\tau\sum_{i=1}^n \langle w_i, \delta u_i\rangle = \sum_{i=1}^n \langle w_i, u_i - u_{i-1}\rangle \\
&\ge \sum_{i=1}^n \big(\psi(u_i) - \psi(u_{i-1})\big) = \psi(u_n) - \psi(u^0) = \psi(u_n) - c.
\end{aligned} \tag{4.16}$$

Let us consider the right hand side of (4.14). Owing to Remark 4.1, we easily have

$$\begin{aligned}
&\tau\sum_{i=1}^n \langle \widetilde{f}_{1,i}, \delta u_i\rangle \le \left(\tau\sum_{i=1}^n |\widetilde{f}_{1,i}|_*^{p'}\right)^{1/p'} \left(\tau\sum_{i=1}^n |\delta u_i|^p\right)^{1/p} \\
&\le \frac{\alpha}{4}\tau\sum_{i=1}^n |\delta u_i|^p + c\tau\sum_{i=1}^n |\widetilde{f}_{1,i}|_*^{p'} \le \frac{\alpha}{4}\tau\sum_{i=1}^n |\delta u_i|^p + c.
\end{aligned} \tag{4.17}$$

The next term is less trivial. We have

$$\begin{aligned}
&\tau\sum_{i=1}^n \langle \widetilde{f}_{2,i}, \delta u_i\rangle = \sum_{i=1}^n \langle \widetilde{f}_{2,i}, u_i - u_{i-1}\rangle \\
&= \sum_{i=1}^n \big(\langle \widetilde{f}_{2,i}, u_i\rangle - \langle \widetilde{f}_{2,i-1}, u_{i-1}\rangle - \langle \widetilde{f}_{2,i} - \widetilde{f}_{2,i-1}, u_{i-1}\rangle\big)
\end{aligned}$$

whence

$$\tau \sum_{i=1}^{n} \langle \widetilde{f}_{2,i}, \delta u_i \rangle \le \langle \widetilde{f}_{2,n}, u_n \rangle - c - \tau \sum_{i=1}^{n} \langle \delta \widetilde{f}_{2,i}, u_{i-1} \rangle \tag{4.18}$$

and we have to estimate the right hand side. For the first term, we owe to the convexity assumption on ψ and use its conjugate function $\psi^* : H^* \to (-\infty, +\infty]$ this way

$$\langle \widetilde{f}_{2,n}, u_n \rangle = \langle 2\widetilde{f}_{2,n}, \tfrac{1}{2} u_n + \tfrac{1}{2} u^0 \rangle - \langle \widetilde{f}_{2,n}, u^0 \rangle \le \psi(\tfrac{1}{2} u_n + \tfrac{1}{2} u^0) + \psi^*(2\widetilde{f}_{2,n}) + c \|\widetilde{f}_{2,n}\|_*$$
$$\le \frac{1}{2}\psi(u_n) + \frac{1}{2}\psi(u^0) + \psi^*(2\widetilde{f}_{2,n}) + c\|\widetilde{f}_{2,n}\|_* \le \frac{1}{2}\psi(u_n) + c + \psi^*(2\widetilde{f}_{2,n}) + c\|\widetilde{f}_{2,n}\|_*.$$

On the other hand, we have $\|\widetilde{f}_{2,n}\|_* \le \|\widetilde{f}_2\|_{L^\infty(0,T;V^*)} \le c$ (by (4.5)). Moreover, the coerciveness assumption (2.10) implies that ψ^* is bounded on every bounded subset of V^*. Therefore, the above inequality becomes

$$\langle \widetilde{f}_{2,n}, u_n \rangle \le \frac{1}{2}\, \psi(u_n) + c. \tag{4.19}$$

Next, we estimate the last sum of (4.18). We have

$$-\tau \sum_{i=1}^{n} \langle \delta \widetilde{f}_{2,i}, u_{i-1} \rangle \le \tau \sum_{i=1}^{n} \big(\psi^*(\delta \widetilde{f}_{2,i}) + \psi(u_{i-1}) \big)$$

and the first term in the sum is bounded since $\|\delta \widetilde{f}_{2,i}\|_* \le \|\widetilde{f}_2'\|_{L^\infty(0,T;V^*)}$, (4.6) holds, and ψ^* is bounded on bounded sets. Hence, we derive that

$$-\tau \sum_{i=1}^{n} \langle \delta \widetilde{f}_{2,i}, u_{i-1} \rangle \le c + \tau \sum_{i=1}^{n} \psi(u_{i-1}) = c + \tau \sum_{i=1}^{n-1} \psi(u_i). \tag{4.20}$$

Finally, we deal with the convolution term of (4.14). We have

$$\tau \sum_{i=1}^{n} \langle (\rho^\tau *_\tau \zeta^\tau)_i, \delta u_i \rangle \le \frac{\alpha}{4}\, \tau \sum_{i=1}^{n} |\delta u_i|^p + c\tau \sum_{i=1}^{n} |(\rho^\tau *_\tau \zeta^\tau)_i|_*^{p'} \tag{4.21}$$

and we now estimate the last term by using the discrete Young Theorem 3.2 and the boundedness of A given by (2.8). We have

$$\tau \sum_{i=1}^{n} |(\rho^\tau *_\tau \zeta^\tau)_i|_*^{p'} \le \tau \sum_{i=1}^{n} \|\overline{(\rho^\tau *_\tau \zeta^\tau)}_\tau\|_{L^\infty(0,i\tau;H^*)}^{p'}$$
$$\le \tau \sum_{i=1}^{n} \|\overline{\rho}_\tau\|_{L^p(0,i\tau)}^{p'} \|\overline{\zeta}_\tau\|_{L^{p'}(0,i\tau;H^*)}^{p'} \le \tau \sum_{i=1}^{n} \|\overline{\rho}_\tau\|_{L^p(0,T)}^{p'} \|\overline{\zeta}_\tau\|_{L^{p'}(0,i\tau;H^*)}^{p'}$$
$$\le c\tau \sum_{i=1}^{n} \tau \sum_{j=1}^{i} |\zeta_j|_*^{p'} \le c\tau \sum_{i=1}^{n} \tau \sum_{j=1}^{i} c\,(1 + |\delta u_j|^p) \le c + c\tau \sum_{i=1}^{n} \tau \sum_{j=1}^{i} |\delta u_j|^p. \tag{4.22}$$

At this point, we collect the equality (4.14) and all the estimates (4.15)–(4.22) and rearrange. We deduce

$$\tau \sum_{i=1}^{n} |\delta u_i|^p + \psi(u_n) \le c + c\tau \sum_{i=1}^{n} \tau \sum_{j=1}^{i} |\delta u_j|^p + c\tau \sum_{i=1}^{n-1} \psi(u_i)$$
$$= c + c\tau \sum_{i=1}^{n-1} \tau \sum_{j=1}^{i} |\delta u_j|^p + c\tau \sum_{i=1}^{n-1} \psi(u_i) + c\tau^2 \sum_{j=1}^{n} |\delta u_j|^p.$$

As the last constant c depends just on the structure of the problem and on the data, we can choose τ_0 having the same dependencies of c such that $c\tau_0 \le 1/2$. Hence, assuming $\tau \le \tau_0$, we have $c\tau^2 \le \tau/2$ and we can apply the discrete Gronwall lemma. We conclude that

$$\tau \sum_{i=1}^{n} |\delta u_i|^p + \psi(u_n) \le c \tag{4.23}$$

and using (2.10) and the first of (2.8), we immediately derive that

$$\|u_n\| \le c \quad \text{and} \quad \tau \sum_{i=1}^{n} |\zeta_i|_*^{p'} \le c. \tag{4.24}$$

Now, we read both the discrete problem and the above estimates in terms of the interpolants. Then (4.12) and (4.13) become

$$\overline{\zeta}_\tau + \overline{w}_\tau + \overline{(k^\tau *_\tau w^\tau)}_\tau = \overline{f}_{1,\tau} + \overline{f}_{2,\tau} \ \text{ or } \ \overline{\zeta}_\tau + \overline{w}_\tau = \overline{\widetilde{f}}_{1,\tau} + \overline{\widetilde{f}}_{2,\tau} + \overline{(\rho^\tau *_\tau \zeta^\tau)}_\tau \tag{4.25}$$

$$\text{and} \quad \overline{\zeta}_\tau \in A(\widehat{u}'_\tau), \quad \text{and} \quad \overline{w}_\tau \in B(\overline{u}_\tau) \tag{4.26}$$

while (4.23)–(4.24) yield

$$\|\widehat{u}_\tau\|_{L^\infty(0,T;V)} = \|\overline{u}_\tau\|_{L^\infty(0,T;V)} \le c, \quad \|\widehat{u}_\tau\|_{W^{1,p}(0,T;H)} + \|\overline{\zeta}_\tau\|_{L^{p'}(0,T;H^*)} \le c. \tag{4.27}$$

Moreover, (3.3) and (4.23) imply

$$\|\widehat{u}_\tau - \overline{u}_\tau\|_{L^\infty(0,T;H)} \le c\tau^{1/p'}.$$

On the other hand, $\overline{f}_{1,\tau}$ is bounded in $L^p(0,T;H^*)$ due to (4.8), and the same holds for $\overline{(\rho^\tau *_\tau \zeta^\tau)}_\tau$ thanks to the above estimate for $\overline{\zeta}_\tau$, the Young theorem (see (3.21)), and Proposition 3.3. Finally, $\overline{\widetilde{f}}_{2,\tau}$ is bounded in $L^\infty(0,T;V^*)$ due to (4.5). Hence, taking the second of (4.25) into account, we infer that

$$\|\overline{w}_\tau - \overline{\widetilde{f}}_{2,\tau}\|_{L^{p'}(0,T;H^*)} \le c \quad \text{and} \quad \|\overline{w}_\tau\|_{L^{p'}(0,T;V^*)} \le c. \tag{4.28}$$

Therefore, we are ready to use well-known weak and weak star compactness tools. Owing to the strong compactness result [25, Sect. 8, Cor. 4] as well, we have for a subsequence

$$\begin{aligned}
\widehat{u}_\tau \to u \qquad & \text{weakly star in } L^\infty(0,T;V), \\
& \quad \text{weakly in } W^{1,p}(0,T;H), \\
& \quad \text{and strongly in } C^0([0,T];H) && (4.29)\\
\overline{u}_\tau \to u \qquad & \text{weakly star in } L^\infty(0,T;V) \\
& \quad \text{and strongly in } L^\infty(0,T;H) && (4.30)\\
\overline{\zeta}_\tau \to \zeta \qquad & \text{weakly in } L^{p'}(0,T;H^*) && (4.31)\\
\overline{w}_\tau \to u \qquad & \text{weakly in } L^{p'}(0,T;V^*) && (4.32)\\
(\overline{w}_\tau - \overline{\widetilde{f}}_{2,\tau}) \to (w - \widetilde{f}_2) \quad & \text{weakly in } L^{p'}(0,T;H^*). && (4.33)
\end{aligned}$$

Note that (u, ζ, w) fulfills the regularity conditions (2.12) of Theorem 2.1 and the Cauchy condition $u(0) = u^0$. Moreover

$$\zeta + w = \widetilde{f}_1 + \widetilde{f}_2 + \rho * \zeta \quad \text{and} \quad \zeta + w + k * w = f_1 + f_2 \tag{4.34}$$

and it just remains to prove that $\zeta \in A(u')$ and that $w \in B(u)$. The latter is easily obtained. We have indeed

$$\lim_{\tau\to 0}\int_0^T \langle \overline{w}_\tau, \overline{u}_\tau\rangle = \lim_{\tau\to 0}\int_0^T \langle \overline{w}_\tau - \overline{\overline{f}}_{2,\tau}, \overline{u}_\tau\rangle + \lim_{\tau\to 0}\int_0^T \langle \overline{\overline{f}}_{2,\tau}, \overline{u}_\tau\rangle$$
$$= \int_0^T \langle w - \widetilde{f}_2, u\rangle + \int_0^T \langle \widetilde{f}_2, u\rangle = \int_0^T \langle w, u\rangle$$

whence $w \in B(u)$ by [8, Lemma 1.3, p. 42]. We aim to use the same result in order to prove that $\zeta \in A(u')$, i.e., we shall check that

$$\limsup_{\tau\to 0}\int_0^T \langle \overline{\zeta}_\tau, \widehat{u}'_\tau\rangle \le \int_0^T \langle \zeta, u'\rangle. \tag{4.35}$$

To this end, we prepare a lemma.

Lemma 4.3. *Let $\{k_\delta\}$, $\{u_\delta\}$, and $\{g_\delta\}$ be three sequences such that*

$$\begin{aligned} &k_\delta \to k \quad \text{strongly in } L^1(0,T)\\ &u_\delta \to u \quad \text{weakly in } W^{1,p}(0,T;H) \text{ and strongly in } C^0([0,T];H)\\ &g_\delta \to g \quad \text{weakly in } L^{p'}(0,T;H^*)\end{aligned}$$

as δ tends to zero. Then, we have

$$\lim_{\delta\to 0}\int_0^T \langle k_\delta * g_\delta, u'_\delta\rangle = \int_0^T \langle k * g, u'\rangle.$$

Proof. We fix $\varepsilon > 0$ and look for $\delta_\varepsilon > 0$ such that

$$\left|\int_0^T \langle k_\delta * g_\delta, u'_\delta\rangle - \int_0^T \langle k * g, u'\rangle\right| \le \varepsilon \quad \text{for } 0 < \delta \le \delta_\varepsilon. \tag{4.36}$$

We fix $\delta_0 > 0$ and M such that

$$\|g_\delta\|_{L^{p'}(0,T;H^*)} \le M \quad \text{and} \quad \|u_\delta\|_{W^{1,p}(0,T;H)} \le M \quad \text{for } 0 < \delta \le \delta_0$$

and assume $\delta \le \delta_0$ in the sequel. Then, we choose akernel $k^\varepsilon \in C^1[0,T]$ such that $\|k^\varepsilon - k\|_{L^1(0,T)} \le \varepsilon$ and term $\widehat{k}^\varepsilon_\delta$ the piece-wise linear interpolant of k^ε with step δ. Then, we have

$$\begin{aligned}&\left|\int_0^T \langle k_\delta * g_\delta, u'_\delta\rangle - \int_0^T \langle k * g, u'\rangle\right|\\ &\le \int_0^T |\langle (k^\varepsilon - k) * g, u'\rangle| + \int_0^T |\langle (k_\delta - \widehat{k}^\varepsilon_\delta) * g_\delta, u'_\delta\rangle|\\ &+ \left|\int_0^T \big(\langle \widehat{k}^\varepsilon_\delta * g_\delta, u'_\delta\rangle - \int_0^T \langle k^\varepsilon * g, u'\rangle\big)\right|\end{aligned} \tag{4.37}$$

and we now treat each term of the right hand side of (4.37), separately. The first one is easily estimated owing to the Young theorem (see (3.21)). We have indeed

$$\int_0^T |\langle (k^\varepsilon - k) * g, u'\rangle| \le \|k^\varepsilon - k\|_{L^1(0,T)}\|g\|_{L^{p'}(0,T;H^*)}\|u'\|_{L^p(0,T;H)} \le M^2\varepsilon.$$

We have similarly

$$\int_0^T |\langle (k_\delta - \widehat{k}_\delta^\varepsilon) * g_\delta, u_\delta' \rangle| \le \|k_\delta - \widehat{k}_\delta^\varepsilon\|_{L^1(0,T)} \|g_\delta\|_{L^{p'}(0,T;H^*)} \|u_\delta'\|_{L^p(0,T;H)}$$
$$\le M^2 \|k_\delta - \widehat{k}_\delta^\varepsilon\|_{L^1(0,T)}.$$

On the other hand, we have

$$\|k_\delta - \widehat{k}_\delta^\varepsilon\|_{L^1(0,T)} \le \|k_\delta - k\|_{L^1(0,T)} + \|k - k^\varepsilon\|_{L^1(0,T)} + \|k^\varepsilon - \widehat{k}_\delta^\varepsilon\|_{L^1(0,T)}$$
$$\le \|k_\delta - k\|_{L^1(0,T)} + \varepsilon + 2\delta \|(k^\varepsilon)'\|_{L^1(0,T)}$$

since

$$|k^\varepsilon(t) - \widehat{k}_\delta^\varepsilon(t)| \le |k^\varepsilon(t) - k^\varepsilon((i-1)\delta)| + |\widehat{k}_\delta^\varepsilon(t) - k^\varepsilon((i-1)\delta)| \le 2 \int_{(i-1)\delta}^{i\delta} |(k^\varepsilon)'(s)|\, ds$$

for $(i-1)\delta < t \le i\delta$. Moreover, $\{k_\delta\}$ converges to k strongly in $L^1(0,T)$ by assumption. Hence

$$\int_0^T |\langle (k_\delta - \widehat{k}_\delta^\varepsilon) * g_\delta, u_\delta' \rangle| \le 3M^2 \varepsilon \quad \text{whenever } \delta \le \delta_\varepsilon^*$$

for some $\delta_\varepsilon^* > 0$. Therefore, in order to conclude, it is sufficient to prove that

$$\lim_{\delta \to 0} \int_0^T \langle \widehat{k}_\delta^\varepsilon * g_\delta, u_\delta' \rangle = \int_0^T \langle k^\varepsilon * g, u' \rangle. \tag{4.38}$$

Indeed, this will yield (4.36) with some δ_ε and, say, $(4M^2+1)\varepsilon$ instead of ε. As k^ε is smooth, we can integrate by parts as follows

$$\int_0^T \langle \widehat{k}_\delta^\varepsilon * g_\delta, u_\delta' \rangle = \langle (\widehat{k}_\delta^\varepsilon * g_\delta)(T), u_\delta(T) \rangle - \int_0^T \langle (\widehat{k}_\delta^\varepsilon * g_\delta)', u_\delta \rangle$$
$$= \langle (\widehat{k}_\delta^\varepsilon * g_\delta)(T), u_\delta(T) \rangle - \widehat{k}_\delta^\varepsilon(0) \int_0^T \langle g_\delta, u_\delta \rangle - \int_0^T \langle (\widehat{k}_\delta^\varepsilon)' * g_\delta, u_\delta \rangle \tag{4.39}$$

and an analogous formula holds for the right hand side of (4.38). Hence, we show that each term of (4.39) converges to the corresponding term of such a formula. We have

$$|\langle (\widehat{k}_\delta^\varepsilon * g_\delta)(T), u_\delta(T) \rangle - \langle (k^\varepsilon * g)(T), u(T) \rangle|$$
$$\le |((\widehat{k}_\delta^\varepsilon - k^\varepsilon) * g_\delta)(T)|_* |u_\delta(T)| + |\langle (k^\varepsilon * g_\delta)(T), u_\delta(T) \rangle - \langle (k^\varepsilon * g)(T), u(T) \rangle|$$
$$\le M^2 \|\widehat{k}_\delta^\varepsilon - k^\varepsilon\|_{L^p(0,T)} + |\langle (k^\varepsilon * g_\delta)(T), u_\delta(T) \rangle - \langle (k^\varepsilon * g)(T), u(T) \rangle|.$$

So, as $\{\widehat{k}_\delta^\varepsilon\}$ converges to k^ε strongly in $L^p(0,T)$ (even much better) as $\delta \to 0$ and $\{u_\delta(T)\}$ converges to $u(T)$ strongly in H, it suffices to prove that $(k^\varepsilon * g_\delta)(T)$ converges to $(k^\varepsilon * g)(T)$ weakly in H^*. This means that

$$\lim_{\delta \to 0} \int_0^T \langle g_\delta(s), k^\varepsilon(T-s)v \rangle\, ds = \int_0^T \langle g(s), k^\varepsilon(T-s)v \rangle\, ds \quad \text{for every } v \in H$$

and it is true, since $g_\delta \to g$ weakly in $L^{p'}(0,T;H^*)$. Thus, we have shown that the first term on the right hand side of (4.39) converges to the desired limit. The second term is trivial to deal with, and the last one properly converges. Indeed, $u_\delta \to u$ strongly in $C^0([0,T];H)$ and $(\widehat{k}^\varepsilon_\delta)' * g_\delta \to (k^\varepsilon)' * g$ weakly in $L^{p'}(0,T;H^*)$, since $g_\delta \to g$ weakly in $L^{p'}(0,T;H^*)$ and $(\widehat{k}^\varepsilon_\delta)' \to (k^\varepsilon)'$ strongly in $L^1(0,T)$ (even uniformly) as $\delta \to 0$, as k^ε is C^1. ∎

Let us come back to the proof of Theorem 2.1. As we said before, it remains to check (4.35). We set $g^\tau := \widetilde{f}_2^\tau - w^\tau$ and $g := \widetilde{f}_2 - w$ for convenience. By (3.9)–(3.10) we have

$$f_2 = \widetilde{f}_2 + k * \widetilde{f}_2 \quad \text{and} \quad f_2^\tau = \widetilde{f}_2^\tau + k^\tau *_\tau \widetilde{f}_2^\tau$$

whence, we can write

$$\zeta = f_1 - w + \widetilde{f}_2 + k * g \quad \text{and} \quad \overline{\zeta}_\tau = \overline{f}_{1,\tau} - \overline{w}_\tau + \overline{\overline{f}}_{2,\tau} + \overline{(k^\tau *_\tau g^\tau)}_\tau.$$

So, we have to compare each term coming from the above right hand sides after testing by u' and by $\widehat{u}'_\tau$, respectively. The integral involving $\overline{f}_{1,\tau}$ converges properly, due to (4.9) and (4.29). The same holds for the one regarding $\overline{\overline{f}}_{2,\tau}$, thanks to (4.10) and (4.30). Next, we observe that (4.29) implies that $\widehat{u}_\tau(T)$ converges to $u(T)$ weakly in V. Hence, using the definition of subdifferential and the l.s.c. of ψ, we have

$$\begin{aligned}
&\limsup_{\tau\to 0} \int_0^T \langle -\overline{w}_\tau, \widehat{u}'_\tau \rangle = -\liminf_{\tau\to 0} \tau \sum_{i=1}^N \langle w_i, \delta u_i \rangle = -\liminf_{\tau\to 0} \sum_{i=1}^N \langle w_i, u_i - u_{i-1} \rangle \\
&\le -\liminf_{\tau\to 0} \sum_{i=1}^N \bigl(\psi(u_i) - \psi(u_{i-1})\bigr) = \psi(u^0) - \liminf_{\tau\to 0} \psi(\widehat{u}_\tau(T)) \\
&\le \psi(u^0) - \psi(u(T)) = -\int_0^T \langle w, u' \rangle.
\end{aligned}$$

Finally, we deal with the last term. We split it this way

$$\int_0^T \langle \overline{(k^\tau *_\tau g^\tau)}_\tau, \widehat{u}'_\tau \rangle = \int_0^T \langle \overline{(k^\tau *_\tau g^\tau)}_\tau - \overline{k}_\tau * \overline{g}_\tau, \widehat{u}'_\tau \rangle + \int_0^T \langle \overline{k}_\tau * \overline{g}_\tau, \widehat{u}'_\tau \rangle$$

and observe that the first integral on the right hand side tends to zero. We have indeed

$$\left| \int_0^T \langle \overline{(k^\tau *_\tau g^\tau)}_\tau - \overline{k}_\tau * \overline{g}_\tau, \widehat{u}'_\tau \rangle \right| \le c\tau \|\overline{g}_\tau\|_{L^{p'}(0,T;H^*)} \|\widehat{u}'_\tau\|_{L^p(0,T;H)}$$

by Proposition 3.3. On the other hand, we can apply Lemma 4.3 with $\delta = \tau$, $k_\tau = \overline{k}_\tau$, $u_\tau = \widehat{u}_\tau$, and $g_\tau = \overline{g}_\tau$, owing to Proposition 3.5 and to the convergences (4.29) and (4.33). This yields

$$\lim_{\tau\to 0} \int_0^T \langle \overline{k}_\tau * \overline{g}_\tau, \widehat{u}'_\tau \rangle = \int_0^T \langle k * g, u' \rangle.$$

Therefore, (4.35) is actually true and the whole proof of Theorem 2.1 is complete.

Remark 4.4. As said in Remark 2.2, the existence result given by Theorem 2.1 can be extended to the case $k \in L^\infty(0,T)$. Here is the outline of the proof. Given $k \in L^\infty(0,T)$, we approximate it with a family $\{k_\delta\}_{\delta\in(0,1)}$ of smooth kernels such that

$$\begin{aligned}
&k_\delta \to k \quad \text{strongly in } L^1(0,T) \quad \text{as } \delta \to 0 \\
&\|k_\delta\|_{L^\infty(0,T)} \le \|k\|_{L^\infty(0,T)} \quad \text{for any } \delta \in (0,1).
\end{aligned}$$

Then, we consider the problem (P_δ) obtained by replacing k with k_δ in (2.5)–(2.7), namely (with a concise notation for convenience)

$$\zeta_\delta + w_\delta + k_\delta * w_\delta \ni f, \quad \zeta_\delta \in A(u'_\delta), \quad w_\delta \in B(u_\delta), \quad \text{and} \quad u_\delta(0) = u^0 \tag{4.40}$$

and solve it with the procedure we used in the above proof. Thus, we find a solution $(u_\delta, \zeta_\delta, w_\delta)$ to (4.40) that is a limit point of solutions to the corresponding discrete problems. As we stressed in Remark 4.2, the values of the constants c we have found in the a priori estimates depend on (a bound of) the L^∞ norm of the kernel (here k_δ) rather than on the BV norm of it. Therefore, such a priori estimates are uniform with respect to δ. Moreover, they are conserved in the limit as $\tau \to 0$, i.e., they hold for $(u_\delta, \zeta_\delta, w_\delta)$ as well, and we can find convergences analogous to (4.29)–(4.33) for $(u_\delta, \zeta_\delta, w_\delta)$ to some (u, ζ, w) (for a subsequence). Clearly, the regularity conditions (2.12), equation (2.5), and the Cauchy condition (2.7) are satisfied. Moreover, we have $w \in B(u)$ since

$$\begin{aligned}\lim_{\delta\to 0} \int_0^T \langle w_\delta, u_\delta \rangle &= \lim_{\delta\to 0} \int_0^T \langle w_\delta - \widetilde{f}_2, u_\delta \rangle + \lim_{\delta\to 0} \int_0^T \langle \widetilde{f}_2, u_\delta \rangle \\ &= \int_0^T \langle w - \widetilde{f}_2, u \rangle + \int_0^T \langle \widetilde{f}_2, u \rangle = \int_0^T \langle w, u \rangle\end{aligned}$$

as before. Finally, we have $\zeta \in A(u')$, as we sketch. We use [8, Lemma 1.3, p. 42] once more as follows. We have

$$\zeta_\delta = \widetilde{f}_1 - w_\delta + \widetilde{f}_2 + k_\delta * (\widetilde{f}_2 - w_\delta) \quad \text{and} \quad \zeta = \widetilde{f}_2 - w + \widetilde{f}_2 + k * (\widetilde{f}_2 - w)$$

and we test such equations by u'_δ and by u', respectively, and integrate over $(0, T)$. Then, we compare the corresponding terms, separately. The first one clearly converges to the desired limit. For the second term we have

$$\begin{aligned}\limsup_{\delta\to 0} \int_0^T \langle -w_\delta, u'_\delta \rangle &= -\liminf_{\delta\to 0} \psi(u_\delta(T)) + \psi(u^0) \\ &\le -\psi(u(T)) + \psi(u^0) = \int_0^T \langle -w, u' \rangle.\end{aligned}$$

The next integral is easily treated with an integration by parts, namely

$$\begin{aligned}\lim_{\delta\to 0} \int_0^T \langle \widetilde{f}_2, u'_\delta \rangle &= \lim_{\delta\to 0} \langle \widetilde{f}_2(T), u_\delta(T) \rangle - \langle \widetilde{f}_2(0), u^0 \rangle - \lim_{\delta\to 0} \int_0^T \langle \widetilde{f}'_2, u_\delta \rangle \\ &= \langle \widetilde{f}_2(T), u(T) \rangle - \langle \widetilde{f}_2(0), u^0 \rangle - \int_0^T \langle \widetilde{f}'_2, u \rangle = \int_0^T \langle \widetilde{f}_2, u' \rangle\end{aligned}$$

since $\{u_\delta(t)\}$ converges to $u(t)$ weakly in V for every $t \in [0, T]$. Finally, Lemma 4.3 applied with $g_\delta := \widetilde{f}_2 - w_\delta$ yields

$$\lim_{\delta\to 0} \int_0^T \langle k_\delta * (\widetilde{f}_2 - w_\delta), u'_\delta \rangle = \int_0^T \langle k * (\widetilde{f}_2 - w), u' \rangle.$$

Therefore, we can conclude that

$$\limsup_{\delta\to 0} \int_0^T \langle \zeta_\delta, u'_\delta \rangle \le \int_0^T \langle \zeta, u' \rangle$$

and derive that $\zeta \in A(u')$.

5. Proof of Theorem 2.3. As in the previous proof, we first make a proper choice of the datum of the discrete problem and then start estimating. As f is smooth (see (2.18)), we behave as we did before for f_2, i.e., we define $f^\tau = (f_i) \in (V^*)^{N+1}$ by

$$f_i := f(i\tau) \quad \text{for } i = 0, \dots, N. \tag{5.1}$$

Hence, we have bounds and convergences similar to those of Remark 4.1. We shall use just the following ones

$$\|\widehat{f}_\tau'\|_{L^2(0,T;V^*)} = \tau \sum_{i=1}^N \|\delta f_i\|_*^2 \le \|f'\|^2_{L^2(0,T;V^*)} \tag{5.2}$$

$$\overline{\overline{f}}_\tau \to \widetilde{f} \quad \text{strongly in } C^0([0,T];V^*). \tag{5.3}$$

Moreover, taking the compatibility condition contained in (2.18) into account, we set $w_0 := B(u^0)$ and recall that

$$\zeta_0 := f_0 - w_0 \in D(\varphi^*). \tag{5.4}$$

As $u_0 = u^0$ and $(k^\tau *_\tau w^\tau)_0 = 0$, we have

$$\zeta_i + w_i + (k^\tau *_\tau w^\tau)_i = f_i \quad \text{and} \quad w_i \in B(u_i) \quad \text{for } i = 0, \dots, N \tag{5.5}$$

$$\zeta_0 \in D(\varphi^*) \quad \text{and} \quad \zeta_i \in A(\delta u_i) \quad \text{for } i = 1, \dots, N. \tag{5.6}$$

¿From (5.5), we derive that

$$\delta\zeta_i + \delta w_i + \delta(k^\tau *_\tau w^\tau)_i = \delta f_i \quad \text{for } i = 1, \dots, N \tag{5.7}$$

and we test it by $u_i - u_{i-1}$. Then, we sum over $1 \le i \le n$ for an arbitrary $n \le N$. Hence, we have

$$\sum_{i=1}^n \langle \delta\zeta_i, u_i - u_{i-1}\rangle + \sum_{i=1}^n \langle \delta w_i, u_i - u_{i-1}\rangle = \sum_{i=1}^n \langle \delta f_i, u_i - u_{i-1}\rangle - \sum_{i=1}^n \langle \delta(k^\tau *_\tau w^\tau)_i, u_i - u_{i-1}\rangle \tag{5.8}$$

and we now estimate each term of (5.8), separately. As far as the symbols c and c_σ for constants are concerned, we still follow the general notation explained at the end of Sect. 3. In the sequel, σ is a positive parameter, whose value will be chosen later.

In view of (2.14) and of the relation between A and φ^*, we have $\delta u_i \in \partial\varphi^*(\zeta_i)$. Moreover, we recall (2.20) (see Remark 2.4). Hence, we have

$$\sum_{i=1}^n \langle \delta\zeta_i, u_i - u_{i-1}\rangle = \sum_{i=1}^n \langle \zeta_i - \zeta_{i-1}, \delta u_i\rangle \ge \sum_{i=1}^n \Big(\varphi^*(\zeta_i) - \varphi^*(\zeta_{i-1})\Big) = \varphi^*(\zeta_n) - \varphi^*(\zeta_0) \ge \alpha|\zeta_n|_*^2 - c. \tag{5.9}$$

Next, owing to the strong monotonicity assumption (2.17), we have

$$\sum_{i=1}^n \langle \delta w_i, u_i - u_{i-1}\rangle = \frac{1}{\tau}\sum_{i=1}^n \langle w_i - w_{i-1}, u_i - u_{i-1}\rangle \ge \frac{\alpha}{\tau}\sum_{i=1}^n \|u_i - u_{i-1}\|^2 = \alpha\tau \sum_{i=1}^n \|\delta u_i\|^2. \tag{5.10}$$

Concerning the first term on the right hand side, we take (5.2) into account and have

$$\begin{aligned}\sum_{i=1}^{n}\langle \delta f_i, u_i - u_{i-1}\rangle &= \tau\sum_{i=1}^{n}\langle \delta f_i, \delta u_i\rangle \\ &\le \sigma\tau\sum_{i=1}^{n}\|\delta u_i\|^2 + c_\sigma\tau\sum_{i=1}^{n}\|\delta f_i\|_*^2 \le \sigma\tau\sum_{i=1}^{n}\|\delta u_i\|^2 + c_\sigma. \qquad (5.11)\end{aligned}$$

Finally, we treat the convolution term using the discrete derivative formula (3.5) this way

$$\begin{aligned}-\sum_{i=1}^{n}\langle \delta(k^\tau *_\tau w^\tau)_i, u_i - u_{i-1}\rangle &= -\tau\sum_{i=1}^{n}\langle k_0 w_i + (\delta k^\tau *_\tau w^\tau)_i, \delta u_i\rangle \\ &\le \sigma\tau\sum_{i=1}^{n}\|\delta u_i\|^2 + c_\sigma\tau\sum_{i=1}^{n}\|w_i\|_*^2 + c_\sigma\tau\sum_{i=1}^{n}\|(\delta k^\tau *_\tau w^\tau)_i\|_*^2. \qquad (5.12)\end{aligned}$$

Now, the discrete Young theorem (see Proposition 3.2) gives

$$\begin{aligned}&\tau\sum_{i=1}^{n}\|(\delta k^\tau *_\tau w^\tau)_i\|_*^2 = \|\overline{(\delta k^\tau *_\tau w^\tau)}_\tau\|_{L^2(0,n\tau;V^*)}^2 \\ &\le \|\overline{(\delta k^\tau)}_\tau\|_{L^1(0,T)}^2 \|\overline{w}_\tau\|_{L^2(0,n\tau;V^*)}^2 = \sum_{i=1}^{n}|k_i - k_{i-1}|\cdot\tau\sum_{i=1}^{n}\|w_i\|_*^2 \le c\tau\sum_{i=1}^{n}\|w_i\|_*^2.\end{aligned}$$

On the other hand, it is clear that $w_i = w_0 + \tau\sum_{j=1}^{i}\delta w_j$. Hence, using the Lipschitz continuity assumption (2.16), we easily obtain

$$\|w_i\|_*^2 \le 2\|w_0\|_*^2 + 2\Big(\tau\sum_{j=1}^{i}\|\delta w_j\|_*\Big)^2 \le 2\|w_0\|_*^2 + 2n\tau^2\sum_{j=1}^{i}\|\delta w_j\|_*^2 \le c + c\tau\sum_{j=1}^{i}\|\delta u_j\|^2.$$

Therefore, we estimate the sum of such contributions as follows

$$\tau\sum_{i=1}^{n}\|w_i\|_*^2 \le c + c\tau^2\sum_{i=1}^{n}\sum_{j=1}^{i}\|\delta u_j\|^2 = c + c\tau^2\sum_{j=1}^{n}\|\delta u_j\|^2 + c\tau\sum_{i=1}^{n-1}\tau\sum_{j=1}^{i}\|\delta u_j\|^2$$

and (5.12) becomes

$$\begin{aligned}&-\sum_{i=1}^{n}\langle \delta(k^\tau *_\tau w^\tau)_i, u_i - u_{i-1}\rangle \\ &\le \sigma\tau\sum_{i=1}^{n}\|\delta u_i\|^2 + c_\sigma + c_\sigma\tau^2\sum_{j=1}^{n}\|\delta u_j\|^2 + c_\sigma\tau\sum_{i=1}^{n-1}\tau\sum_{j=1}^{i}\|\delta u_j\|^2. \qquad (5.13)\end{aligned}$$

At this point, we collect (5.8)–(5.11) and (5.13). Next, we choose first σ and then τ_0 small enough and conclude that

$$|\zeta_n|_*^2 + \tau\sum_{i=1}^{n}\|\delta u_i\|^2 \le c + c\tau\sum_{i=1}^{n-1}\tau\sum_{j=1}^{i}\|\delta u_j\|^2 \quad \text{provided that } \tau \le \tau_0.$$

Finally, we apply the discrete Gronwall lemma (see (3.23)) and obtain the desired estimate

$$|\zeta_n|_*^2 + \tau \sum_{i=1}^{n} \|\delta u_i\|^2 \le c \quad \text{for } n = 1, \ldots, N. \tag{5.14}$$

We easily derive further estimates from (5.14). We proceed as follows. Taking firstly the Lipschitz continuity assumption (2.16) into account and then applying the derivative formula and the discrete Young theorem once more, we infer that

$$\tau \sum_{i=1}^{N} \|\delta w_i\|_*^2 + \tau \sum_{i=1}^{N} \|\delta(k^\tau *_\tau w^\tau)_i\|_*^2 \le c. \tag{5.15}$$

Next, owing now to (5.2) for the forcing term of equation (5.7), we conclude by comparison that

$$\tau \sum_{i=1}^{N} \|\delta\zeta_i\|_*^2 \le c. \tag{5.16}$$

Now, as in the previous section, we read the above estimates in terms of the interpolants and find weakly, weakly star, and strongly convergent subsequences. In fact, (5.14)–(5.16) mean

$$\|\overline{\zeta}_\tau\|_{L^\infty(0,T;H^*)} = \|\widehat{\zeta}_\tau\|_{L^\infty(0,T;H^*)} \le c, \quad \|\widehat{u}_\tau'\|_{L^2(0,T;V)} \le c,$$
$$\|\widehat{w}_\tau'\|_{L^2(0,T;V^*)} \le c, \quad \text{and} \quad \|\widehat{\zeta}_\tau'\|_{L^2(0,T;V^*)} \le c$$

whence we immediately deduce

$$\|\overline{u}_\tau\|_{L^\infty(0,T;V)} = \|\widehat{u}_\tau\|_{L^\infty(0,T;V)} \le c \quad \text{and} \quad \|\overline{w}_\tau\|_{L^\infty(0,T;V^*)} = \|\widehat{w}_\tau\|_{L^\infty(0,T;V^*)} \le c.$$

Hence, owing to the compactness results already used and to (3.3), we have

$$\widehat{u}_\tau \to u \quad \text{weakly in } H^1(0,T;V) \text{ and strongly in } C^0([0,T];H) \tag{5.17}$$
$$\overline{u}_\tau \to u \quad \text{weakly star in } L^\infty(0,T;V) \text{ and strongly in } L^\infty(0,T;H) \tag{5.18}$$
$$\widehat{\zeta}_\tau \to \zeta \quad \text{weakly in } H^1(0,T;V^*) \text{ and strongly in } C^0([0,T];V^*) \tag{5.19}$$
$$\overline{\zeta}_\tau \to \zeta \quad \text{weakly star in } L^\infty(0,T;H^*) \text{ and strongly in } L^\infty(0,T;V^*) \tag{5.20}$$
$$\widehat{w}_\tau \to w \quad \text{weakly in } H^1(0,T;V^*) \tag{5.21}$$
$$\overline{w}_\tau \to w \quad \text{weakly star in } L^\infty(0,T;V^*). \tag{5.22}$$

Moreover, the discrete scheme reads

$$\overline{\zeta}_\tau + \overline{w}_\tau + \overline{(k^\tau *_\tau w^\tau)}_\tau = \overline{f}_\tau, \quad \overline{\zeta}_\tau \in A(\widehat{u}_\tau'), \quad \text{and} \quad \overline{w}_\tau \in B(\overline{u}_\tau) \tag{5.23}$$

so that the above convergences and Proposition 3.3 imply that equation (2.5) and the Cauchy condition (2.7) are satisfied. Thus, it remains to identify the limits of the nonlinear terms, i.e., to check relations (2.6). The weak convergence (5.17) and the strong convergence (5.20) immediately imply that

$$\lim_{\tau \to 0} \int_0^T \langle \overline{\zeta}_\tau, \widehat{u}_\tau' \rangle = \int_0^T \langle \zeta, u' \rangle.$$

As $\overline{\zeta}_\tau \in A(\widehat{u}'_\tau)$ for every τ, we deduce that $\zeta \in A(u')$ by [8, Lemma 1.3, p. 42]. In order to prove that $w \in B(u)$, we use the transformed equations

$$\overline{\zeta}_\tau + \overline{w}_\tau = \overline{\overline{f}}_\tau + \overline{(\rho^\tau *_\tau \zeta^\tau)}_\tau \quad \text{and} \quad \zeta + w = \widetilde{f} + \rho * \zeta$$

and observe that (5.20), Proposition 3.5, and Corollary 3.4 imply that

$$(\overline{w}_\tau - \overline{\overline{f}}_\tau) \to (w - \widetilde{f}) \quad \text{weakly in } L^2(0,T;H^*).$$

Therefore, using (5.3), and both convergences (5.18), we obtain

$$\lim_{\tau\to 0} \int_0^T \langle \overline{w}_\tau, \overline{u}_\tau \rangle = \lim_{\tau\to 0} \int_0^T \langle \overline{w}_\tau - \overline{\overline{f}}_\tau, \overline{u}_\tau \rangle + \lim_{\tau\to 0} \int_0^T \langle \overline{\overline{f}}_\tau, \overline{u}_\tau \rangle$$
$$= \int_0^T \langle w - \widetilde{f}, u \rangle + \int_0^T \langle \widetilde{f}, u \rangle = \int_0^T \langle w, u \rangle$$

and conclude that $w \in B(u)$ as well. This completes the proof.

Acknowledgment. The authors would like to acknowledge financial support from the MIUR-COFIN 2004 research program on "Mathematical Modeling and Analysis of Free Boundary Problems".

REFERENCES

[1] R. A. Adams *Sobolev spaces*, Pure and Applied Mathematics, Vol. 65, Academic Press, New York-London, 1975.

[2] S. Aizicovici, P. Colli, and M. Grasselli, *On a class of degenerate nonlinear Volterra equations*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur., 132 (1998), 135–152.

[3] ———, *Doubly nonlinear evolution equations with memory*, Funkcial. Ekvac., 44 (2001), 1:19–51.

[4] S. Aizicovici and V.-M. Hokkanen, *Doubly nonlinear equations with unbounded operators*, Nonlinear Anal., 58 (2004), 5-6:591–607.

[5] ———, *Doubly nonlinear periodic problems with unbounded operators*, J. Math. Anal. Appl., 292 (2004), 2:540–557.

[6] H. W. Alt and S. Luckhaus, *Quasilinear elliptic-parabolic differential equations*, Math. Z., 183 (1983), 3:311–341.

[7] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[8] V. Barbu *Nonlinear semigroups and differential equations in Banach spaces.* Noordhoff International Publishing, Leyden, 1976.

[9] V. Barbu, *Existence for a nonlinear Volterra equation in Hilbert spaces*, SIAM J. Math. Anal., 10 (1979), 3:552–569.

[10] V. Barbu, P. Colli, G. Gilardi, and M. Grasselli, *Existence, uniqueness, and longtime behavior for a nonlinear Volterra integrodifferential equation*, Differential Integral Equations, 13 (2000), 10-12:1233–1262.

[11] F. Bernis, *Existence results for doubly nonlinear higher order parabolic equations on unbounded domains*, Math. Ann., 279 (1988), 3:373–394.

[12] P. Colli, *On some doubly nonlinear evolution equations in Banach spaces*, Japan J. Indust. Appl. Math., 9 (1992), 2:181–203.

[13] P. Colli and M. Grasselli, *Nonlinear hyperbolic problems modelling transition dynamics with memory*, In *Nonlinear analysis and applications (Warsaw, 1994)*, volume 7 of *GAKUTO Internat. Ser. Math. Sci. Appl.*, 79–99. Gakkōtosho, Tokyo, 1996.

[14] ———, *Degenerate nonlinear Volterra integrodifferential equations*, In *Volterra equations and applications (Arlington, TX, 1996)*, volume 10 of *Stability Control Theory Methods Appl.*, 187–195. Gordon and Breach, Amsterdam, 2000.

[15] P. Colli and A. Visintin, *On a class of doubly nonlinear evolution equations*, Comm. Partial Differential Equations, 15 (1990), 5:737–756.
[16] E. Di Benedetto and R. E. Showalter, *Implicit degenerate evolution equations and applications*, SIAM J. Math. Anal., 12 (1981), 5:731–751.
[17] H. Gajewski and I. V. Skrypnik, *To the uniqueness problem for nonlinear parabolic equations*, Discrete Contin. Dyn. Syst., 10 (2004) 1-2:315–336.
[18] G. Gilardi and U. Stefanelli, *Time-discretization and global solution for a doubly nonlinear Volterra equation*, J. Differential Equations, (2006), to appear.
[19] O. Grange and F. Mignot, *Sur la résolution d'une équation et d'une inéquation paraboliques non linéaires*, J. Functional Analysis, 11 (1972), 77–92.
[20] V.-M. Hokkanen, *An implicit nonlinear time dependent equation has a solution*, J. Math. Anal. Appl., 161 (1991), 1:117–141.
[21] ———, *Existence for nonlinear time dependent Volterra equations in Hilbert spaces*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat., 38 (1992), 1:29–49.
[22] ———, *On nonlinear Volterra equations in Hilbert spaces*, Differential Integral Equations, 5 (1992), 3:647–669.
[23] J. W. Jerome, *Approximation of nonlinear evolution systems.* Number 164 in Math. Sci. Engrg. Academic Press, Orlando, 1983.
[24] E. Maitre and P. Witomski, *A pseudo-monotonicity adapted to doubly nonlinear elliptic-parabolic equations*, Nonlinear Anal., 50 (2002), 2:223–250.
[25] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. (4), 146 (1987), 65–96.
[26] U. Stefanelli, *Well-posedness and time discretization of a nonlinear Volterra integrodifferential equation*, J. Integral Equations Appl., 13 (2001), 3:273–304.
[27] ———, *On a class of doubly nonlinear nonlocal evolution equations*, Differential Integral Equations, 15 (2002), 8:897–922.
[28] ———, *On some nonlocal evolution equations in Banach spaces*, J. Evol. Equ., 4 (2004), 1:1–26.