On a conserved phase field model with irregular potentials and dynamic boundary conditions *

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Abstract

A system coupling the heat equation for temperature and the Cahn-Hilliard equation for the conserved order parameter is studied. The phase dynamic is possibly singular due to irregular potentials. A well-posedness result is proved for a boundary value problem involving dynamic boundary conditions for the order parameter.

Sunto

Si considera un sistema di tipo “campo di fase conservativo” nel quale l’equazione del calore per la temperatura è accoppiata all’equazione di Cahn-Hilliard per il parametro d’ordine. In quest’ultima si consentono potenziali irregolari. Si dimostra un risultato di buona positura per un problema ai limiti in cui intervengono condizioni di tipo dinamico per il parametro d’ordine.

Key words: Phase field models, Cahn-Hilliard equation, dynamic boundary conditions, irregular potentials, phase separation.

AMS (MOS) Subject Classification: 35K55 (35K50, 82C26)

1 Introduction

In the last decades, several models for two phase systems have been introduced and the literature contains a number of results. In particular, when phase separation is considered, the Cahn-Hilliard equation (see [2]) plays a central role and a system like

$$\partial_t\vartheta + \lambda'(\chi)\,\partial_t\chi + \operatorname{div}\mathbf{q} = f \tag{1.1}$$

$$\partial_t\chi - \Delta w = 0 \tag{1.2}$$

$$w = -\Delta\chi + \mathcal{W}'(\chi) - \lambda'(\chi)\,\vartheta \tag{1.3}$$

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is accepted as a good model. In such equations, ϑ, $\mathbf{q}$, f, χ, $\lambda'(\chi)$, and w denote the temperature, the heat flux, a source term, the order parameter, the latent heat density (which is generally allowed to depend on χ), and the chemical potential, respectively, and it is understood that (1.1)–(1.3) hold in a domain $\Omega \subset \mathbb{R}^3$ and in a given time interval $(0,T)$. Moreover, $\mathcal{W}$ is a double well potential. Finally, some constants have been normalized to 1, since we are interested in the mathematical aspects of the system. A typical choice of $\mathcal{W}$ is the following

$$\mathcal{W}(r) = (r^2-1)^2 \quad \text{for } r \in \mathbb{R} \tag{1.4}$$

whose wells are located at $r = \pm 1$, and the most part of the literature regards such a potential or, more generally, everywhere defined smooth potentials. However, the so-called logarithmic potential, namely

$$\mathcal{W}(r) = \int_0^r \left(-2cs + \ln \frac{1+s}{1-s} \right) ds \quad \text{for } |r| < 1 \tag{1.5}$$

is thermodynamically relevant. Note that it actually provides a double well if $c > 1$. More generally, one can consider potentials containing constraints on the order parameter, and an example is the following

$$\mathcal{W} = \text{indicator function of } [-1,1] + \text{smooth function.} \tag{1.6}$$

Note that (1.6) can be seen as a generalized double well potential if its smooth part is concave.

As far as the boundary conditions on $\Gamma := \partial\Omega$ are concerned, there is some freedom in choosing some of them (namely, those for ϑ and χ that have to be coupled to (1.1) and (1.3)), while the homogenous Neumann boundary condition $\partial_n w = 0$ is essentially mandatory for (1.2), since it implies conservation for the total mass of χ. If such a boundary condition holds, an integration of (1.2) over Ω and the divergence theorem yield indeed

$$\partial_t \int_\Omega \chi = \int_\Omega \partial_t \chi = 0, \quad \text{whence} \quad \int_\Omega \chi(t) = \int_\Omega \chi_0 \quad \text{for every } t \in [0,T]. \tag{1.7}$$

For that reason, one speaks of a conserved model. Nevertheless, in most works, the Neumann boundary is considered for χ as well. On the other hand, physicists have recently proposed to endow (1.3) with the so-called dynamic boundary conditions, namely

$$\partial_t \chi + (\partial_n \chi)|_\Gamma - \Delta_\Gamma \chi + \mathcal{W}'_\Gamma(\chi) = f_\Gamma \tag{1.8}$$

where ∂_n is the outer normal derivative and Δ_Γ is the Laplace-Beltrami operator. Furthermore, $\mathcal{W}_\Gamma$ is a given boundary potential and f_Γ is a forcing term. However, one can consider a boundary condition like

$$\partial_t \chi + (\partial_n \chi)|_\Gamma + \kappa_\Gamma (\chi - \chi_\Gamma) = 0 \tag{1.9}$$

as well, where κ_Γ is a positive constant and χ_Γ is a given function on the boundary. This is a dynamic version of the Robin boundary condition and can be seen as a particular case of (1.8) if one forgets the Laplace-Beltrami operator there.

Problems involving the Cahn-Hilliard equations endowed with the dynamic boundary condition (1.8) have been studied just in the last years, and generally by taking the choice (1.4) or assuming a big regularity for the double well potential (see [3], [8], [10], [12], [14]). Moreover, a phase field system of type (1.11)–(1.13) with a smooth $\mathcal{W}$ and the dynamic boundary condition (1.8) is studied in [11]. Very recently, well-posedness results are proved in [5] for the Cahn-Hilliard equations with constant temperature and dynamic boundary conditions, without assuming the potentials to be smooth. We will widely refer to [5], and quote it here for a list of references on the whole subject.

In [5], the only assumption on $\mathcal{W}$ is that it can be written as the sum of a convex, proper, lower semicontinuous function and a smoother perturbation, essentially (important further assumptions are compatibility conditions on the structure of (1.8) with respect to $\mathcal{W}$, indeed). In particular, both potentials (1.5) and (1.6) are included. Furthermore, both (1.8) and (1.9) are allowed as possible boundary conditions and even the so-called viscous Cahn-Hilliard equation (obtained by adding $\partial_t \chi$ to the right-hand side of (1.13), see, e.g., [6] and [9]) is considered.

In this paper, we study the corresponding non-isothermal case. In order to avoid any further complication, we assume the simplest Fourier law for the heat flux and a Robin boundary conditions for temperature, namely

$$\mathbf{q} = -\nabla \vartheta \quad \text{and} \quad \mathbf{q} \cdot \mathbf{n} = \kappa(\vartheta - \vartheta_\Gamma) \tag{1.10}$$

where $\mathbf{n}$ is the outward normal unit vector on Γ and the positive constant κ and the function ϑ_Γ are prescribed. Therefore, the system we are interested in is the following

$$\partial_t \big(\vartheta + \lambda(\chi)\big) - \Delta\vartheta = f \quad \text{in } \Omega \tag{1.11}$$
$$\partial_t \chi - \Delta w = 0 \quad \text{in } \Omega \tag{1.12}$$
$$w = \tau\, \partial_t \chi - \Delta \chi + \beta(\chi) + \pi(\chi) - \lambda'(\chi)\, \vartheta \quad \text{in } \Omega \tag{1.13}$$
$$\partial_n \vartheta + \kappa(\vartheta - \vartheta_\Gamma) = 0 \quad \text{and} \quad \partial_n w = 0 \quad \text{on } \Gamma \tag{1.14}$$
$$\partial_t \chi + (\partial_n \chi)|_\Gamma - \nu \Delta_\Gamma \chi + \beta_\Gamma(\chi) + \pi_\Gamma(\chi) = f_\Gamma \quad \text{on } \Gamma \tag{1.15}$$
$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \chi(0) = \chi_0 \quad \text{in } \Omega. \tag{1.16}$$

Such a system is related to the former as follows. As in [5], we have split $\mathcal{W}'$ and $\mathcal{W}'_\Gamma$ as $\mathcal{W}' = \beta + \pi$ and $\mathcal{W}'_\Gamma = \beta_\Gamma + \pi_\Gamma$, respectively, where β and β_Γ are monotone, possibly non-smooth and even multi-valued, while π and π_Γ are Lipschitz continuous perturbations, and we are interested in keeping β and β_Γ as general as possible (especially β). We note that, if β and β_Γ actually are multi-valued, equations (1.12)–(1.13) have to be read as differential inclusions. Furthermore, τ and ν are nonnegative parameters. Thus, we obtain the viscous Cahn-Hilliard equations if $\tau > 0$. On the other hand, the choice $\nu = 0$ corresponds to forget the Laplace-Beltrami operator in the dynamic boundary condition (1.15). Hence, (1.9) is included as a particular case. Finally, the initial data ϑ_0 and χ_0 are given.

The present paper deals with well-posedness for the above problem. We can show uniqueness in a very general framework and global existence under further assumptions on the nonlinearities. We still allow β and β_Γ to be essentially arbitrary, but we assume some compatibility between them as in [5]. Roughly speaking, we suppose that $\beta_\Gamma(r)$ does not grow faster than $\beta(r)$ as r approaches the boundary of the domain of β (or for large $|r|$ if β is everywhere defined) and that β_Γ dominates the boundary contributions of (1.15) given by π_Γ and f_Γ.

2 Main results

In this section, we precisely state the problem we are going to deal with and the results we prove in the subsequent sections. The notation given in the Introduction is still kept as far as Ω, Γ, ∂_n, and T are concerned. More precisely, we assume $\Omega \subset \mathbb{R}^3$ (but the cases $\Omega \subset \mathbb{R}^d$ with

$1 \leq d \leq 3$ could be considered as well), to be bounded, connected, and smooth, and write $|\Omega|$ for its Lebesgue measure. Similarly, $|\Gamma|$ denotes the 2-dimensional measure of Γ. We set for convenience

$$Q_t := \Omega \times (0,t) \quad \text{and} \quad \Sigma_t := \Gamma \times (0,t) \quad \text{for every } t \in (0,T] \tag{2.1}$$

$$Q := Q_T\,, \quad \text{and} \quad \Sigma := \Sigma_T\,. \tag{2.2}$$

The main features of the structure of our system are described below and further assumptions will be made later on. We are given functions λ, $\widehat{\beta}$, $\widehat{\beta}_\Gamma$, π, π_Γ and constants τ, ν, κ satisfying the conditions listed below

$$\lambda : \mathbb{R} \to \mathbb{R} \text{ is of class } C^2 \text{ and } \lambda' \text{ is Lipschitz continuous} \tag{2.3}$$

$$\widehat{\beta}, \widehat{\beta}_\Gamma : \mathbb{R} \to [0,+\infty] \text{ are convex, proper, and l.s.c. and } \widehat{\beta}(0) = \widehat{\beta}_\Gamma(0) = 0 \tag{2.4}$$

$$\pi, \pi_\Gamma : \mathbb{R} \to \mathbb{R} \text{ are Lipschitz continuous} \tag{2.5}$$

$$\tau, \nu \geq 0 \quad \text{and} \quad \kappa > 0. \tag{2.6}$$

We define the maximal monotone graphs β and β_Γ in $\mathbb{R} \times \mathbb{R}$ by

$$\beta := \partial\widehat{\beta} \quad \text{and} \quad \beta_\Gamma := \partial\widehat{\beta}_\Gamma \tag{2.7}$$

and note that $\beta(0) \ni 0$ and $\beta_\Gamma(0) \ni 0$. Furthermore, we observe that both β and β_Γ might have effective domains $D(\beta)$ and $D(\beta_\Gamma)$, respectively. In the sequel, for any maximal monotone graph $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$, we use the notation (see, e.g., [1, p. 28])

$$\gamma^\circ(r) \text{ is the element of } \gamma(r) \text{ having minimum modulus} \tag{2.8}$$

$$\gamma^Y_\varepsilon \text{ is the Yosida regularization of } \gamma \text{ at level } \varepsilon, \text{ for } \varepsilon > 0 \tag{2.9}$$

and still employ the symbol γ (and, e.g., γ^Y_ε as a particular case) for the maximal monotone operator induced by γ on any L^2-space. Next, we set

$$\begin{aligned} &V := H^1(\Omega), \quad H := L^2(\Omega), \quad H_\Gamma := L^2(\Gamma) \\ &V_\Gamma := H^1(\Gamma) \quad \text{if } \nu > 0 \quad \text{and} \quad V_\Gamma := H^{1/2}(\Gamma) \quad \text{if } \nu = 0 \\ &\mathcal{V} := \{ v \in V : \ v|_\Gamma \in V_\Gamma \} \end{aligned} \tag{2.10}$$

the latter being endowed with the graph norm. Note that $\mathcal{V} = V$ if $\nu = 0$. Throughout the whole paper, $H^k(\Omega)$ and $H^k(\Gamma)$ denote the usual Sobolev spaces with real index (see [7, Chapt. 1] for the general theory and for the notation used here) and the symbols $\|\cdot\|_{k,\Omega}$ and $\|\cdot\|_{k,\Gamma}$ stand for their norms (the standard ones if k is a nonnegative integer). In particular, $\|\cdot\|_{1,\Omega}$, $\|\cdot\|_{0,\Omega}$, and $\|\cdot\|_{0,\Gamma}$ are the norms in V, H, and H_Γ, respectively. On the contrary, we write $\|\cdot\|_{V_\Gamma}$ for the norm in V_Γ since the definition of such a space depends on ν. For the sake of simplicity, the same notation will be used for both a space and any power of it. We recall the optimal trace theorem for V, namely, the inequality

$$\|v|_\Gamma\|_{1/2,\Gamma} \leq M_\Omega \|v\|_{1,\Omega} \quad \text{for every } v \in V \tag{2.11}$$

where M_Ω depends on Ω, only. Finally, the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the duality pairing between V^* and V. In the sequel, it is understood that H is embedded in V^* in the usual way, i.e., in order that $\langle u, v\rangle = (u,v)$, the inner product of H, for every $u \in H$ and $v \in V$.

At this point, we can describe our problem, which consists in the variational formulation of system (1.11)–(1.16), obtained by testing the equations and formally integrating by parts.

However, as β and β_Γ might be multi-valued, we have to consider selections ξ and ξ_Γ of $\beta(\chi)$ and of $\beta_\Gamma(\chi)$, in addition. Moreover, we have to give our assumptions on the data. Inspired by the concrete case of the Introduction and starting from given (suitably smooth) functions f and ϑ_Γ, we construct a V^*-valued function F by the formula

$$\langle F(t), v\rangle := \int_\Omega f(t)\, v + \kappa \int_\Omega \vartheta_\Gamma(t)\, v \quad \text{for a.a. } t \in (0,T) \text{ and } v \in V. \tag{2.12}$$

However, we can consider a more general F and, just to start, we require that

$$F \in L^2(0,T;V^*), \quad f_\Gamma \in L^2(0,T;H_\Gamma), \quad \vartheta_0 \in H, \quad \text{and} \quad \chi_0 \in V \tag{2.13}$$

but we note that the first of (2.13) holds for F defined by (2.12) whenever $f \in L^2(0,T;H)$ and $\vartheta_\Gamma \in L^2(0,T;H_\Gamma)$. At this point, we are ready to state the problem we are going to deal with. We look for a quintuplet $(\vartheta, \chi, w, \xi, \xi_\Gamma)$ such that

$$\vartheta \in L^2(0,T;V) \cap H^1(0,T;V^*) \tag{2.14}$$
$$\chi \in L^2(0,T;V) \cap H^1(0,T;V^*) \tag{2.15}$$
$$\partial_t \lambda(\chi) \in L^2(0,T;V^*) \quad \text{and} \quad \tau\, \partial_t \chi \in L^2(0,T;H) \tag{2.16}$$
$$\chi|_\Gamma \in L^2(0,T;V_\Gamma) \cap H^1(0,T;H_\Gamma) \tag{2.17}$$
$$w \in L^2(0,T;V) \tag{2.18}$$
$$\xi \in L^2(0,T;H) \quad \text{and} \quad \xi \in \beta(\chi) \quad \text{a.e. in } Q \tag{2.19}$$
$$\xi_\Gamma \in L^2(0,T;H_\Gamma) \quad \text{and} \quad \xi_\Gamma \in \beta_\Gamma(\chi|_\Gamma) \quad \text{a.e. on } \Sigma \tag{2.20}$$
$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \chi(0) = \chi_0 \tag{2.21}$$

and satisfying for a.a. $t \in (0,T)$

$$\langle \partial_t \big(\vartheta(t) + \lambda(\chi(t))\big), v\rangle + \int_\Omega \nabla\vartheta(t) \cdot \nabla v + \kappa \int_\Omega \vartheta(t)\, v = \langle F(t), v\rangle \tag{2.22}$$

$$\langle \partial_t \chi(t), v\rangle + \int_\Omega \nabla w(t) \cdot \nabla v = 0 \tag{2.23}$$

$$\begin{aligned}\int_\Omega w(t) v &= \int_\Omega \tau\, \partial_t\chi(t)\, v + \int_\Gamma \partial_t\chi(t)\, v + \int_\Omega \nabla\chi(t)\cdot\nabla v + \int_\Gamma \nu \nabla_\Gamma \chi(t) \cdot \nabla_\Gamma v \\ &\quad + \int_\Omega \big(\xi(t) + \pi(\chi(t)) - \lambda'(\chi(t))\, \vartheta(t)\big)\, v + \int_\Gamma \big(\xi_\Gamma(t) + \pi_\Gamma(\chi(t)) - f_\Gamma(t)\big)\, v\end{aligned} \tag{2.24}$$

for every $v \in V$, every $v \in V$, and every $v \in \mathcal{V}$, respectively. We note that the term of (2.24) that invoves λ' is meaningful. Indeed, all the factors of the product belong to $L^6(\Omega)$ thanks to the Sobolev embedding $V \subset L^6(\Omega)$ and to the Lipschitz continuity of λ' (see (2.3)).

Remark 2.1. Note that, by testing (2.23) by the constant $1/|\Omega|$, we obtain

$$\partial_t(\chi(t)_\Omega) = 0 \quad \text{for a.a. } t \in (0,T) \quad \text{and} \quad \chi(t)_\Omega = (\chi_0)_\Omega \quad \text{for every } t \in [0,T] \tag{2.25}$$

where, more generally, we set

$$v^*_\Omega := \frac{1}{|\Omega|}\, \langle v^*, 1\rangle \quad \text{for } v^* \in V^*. \tag{2.26}$$

Clearly, (2.26) gives the usual mean value when applied to elements of H.

Now we recall definitions (2.2) and (2.10) and state our results. The simplest one regards uniqueness, since no further assumptions on the structure and on the data are needed.

Theorem 2.2. *Assume* (2.3)–(2.7) *and* (2.13). *Then, any two solutions to problem* (2.22)–(2.24) *satisfying the regularity requirements* (2.14)–(2.20) *and the Cauchy condition* (2.21) *have the same first and second components.*

Remark 2.3. If β is single-valued, the component ξ is uniquely determined as well. Then, a comparison in (2.24) with $v \in H_0^1(\Omega)$ shows that the same happens for the component w. Hence, by writing (2.24) once more with such an information, we see that even the component ξ_Γ is uniquely determined and we have a full uniqueness result. On the contrary, if β is multi-valued, uniqueness for all the components might fail, and an example in such a direction is given in [5] for the case of a constant temperature. Thus, the above result is essentially optimal. Furthermore, we remark that a continuous dependence result holds as well. Namely, with a self-explaining notation and setting $\Theta_i(t) = \int_0^t \vartheta_i(s)\,ds$, we have

$$\begin{aligned}
&\|\vartheta_1 - \vartheta_2\|^2_{L^2(0,T;H)} + \|\Theta_1 - \Theta_2\|^2_{L^\infty(0,T;V)} \\
&\quad + \|\chi_1 - \chi_2\|^2_{L^\infty(0,T;V^*)\cap L^2(0,T;\mathcal{V})} + \tau\|\chi_1 - \chi_2\|^2_{L^\infty(0,T;H)} \\
&\le c\Big\{\|\vartheta_{0,1} - \vartheta_{0,2}\|^2_{0,\Omega} + \|\chi_{0,1} - \chi_{0,2}\|^2_* + \tau\|\chi_{0,1} - \chi_{0,2}\|^2_{0,\Omega} + \|\chi_{0,1} - \chi_{0,2}\|^2_{0,\Gamma} \\
&\qquad + \|F_1 - F_2\|^2_{L^2(0,T;V^*)} + \|f_{\Gamma,1} - f_{\Gamma,2}\|^2_{L^2(0,T;H_\Gamma)}\Big\}
\end{aligned} \tag{2.27}$$

provided that $\chi_{0,1}$ and $\chi_{0,2}$ have the same mean value and that ϑ_i belong to a fixed ball B of the space $L^\infty(0,T;H)$ for $i = 1,2$. Inequality (2.27) can be proved by touching up our uniqueness proof of Section 3 and the constant c depends on the prescribed ball B. However, we point out that an estimate of the $L^\infty(0,T;H)$-norm of the component ϑ of the solution can be obtained in terms of suitable norms of the data if the assumption of our existence result we are going to state are fulfilled, as the argument of the forthcoming Section 5 clearly shows. Hence, in such a case, the value of c in (2.27) actually depends on an upper bound for some norms of the data.

Existence is ensured just under further assumptions. We recall that our aim is to keep the maximal monotone operators as general as we can, mainly, and we are able do that under suitable conditions (see the forthcoming Remark 2.4 for comments). As far as the data are concerned, we assume that

$$F \in L^2(0,T;H) + H^1(0,T;V^*) \quad \text{and} \quad f_\Gamma \in H^1(0,T;H_\Gamma) \cap L^\infty(\Sigma) \tag{2.28}$$
$$\vartheta_0 \in V, \quad \chi_0 \in H^2(\Omega), \quad \partial_n\chi_0 = 0, \quad \text{and} \quad \nu\,\chi_0|_\Gamma \in H^2(\Gamma) \tag{2.29}$$
$$\widehat{\beta}(\chi_0) \in L^1(\Omega), \quad \widehat{\beta}(\chi_0|_\Gamma) \in L^1(\Gamma), \quad \text{and} \quad \widehat{\beta}_\Gamma(\chi_0|_\Gamma) \in L^1(\Gamma) \tag{2.30}$$
$$\text{the mean value of } \chi_0 \text{ belongs to the interior of } D(\beta). \tag{2.31}$$

Moreover, we assume that (see (2.8)–(2.9) for the notation)

$$\beta^\circ(\chi_0) \in H \quad \text{and} \quad \beta^\circ_\Gamma(\chi_0|_\Gamma) \in H_\Gamma \tag{2.32}$$
$$-\Delta\chi_0 + \beta^Y_\varepsilon(\chi_0) - \lambda'(\chi_0)\,\vartheta_0 \quad \text{remains bounded in } V \text{ as } \varepsilon \to 0^+ \text{ if } \tau = 0. \tag{2.33}$$

As far as the structure of the system is concerned, we need some compatibility condition on the main nonlinearitis and on the perturbation π_Γ on the boundary. Namely, we assume that

$$D(\beta_\Gamma) \supseteq D(\beta) \quad \text{and} \quad \beta_\Gamma(0) = \{0\} \tag{2.34}$$

and that real constants α, C_Γ, σ, L_Γ, M_Γ, and $r_\pm$ exist such that

$$\begin{aligned}
&\alpha > 0, \quad C_\Gamma \geq 0, \quad \sigma \in (0,1), \quad L_\Gamma > \sup |\pi_\Gamma'| \\
&\quad \text{and} \quad M_\Gamma > |\pi_\Gamma(0)| + \|f_\Gamma\|_{L^\infty(\Gamma)} && (2.35) \\
&r_- \leq 0 \leq r_+ \,, \ \text{and } r_\pm \text{ belong to the interior of } D(\beta) && (2.36) \\
&|\beta^\circ(r)| \geq \alpha |\beta_\Gamma^\circ(r)| - C_\Gamma \quad \text{for every } r \in D(\beta) && (2.37) \\
&\sigma |\beta_\Gamma^\circ(r)| \geq L_\Gamma |r| + M_\Gamma \quad \text{for every } r \in D(\beta_\Gamma) \setminus (r_-, r_+). && (2.38)
\end{aligned}$$

Remark 2.4. We comment some of the above assumptions a little. The first of (2.28) holds if F is defined by (2.12) provided that $f \in L^2(0,T;H)$ and $\vartheta_\Gamma \in H^1(0,T;H_\Gamma)$. In such a case, formula (2.12) itself suggests how to split F as in (2.28). Assumptions (2.30) and (2.32) are not independent. The latter implies the first and third of the former, indeed, and we have written all of them just for convenience. If $\chi_0 \in H^3(\Omega)$, then (2.33) essentially requires β to be rather smooth on the range of χ_0. For instance, if β comes from the logarithmic potential (1.5) (or from the singular potential (1.6)), this simply means that $\sup |\chi_0| < 1$. Assumption (2.37) is the main compatibility condition and its meaning is rather clear. However, we note that it is satisfied whenever $D(\beta)$ is bounded and its clusure is contained in the interior of $D(\beta_\Gamma)$, in particular if β is singular like in (1.5) and β_Γ is not. As far as (2.38) is concerned, let us just say that it is satisfied in the following two significant cases. The first one corresponds to $D(\beta) = D(\beta_\Gamma) = \mathbb{R}$ and to a β_Γ that is stricly superlinear at infinity. In the second one, we think of a bounded $D(\beta)$, say $D(\beta) = (-1,1)$ as in the case of the logaritmic potential (1.5), and of a β_Γ everywhere defined. Then, (2.38) essentially requires that the nonlinear boundary term has a decomposition $\beta_\Gamma + \pi_\Gamma$ such that $|\beta_\Gamma(r)|$ is large enough for $|r|$ close to 1. This is satisfied, in particular, if the dynamic boundary condition has the form (1.9) with $\sup |\chi_\Gamma| < 1$.

Even though a different existence theorem could be proved as in [5] by requiring just growth conditions on β and β_Γ, we confined ourselves to state the following result.

Theorem 2.5. *Assume that* (2.3)–(2.7) *and* (2.28)–(2.38) *are fulfilled. Then, there exists a quintuplet* $(\vartheta, \chi, w, \xi, \xi_\Gamma)$ *satisfying* (2.14)–(2.21) *and solving problem* (2.22)–(2.24).

Remark 2.6. Actually, the solution we construct in our proof is smoother than required in (2.14)–(2.20) (see the forthcoming Remark 5.1).

We recall some facts at once (see (2.26) for the notation regarding mean values). First of all, the Poincaré type inequalities

$$\|v\|_{1,\Omega}^2 \leq M_\Omega (\|\nabla v\|_{0,\Omega}^2 + |v_\Omega|^2) \quad \text{and} \quad \|v\|_{1,\Omega}^2 \leq M_\Omega (\|\nabla v\|_{0,\Omega}^2 + \|v\|_{0,\Gamma}^2) \tag{2.39}$$

hold true for every $v \in V$ and for some constant M_Ω depending on Ω, only. Next, we define the operator $\mathcal{N} : \operatorname{dom} \mathcal{N} \subset V^* \to V$ and the norm $\|\cdot\|_*$ on V^* as follows. We set

$$\operatorname{dom} \mathcal{N} := \{ v^* \in V^* : \ v^*_\Omega = 0 \} \tag{2.40}$$

and, for $v^* \in \operatorname{dom} \mathcal{N}$, we term $\mathcal{N} v^*$ the unique element of V that satisfies

$$\mathcal{N} v^* \in V, \quad (\mathcal{N} v^*)_\Omega = 0, \quad \text{and} \quad \int_\Omega \nabla \mathcal{N} v^* \cdot \nabla z = \langle v^*, z \rangle \quad \text{for every } z \in V. \tag{2.41}$$

Moreover, for $v^* \in V^*$, we set

$$\|v^*\|_*^2 := \|\nabla \mathcal{N}(v^* - v^*_\Omega)\|_{0,\Omega}^2 + |v^*_\Omega|^2. \tag{2.42}$$

Then, the identities

$$\langle v^*, \mathcal{N}v^* \rangle = \|v^*\|_*^2 \quad \text{for every } v^* \in \operatorname{dom}\mathcal{N} \tag{2.43}$$

$$2\langle \partial_t v^*, \mathcal{N}v^* \rangle = \frac{d}{dt}\,\|v^*\|_*^2 \quad \text{a.e. in } (0,T) \quad \text{for every } v^* \in H^1(0,T;\operatorname{dom}\mathcal{N}) \tag{2.44}$$

hold true as well as the inequalities

$$\frac{1}{M_\Omega}\,\|v^*\|_{V^*} \le \|v^*\|_* \le M_\Omega \|v^*\|_{V^*} \quad \text{for } v^* \in V^* \tag{2.45}$$

where M_Ω depends on Ω, only. Moreover, we point out the easy inequalities

$$ab \le \delta a^2 + c_\delta\, b^2 \tag{2.46}$$

$$\|v(t)\|_{0,\Omega}^2 \le \|v(0)\|_{0,\Omega}^2 + \delta \int_0^t \|\partial_t v(s)\|_*^2\, ds + c_\delta \int_0^t \|v(s)\|_{1,\Omega}^2\, ds \tag{2.47}$$

$$\|v(t)\|_{0,\Gamma}^2 \le \|v(0)\|_{0,\Gamma}^2 + \delta \int_0^t \|\partial_t v(s)\|_{0,\Gamma}^2\, ds + c_\delta \int_0^t \|v(s)\|_{0,\Gamma}^2\, ds \tag{2.48}$$

$$\|v\|_{L^4(\Omega)}^2 \le \delta \|\nabla v\|_{0,\Omega}^2 + c_\delta \|v\|_*^2 \tag{2.49}$$

which hold for every $a, b \ge 0$, every $v \in L^2(0,T;V) \cap H^1(0,T;V^*)$, every $v \in H^1(0,T;H_\Gamma)$, and every $v \in V$, respectively. Moreover, t varies a.e. in $(0,T)$ in (2.47)–(2.48) and $\delta > 0$ is arbitrary in all of them, while c_δ depends on δ in (2.46) and even on Ω in (2.47)–(2.49).

By the way, in the sequel, we avoid specifying the precise values of the constants and use the general symbol c_δ for a situation like the latter. More generally, throughout the paper, the symbol c stands for different constants which depend only on Ω, on the final time T, and on the constants and the norms of the functions involved in the assumptions of our statements, and a notation like c_δ allows the constant to depend on δ, in addition. The values of c and c_δ might change from line to line and even in the same chain of inequalities. We notice that δ will always denote a positive parameter whose value is choosen when it is convenient to do it.

3 Uniqueness

In this section, we prove Theorem 2.2. We take two solutions, label their components with subscripts 1 and 2, set $\vartheta := \vartheta_1 - \vartheta_2$, and introduce an analogous notation for the other components for convenience. Moreover, we set $\Theta(t) = \int_0^t \vartheta(s)\, ds$ for $t \in [0,T]$. Therefore, if we write (2.22) for both solutions, take the difference, and integrate with respect to time, we obtain

$$\int_\Omega \vartheta(t)\, v + \int_\Omega \nabla\Theta(t) \cdot \nabla v + \kappa \int_\Gamma \Theta(t)\, v = -\int_\Omega \bigl(\lambda(\chi_1(t)) - \lambda(\chi_2(t))\bigr)\, v \tag{3.1}$$

for every $t \in [0,T]$ and every $v \in V$. At this point, we test (3.1) by $\vartheta = \partial_t\Theta$. At the same time, we write (2.23) and (2.24) for both solutions and test the differences by $\mathcal{N}\chi$ and $-\chi$, respectively, by observing that χ has zero mean value thanks to (2.25) applied to both solutions. Finally, we add the obtained equalities to each other, integrate over $(0,t)$ where $t \in (0,T]$ is arbitrary, and rearrange a little. Then, we have

$$
\begin{aligned}
&\int_{Q_t} |\vartheta|^2 + \frac{1}{2}\int_\Omega |\nabla\Theta(t)|^2 + \frac{\kappa}{2}\int_\Gamma |\Theta(t)|^2 \\
&\quad + \int_0^t \langle \partial_t \chi(s), \mathcal{N}\chi(s)\rangle \, ds + \int_{Q_t} \nabla w \cdot \nabla \mathcal{N}\chi - \int_{Q_t} w\chi \\
&\quad + \frac{\tau}{2}\int_\Omega |\chi(t)|^2 + \frac{1}{2}\int_\Gamma |\chi(t)|^2 + \int_{Q_t} |\nabla\chi|^2 + \nu \int_{\Sigma_t} |\nabla_\Gamma \chi|^2 + \int_{Q_t} \xi\chi + \int_{\Sigma_t} \xi_\Gamma \chi \\
&= \int_{Q_t} \bigl(\pi(\chi_2) - \pi(\chi_1)\bigr)\,\chi + \int_{\Sigma_t} \bigl(\pi_\Gamma(\chi_2) - \pi_\Gamma(\chi_1)\bigr)\,\chi \\
&\quad + \int_{Q_t} \Bigl\{ \bigl(\lambda'(\chi_1)\,\vartheta_1 - \lambda'(\chi_2)\,\vartheta_2\bigr)\,\chi - \bigl(\lambda(\chi_1) - \lambda(\chi_2)\bigr)\,\vartheta \Bigr\}. \qquad (3.2)
\end{aligned}
$$

Now, we use (2.44) for the fourth term on the left-hand side and cancel the subsequent two integrals accounting for (2.41). Moreover, we observe that the last two integrals on the left-hand side are nonnegative since β and β_Γ are monotone. Furthermore, we rewrite the integrand of the last term of (3.2) by using the Taylor formula this way

$$
\begin{aligned}
&\bigl(\lambda'(\chi_1)\,\vartheta_1 - \lambda'(\chi_2)\,\vartheta_2\bigr)\,\chi - \bigl(\lambda(\chi_1) - \lambda(\chi_2)\bigr)\,\vartheta \\
&= \vartheta_1 \bigl\{\lambda(\chi_2) - \lambda(\chi_1) - \lambda'(\chi_1)(\chi_2 - \chi_1)\bigr\} + \vartheta_2\bigl\{\lambda(\chi_1) - \lambda(\chi_2) - \lambda'(\chi_2)(\chi_1 - \chi_2)\bigr\} \\
&= \vartheta_1\, \frac{\lambda''(\zeta_1)}{2}\,|\chi|^2 + \vartheta_2\,\frac{\lambda''(\zeta_2)}{2}\,|\chi|^2
\end{aligned}
$$

for some functions ζ_1, ζ_2 between χ_1 and χ_2. On the other hand, λ'' is bounded (cf. (2.3)) and (2.14) applied to both solutions yields $\vartheta_1, \vartheta_2 \in L^\infty(0,T;H)$. Hence, by owing to the Hölder inequality as well, we deduce that

$$
\begin{aligned}
&\int_{Q_t} \Bigl\{ \bigl(\lambda'(\chi_1)\,\vartheta_1 - \lambda'(\chi_2)\,\vartheta_2\bigr)\,\chi - \bigl(\lambda(\chi_1) - \lambda(\chi_2)\bigr)\,\vartheta \Bigr\} \\
&\le c\int_{Q_t} \bigl(|\vartheta_1| + |\vartheta_2|\bigr)\,|\chi|^2 \le c\int_0^t \bigl(\|\vartheta_1(s)\|_{0,\Omega} + \|\vartheta_2(s)\|_{0,\Omega}\bigr)\|\chi(s)\|^2_{L^4(\Omega)}\, ds \\
&\le c\int_0^t \|\chi(s)\|^2_{L^4(\Omega)}\, ds.
\end{aligned}
$$

Here and in the subsequent inequalities, the value of c also depends on the solutions we have picked at the beginning, of course. Therefore, if we also account for the Lipschitz continuity of π and π_Γ (see (2.5)) and forget some nonnegative terms on the left-hand side of (3.2), we obtain

$$
\begin{aligned}
&\int_{Q_t} |\vartheta|^2 + \frac{1}{2}\|\chi(t)\|_*^2 + \frac{1}{2}\|\chi(t)\|^2_{0,\Gamma} + \int_{Q_t} |\nabla\chi|^2 + \nu\int_{\Sigma_t} |\nabla_\Gamma\chi|^2 \\
&\le c\int_{Q_t} |\chi|^2 + c\int_{\Sigma_t} |\chi|^2 + c\int_0^t \|\chi(s)\|^2_{L^4(\Omega)}\, ds \le c\int_0^t \|\chi(s)\|^2_{L^4(\Omega)}\, ds + c\int_{\Sigma_t} |\chi|^2.
\end{aligned}
$$

Furthermore, we use (2.49) and get

$$
\int_0^t \|\chi(s)\|^2_{L^4(\Omega)}\, ds \le \delta \int_{Q_t} |\nabla\chi|^2 + c_\delta \int_0^t \|\chi(s)\|_*^2\, ds.
$$

At this point, it suffices to choose δ small enough and apply the Gronwall lemma to obtain $\vartheta = 0$ and $\chi = 0$, that is, $\vartheta_1 = \vartheta_2$ and $\chi_1 = \chi_2$, and the proof is complete.

4 Approximating problems

We approximate problem (2.22)–(2.24) by a smoother one, which depends on the parameter $\varepsilon \in (0,1)$ and is obtained by smoothing the worst nonlinearities β and β_Γ. Moreover, it is more convenient to have a viscous Cahn-Hilliard equations in the approximation and to modify λ as well. As far as the latter is concerned, we introduce a family $\{\lambda_\varepsilon\}$ satisfying

$$\lambda_\varepsilon \text{ is a Lipschitz } \mathrm{C}^2\text{-function} \quad \text{and} \quad |\lambda_\varepsilon(0)| + |\lambda'_\varepsilon(0)| + \sup_{r\in\mathbb{R}} |\lambda''_\varepsilon(r)| \le C_\lambda \tag{4.1}$$

for every $\varepsilon \in (0,1)$ and some constant C_λ and converging to λ in the following sense

$$\lambda_\varepsilon(r) \to \lambda(r) \quad \text{and} \quad \lambda'_\varepsilon(r) \to \lambda'(r) \quad \text{as } \varepsilon \to 0, \quad \text{for every } r \in \mathbb{R}. \tag{4.2}$$

Moreover, following [5], we define the real number τ_ε and the functions $\beta_\varepsilon, \beta_{\Gamma,\varepsilon} : \mathbb{R} \to \mathbb{R}$ by the formulas

$$\tau_\varepsilon := \max\{\tau, \varepsilon\} \tag{4.3}$$

$$\beta_\varepsilon(r) := \beta^Y_\varepsilon(r) \quad \text{for } r \in \mathbb{R} \tag{4.4}$$

$$\begin{aligned}
\beta_{\Gamma,\varepsilon}(r) &:= \beta^Y_{\Gamma,\alpha\varepsilon}(r - \varepsilon C_\Gamma) \quad \text{if } r \le -\varepsilon C_\Gamma \\
&:= \beta^Y_{\Gamma,\alpha\varepsilon}(r + \varepsilon C_\Gamma) \quad \text{if } r \ge \varepsilon C_\Gamma \\
&:= \frac{r}{\varepsilon C_\Gamma}\, \beta^Y_{\Gamma,\alpha\varepsilon}(-2\varepsilon C_\Gamma) \quad \text{if } -\varepsilon C_\Gamma < r < 0 \\
&:= \frac{r}{\varepsilon C_\Gamma}\, \beta^Y_{\Gamma,\alpha\varepsilon}(2\varepsilon C_\Gamma) \quad \text{if } 0 \le r < \varepsilon C_\Gamma
\end{aligned} \tag{4.5}$$

where α and C_Γ are the same as in (2.37) and notation (2.9) for Yosida regularizations is used. Furthermore, we define for convenience $\widehat{\beta}_\varepsilon, \widehat{\beta}_{\Gamma,\varepsilon} : \mathbb{R} \to \mathbb{R}$ by the formulas

$$\widehat{\beta}_\varepsilon(r) := \int_0^r \beta_\varepsilon(s)\, ds \quad \text{and} \quad \widehat{\beta}_{\Gamma,\varepsilon}(r) := \int_0^r \beta_{\Gamma,\varepsilon}(s)\, ds \quad \text{for } r \in \mathbb{R}. \tag{4.6}$$

As the Yosida regularization of a maximal monotone operator is monotone and Lipschitz continous, such a property holds for both β_ε and $\beta_{\Gamma,\varepsilon}$. Moreover, such functions vanish at 0. It follows that $\widehat{\beta}_\varepsilon$ and $\widehat{\beta}_{\Gamma,\varepsilon}$ are nonnegative convex functions with a quadratic growth.

Then, the approximating problem consists in finding a triplet $(\vartheta_\varepsilon, \chi_\varepsilon, w_\varepsilon)$ satisfing the regularity properties and the Cauchy conditions given below

$$\vartheta_\varepsilon \in L^2(0,T;V) \cap H^1(0,T;H) \tag{4.7}$$

$$\chi_\varepsilon \in L^\infty(0,T;V) \cap H^1(0,T;H) \tag{4.8}$$

$$\chi_\varepsilon|_\Gamma \in L^\infty(0,T;V_\Gamma) \cap H^1(0,T;H_\Gamma) \tag{4.9}$$

$$w_\varepsilon \in L^2(0,T;V) \tag{4.10}$$

$$\vartheta_\varepsilon(0) = \vartheta_0 \quad \text{and} \quad \chi_\varepsilon(0) = \chi_0 \tag{4.11}$$

and solving, for a.a. $t \in (0,T)$, the variational equations

$$\int_\Omega \partial_t\big(\vartheta_\varepsilon(t) + \lambda_\varepsilon(\chi_\varepsilon(t))\big)\, v + \int_\Omega \nabla\vartheta_\varepsilon(t)\cdot\nabla v + \kappa \int_\Omega \vartheta_\varepsilon(t)\, v = \langle F(t)\, v\rangle \tag{4.12}$$

$$\int_\Omega \partial_t\chi_\varepsilon(t)\, v + \int_\Omega \nabla w_\varepsilon(t)\cdot\nabla v = 0 \tag{4.13}$$

$$\begin{aligned}\int_\Omega w_\varepsilon(t)\, v &= \tau_\varepsilon \int_\Omega \partial_t\chi_\varepsilon(t)\, v + \int_\Gamma \partial_t\chi_\varepsilon(t)\, v + \int_\Omega \nabla\chi_\varepsilon(t)\cdot\nabla v + \nu \int_\Gamma \nabla_\Gamma\chi_\varepsilon(t)\cdot\nabla_\Gamma v \\ &\quad + \int_\Omega \big(\beta_\varepsilon(\chi_\varepsilon(t)) + \pi(\chi_\varepsilon(t)) - \lambda_\varepsilon'(\chi_\varepsilon)\,\vartheta_\varepsilon\big)\, v \\ &\quad + \int_\Gamma \big(\beta_{\Gamma,\varepsilon}(\chi_\varepsilon(t)) + \pi_\Gamma(\chi_\varepsilon(t)) - f_\Gamma(t)\big)\, v \end{aligned}\tag{4.14}$$

for every $v \in V$, every $v \in V$, and every $v \in \mathcal{V}$, respectively.

The first step regards well-posedness for the above problem. More generally, there holds the result stated below which does not require that the operators involved in the problem are exactly the previous ones. Here, uniqueness is not stressed. However, the same proof of Theorem 2.2 works in the present case.

Theorem 4.1. *In addition to the assumptions of Theorem 2.5, suppose that λ_ε, β_ε, and $\beta_{\Gamma,\varepsilon}$ are Lipschitz continuous and that $\tau_\varepsilon > 0$. Then, there exists a triplet $(\vartheta_\varepsilon, \chi_\varepsilon, w_\varepsilon)$ satisfying* (4.7)–(4.11) *and solving* (4.12)–(4.14).

Proof. We just give a sketch. Following [5], we use a Galerkin scheme and solve a discrete problem. We consider the eigenvalue problems for the Laplace operator with homogeneous Robin boundary conditions and homogeneous Neumann boundary conditions, respectively. More precisely, we look for $\mu', \mu'' \in \mathbb{R}$ and $\eta, e \in V \setminus \{0\}$ such that

$$\int_\Omega \nabla\eta\cdot\nabla v + \kappa\int_\Gamma \eta v = \mu'\int_\Omega \eta v \quad \text{and} \quad \int_\Omega \nabla e\cdot\nabla v = \mu''\int_\Omega ev \quad \text{for every } v \in V.$$

We label (μ_i', η_i) and (μ_i'', e_i) the pairs of eigenvalues and eigenfunctions of the above problems, where i runs over the positive integers, being understood that the sequences of the eigenvalues are nondecreasing and that the sequences of the eigenfunctions are orthonormal anc complete in H. Then, we set

$$V_n' := \operatorname{span}\{\eta_i : i = 1,\dots,n\} \quad \text{and} \quad V_n'' := \operatorname{span}\{e_i : i = 1,\dots,n\} \quad \text{for every } n \ge 1$$

and look for a triplet $(\vartheta_\varepsilon^n, \chi_\varepsilon^n, w_\varepsilon^n)$ satisfying

$$\vartheta_\varepsilon^n \in H^1(0,T;V_n'), \quad \chi_\varepsilon^n \in H^1(0,T;V_n''), \quad \text{and} \quad w_\varepsilon^n \in L^2(0,T;V_n'')$$

and solving, for every $t \in [0,T]$, the following variational equations

$$\int_\Omega \partial_t\bigl(\vartheta_\varepsilon^n(t) + \lambda_\varepsilon(\mathcal{X}_\varepsilon^n(t))\bigr)\, v + \int_\Omega \nabla\vartheta_\varepsilon^n(t)\cdot\nabla v + \kappa\int_\Omega \vartheta_\varepsilon^n(t)\, v = \langle F(t)\, v\rangle \tag{4.15}$$

$$\int_\Omega \partial_t\mathcal{X}_\varepsilon^n(t)\, v + \int_\Omega \nabla w_\varepsilon^n(t)\cdot\nabla v = 0 \tag{4.16}$$

$$\begin{aligned}\int_\Omega w_\varepsilon^n(t)\, v &= \tau_\varepsilon\int_\Omega \partial_t\mathcal{X}_\varepsilon^n(t)\, v + \int_\Gamma \partial_t\mathcal{X}_\varepsilon^n(t)\, v + \int_\Omega \nabla\mathcal{X}_\varepsilon^n(t)\cdot\nabla v + \nu\int_\Gamma \nabla_\Gamma\mathcal{X}_\varepsilon^n(t)\cdot\nabla_\Gamma v \\ &\quad + \int_\Omega \bigl(\beta_\varepsilon(\mathcal{X}_\varepsilon^n(t)) + \pi(\mathcal{X}_\varepsilon^n(t)) - \lambda_\varepsilon'(\mathcal{X}_\varepsilon^n)\,\vartheta_\varepsilon^n\bigr)\, v \\ &\quad + \int_\Gamma \bigl(\beta_{\Gamma,\varepsilon}(\mathcal{X}_\varepsilon^n(t)) + \pi_\Gamma(\mathcal{X}_\varepsilon^n(t)) - f_\Gamma(t)\bigr)\, v\end{aligned} \tag{4.17}$$

for every $v \in V_n'$, every $v \in V_n''$, and every $v \in V_n''$, respectively, and suitable Cauchy conditions. Namely, as initial values ϑ_0^n and $\mathcal{X}_0^n$ for ϑ_ε^n and $\mathcal{X}_\varepsilon^n$, we can take the L^2-projections of ϑ_0 and $\mathcal{X}_0$ on V_n' and on V_n'', respectively. We obtain a Cauchy problem for a system of ordinary differential equations and the main trouble comes from the part of it that is related to the second and third of the above variational equations. Hence, the situation is essentially similar to the one discussed in [5, Sect. 4] with full detail and a well-posedness result can be established.

The next step consists in a priori estimates that allow us to apply well-known compactness results and to let n tend to infinity. In performing them, some bounds and convergence for the initial values are needed. Namely, we have

$$\|\vartheta_0^n\|_{1,\Omega} + \|\mathcal{X}_0^n\|_{1,\Omega} \le c, \quad \vartheta_0^n \to \vartheta_0 \quad \text{strongly in } V, \quad \text{and} \quad \mathcal{X}_0^n \to \mathcal{X}_0 \quad \text{strongly in } \mathcal{V} \tag{4.18}$$

whenever conditions (2.29) hold. Indeed, the latter is proved in [5, Lemma 4.4] and the former can be easily checked by a similar argument. The first a priori estimate is obtained by testing equations (4.15)–(4.17) by ϑ_ε^n, $\mathcal{N}\partial_t\mathcal{X}_\varepsilon^n$, and $-\partial_t\mathcal{X}_\varepsilon^n$, respectively, taking the sum, and integrating over $(0,t)$. As the terms involving λ_ε cancel, we obtain

$$\|\vartheta_\varepsilon^n\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\mathcal{X}_\varepsilon^n\|_{L^\infty(0,T;V)\cap H^1(0,T;H)} + \|\mathcal{X}_\varepsilon^n|_\Gamma\|_{L^\infty(0,T;V_\Gamma)\cap H^1(0,T;H_\Gamma)} \le c_\varepsilon \tag{4.19}$$

by simply proceeding as in the analogous estimate of [5, Sect. 4]. Next, we derive a bound for $\partial_t\vartheta_\varepsilon^n$ in $L^2(0,T;H)$. For the sake of simplicity, we split F as $F_1 + F_2$ with $F_1 \in L^2(0,T;H)$ and $F_2 \in H^1(0,T;V^*)$ at once and allow the value of c_ε to depend on both F_1 and F_2 rather than on their sum F. We test (4.15) by $\partial_t\vartheta_\varepsilon^n$ and integrate over $(0,t)$. We obtain

$$\begin{aligned}&\int_{Q_t} |\partial_t\vartheta_\varepsilon^n|^2 + \frac12\int_\Omega |\nabla\vartheta_\varepsilon^n(t)|^2 + \frac{\kappa}{2}\int_\Gamma |\vartheta_\varepsilon^n(t)|^2 \\ &= \int_0^t \langle F_1(s) + F_2(s) - \lambda_\varepsilon'(\mathcal{X}_\varepsilon^n(s))\,\partial_t\mathcal{X}_\varepsilon^n(s), \partial_t\vartheta_\varepsilon^n(s)\rangle\, ds + \frac12\int_\Omega |\nabla\vartheta_0^n|^2 + \frac{\kappa}{2}\int_\Gamma |\vartheta_0^n|^2.\end{aligned}$$

The terms involving F_1 and λ_ε can be dealt with in a trivial way and those concerning the initial values are bounded by (4.18). Regarding the F_2-term, by owing to (4.18) as well, we have

$$\begin{aligned}&\int_0^t \langle F_2(s), \partial_t\vartheta_\varepsilon^n(s)\rangle\, ds = \langle F_2(t), \vartheta_\varepsilon^n(t)\rangle - \langle F_2(0), \vartheta_\varepsilon^n(0)\rangle - \int_0^t \langle F_2'(s), \vartheta_\varepsilon^n(s)\rangle\, ds \\ &\le \delta\|\vartheta_\varepsilon^n(t)\|_{1,\Omega}^2 + \int_0^t \|\vartheta_\varepsilon^n(s)\|_{1,\Omega}^2\, ds + c_\delta\,.\end{aligned}$$

Hence, the F_2-term can be controlled by the Gronwall lemma for small δ in view of the second of (2.39). We conclude that

$$\|\partial_t \vartheta_\varepsilon^n\|_{L^2(0,T;H)} + \|\vartheta_\varepsilon^n\|_{L^\infty(0,T;V)} \le c_\varepsilon \,. \tag{4.20}$$

At this point, we get a limit triplet $(\vartheta_\varepsilon, \chi_\varepsilon, w_\varepsilon)$ (for a subsequence) by using standard compactness results and there is no difficulty in proving that $(\vartheta_\varepsilon, \chi_\varepsilon, w_\varepsilon)$ solves the approximating problem (4.12)–(4.14). Indeed, we have strong compactness in $L^2(Q)$ for both ϑ_ε^n and χ_ε^n and all the nonlinearities are Lipschitz continuous. □

Remark 4.2. We point out some facts. First of all, if we let λ_ε, β_ε, $\beta_{\Gamma,\varepsilon}$, π, π_Γ, and the data F, f_Γ, ϑ_0, χ_0 vary in some families (depending on some parameter ε' in addition) such that the assumptions of Theorem 4.1 are fulfilled and the inequalities

$$\begin{aligned} &|\lambda_\varepsilon(0)| + \sup|\lambda_\varepsilon'| + \sup|\lambda_\varepsilon''| + \sup\beta_\varepsilon' + \sup\beta_{\Gamma,\varepsilon}' \\ &\quad + |\pi(0)| + \sup|\pi'| + |\pi_\Gamma(0)| + \sup|\pi_\Gamma'| \\ &\quad + \|\vartheta_0\|_{0,\Omega} + \|\chi_0\|_{2,\Omega} + \|F\|_{L^2(0,T;H)+H^1(0,T;V^*)} + \|f_\Gamma\|_{L^2(0,T;H_\Gamma)} \le M = M_\varepsilon \end{aligned} \tag{4.21}$$

are satisfied for some constant M and all the functions and the data of such families, then the corresponding solutions $(\vartheta_\varepsilon, \chi_\varepsilon, w_\varepsilon)$ satisfy

$$\begin{aligned} &\|\vartheta_\varepsilon\|_{L^2(0,T;V)\cap H^1(0,T;V^*)} + \|\chi_\varepsilon\|_{L^\infty(0,T;V)\cap H^1(0,T;H)} \\ &\quad + \|\chi_\varepsilon\|_{L^\infty(0,T;V_\Gamma)\cap H^1(0,T;H_\Gamma)} + \|w_\varepsilon\|_{L^2(0,T;V)} \le M' = M_\varepsilon' \end{aligned} \tag{4.22}$$

where the constant M' depends on ε and M, only. In particular, if we fix ε but perturb the functions λ_ε, π, π_Γ, the approximating monotone functions β_ε and $\beta_{\Gamma,\varepsilon}$ given by (4.4) and (4.5), and the data by replacing them with smoother functions and data, depending on ε', and a bound like (4.21) holds uniformly with respect to ε', then the corresponding solutions satisfy the analogue of (4.22) uniformly with respect to ε'. Furthermore, it is clear that we actually can perturb both the structure and the data and obtain a much smoother problem, depending on the parameter ε' in addition, whose structure and data satisfy an estimate like (4.21) uniformly with respect to ε'. On the other hand, the solution to the perturbed problem is smoother. So, whenever one would like to use some procedure that is not permitted just because of a lack of smoothness of $(\vartheta_\varepsilon, \chi_\varepsilon, w_\varepsilon)$, one can think of working first on the doubly approximating problem. Once the correponding a priori estimate is performed, one lets ε' tend to 0 and obtains the desired estimate on the solution $(\vartheta_\varepsilon, \chi_\varepsilon, w_\varepsilon)$. Indeed, (4.22) implies strong L^2-compactness for ϑ_ε, χ_ε, and $\chi_\varepsilon|_\Gamma$ with respect to ε' for fixed ε, whence no trouble arises in taking the limit in the nonlinearities. Therefore, just the correct further regularity corresponding to some more smoothness for both structure and data should be checked here, and this could be done acting on the discrete solution. The latter is automatically smooth in such a case, indeed. However, we just discuss a technical point regarding the regularity

$$\begin{aligned} &\vartheta_\varepsilon \in L^\infty(0,T;V)\cap H^1(0,T;H), \quad \partial_t\chi_\varepsilon \in L^\infty(0,T;V)\cap H^1(0,T;H), \\ &\text{and} \quad \partial_t\chi_\varepsilon|_\Gamma \in L^\infty(0,T;V_\Gamma)\cap H^1(0,T;H_\Gamma) \end{aligned} \tag{4.23}$$

which actually holds without further assumptions on the data. Such a regularity is formally obtained by differentiating equations (4.13)–(4.14) and then suitably testing the obtained equalities. Therefore, we have to check that all the derivatives we need actually exist at least for the further regularized problem. To this aim, we work on the discrete solution and behave exaclty as in [5]. We differentiate equations (4.16) and (4.17) and test the equalities we get by $\mathcal{N}\partial_t^2\mathcal{X}_\varepsilon^n$ and $-\partial_t^2\mathcal{X}_\varepsilon^n$, repectively. Finally, we take the sum, integrate over $(0,t)$, and add the same integrals to both sides for convenience. We obtain

$$\begin{aligned}
&\int_0^t \|\partial_t^2\mathcal{X}_\varepsilon^n(s)\|_*^2\,ds + \tau_\varepsilon\int_{Q_t}|\partial_t^2\mathcal{X}_\varepsilon^n|^2 + \int_{\Sigma_t}|\partial_t^2\mathcal{X}_\varepsilon^n|^2 \\
&\quad + \frac{1}{2}\,\|\partial_t\mathcal{X}_\varepsilon^n(t)\|_{1,\Omega}^2 + \frac{\nu}{2}\int_\Gamma |\nabla_\Gamma\partial_t\mathcal{X}_\varepsilon^n(t)|^2 + \frac{\nu}{2}\int_\Gamma |\partial_t\mathcal{X}_\varepsilon^n(t)|^2 \\
&= -\frac{1}{2}\int_{Q_t}(\beta_\varepsilon+\pi)'(\mathcal{X}_\varepsilon^n)\,\partial_t\big(|\partial_t\mathcal{X}_\varepsilon^n|^2\big) - \frac{1}{2}\int_{\Sigma_t}(\beta_{\Gamma,\varepsilon}+\pi_\Gamma)'(\mathcal{X}_\varepsilon^n)\,\partial_t\big(|\partial_t\mathcal{X}_\varepsilon^n|^2\big) \\
&\quad + \int_{Q_t}\lambda_\varepsilon'(\mathcal{X}_\varepsilon^n)\,\partial_t\vartheta_\varepsilon^n\,\partial_t^2\mathcal{X}_\varepsilon^n + \int_{Q_t}\lambda_\varepsilon''(\mathcal{X}_\varepsilon)\,\partial_t\mathcal{X}_\varepsilon^n\,\vartheta_\varepsilon^n\,\partial_t^2\mathcal{X}_\varepsilon^n \\
&\quad + \int_{\Sigma_t}\partial_t f_\Gamma\,\partial_t^2\mathcal{X}_\varepsilon^n + \frac{1}{2}\int_\Omega|\nabla\partial_t\mathcal{X}_\varepsilon^n(0)|^2 + \frac{\nu}{2}\int_\Gamma|\nabla_\Gamma\partial_t\mathcal{X}_\varepsilon^n(0)|^2 \\
&\quad + \frac{1}{2}\int_\Omega|\partial_t\mathcal{X}_\varepsilon^n(t)|^2 + \frac{\nu}{2}\int_\Gamma|\partial_t\mathcal{X}_\varepsilon^n(t)|^2.
\end{aligned} \tag{4.24}$$

Hence, the only significant difference with respect to [5, formula (4.47)] are the terms involving λ_ε. So, we deal with such terms, only. The first one is treated in a trivial way by owing to (4.20). As far as the second one is concerned, by using the Hölder inequality, the continuous embedding $V \subset L^4(\Omega)$, and (4.20) once more, we have

$$\begin{aligned}
&\int_{Q_t}\lambda_\varepsilon''(\mathcal{X}_\varepsilon)\,\partial_t\mathcal{X}_\varepsilon^n\,\vartheta_\varepsilon^n\,\partial_t^2\mathcal{X}_\varepsilon^n \le \delta\int_{Q_t}|\partial_t^2\mathcal{X}_\varepsilon^n|^2 + c_\delta c_\varepsilon\int_{Q_t}|\partial_t\mathcal{X}_\varepsilon^n|^2\,|\vartheta_\varepsilon^n|^2 \\
&\le \delta\int_{Q_t}|\partial_t^2\mathcal{X}_\varepsilon^n|^2 + c_\delta c_\varepsilon\int_0^t\|\vartheta_\varepsilon^n(s)\|_{L^4(\Omega)}^2\,\|\partial_t\mathcal{X}_\varepsilon^n(s)\|_{L^4(\Omega)}^2\,ds \\
&\le \delta\int_{Q_t}|\partial_t^2\mathcal{X}_\varepsilon^n|^2 + c_\delta c_\varepsilon\int_0^t\|\vartheta_\varepsilon^n(s)\|_{1,\Omega}^2\,\|\partial_t\mathcal{X}_\varepsilon^n(s)\|_{1,\Omega}^2\,ds \\
&\le \delta\int_{Q_t}|\partial_t^2\mathcal{X}_\varepsilon^n|^2 + c_\delta c_\varepsilon\int_0^t\|\partial_t\mathcal{X}_\varepsilon^n(s)\|_{1,\Omega}^2\,ds
\end{aligned}$$

so that we can control the above term by the Gronwall lemma for small δ.

Remark 4.3. We conclude this section by remarking that

$$\partial_t(\mathcal{X}_\varepsilon(t)_\Omega) = 0 \quad \text{for a.a. } t\in(0,T) \quad \text{and} \quad \mathcal{X}_\varepsilon(t)_\Omega = (\mathcal{X}_0)_\Omega \quad \text{for every } t\in[0,T] \tag{4.25}$$

as for problem (2.22)–(2.24).

5 Existence

In this section, we prove Theorem 2.5 and the starting point is a solution $(\vartheta_\varepsilon, \mathcal{X}_\varepsilon, w_\varepsilon)$ to the approximating problem (4.11)–(4.14). Our method relies on a priori estimates and compactness arguments. As far as the former are concerned, by accounting for Remark 4.2, we proceed formally in the sequel, e.g., by differentiating the equations and using some non-admissible test functions. However, before starting estimating, we collect some auxiliary material without any proof. We quote, e.g., [1] for the classical theory and [5, Sect. 5] for the specific results related to the problem we are dealing with.

By recalling (2.8)–(2.9), we note that the Yosida regularization γ_ε of every maximal monotone operator $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$ is monotone and Lipschitz continuous with constant $1/\varepsilon$. Moreover, we have $\gamma_\varepsilon(0) = 0$ whenever $\gamma(0) \ni 0$ and the inequality $|\gamma_\varepsilon(r)| \le |\gamma^\circ(r)|$ holds true for every $r \in D(\gamma)$ and $\varepsilon > 0$. As far as our case is concerned, we have

$$|\beta_\varepsilon(r)| \le |\beta^\circ(r)| \quad \text{and} \quad |\beta_{\Gamma,\varepsilon}(r)| \le |\beta_\Gamma^\circ(r)| + \frac{2C_\Gamma}{\alpha}$$
$$|\widehat{\beta}_\varepsilon(r)| \le |\widehat{\beta}(r)| \quad \text{and} \quad |\widehat{\beta}_{\Gamma,\varepsilon}(r)| \le |\widehat{\beta}_\Gamma(r)| + \frac{2C_\Gamma}{\alpha}\,|r| \tag{5.1}$$

for $r \in D(\beta)$ or $r \in D(\beta_\Gamma)$, accordingly. Moreover, the following inequalities hold

$$|\beta_\varepsilon(r)| \ge \alpha|\beta_{\Gamma,\varepsilon}(r)| - 2C_\Gamma \quad \text{for every } r \in \mathbb{R}. \tag{5.2}$$
$$\sigma|\beta_{\Gamma,\varepsilon}(r)| \ge (\sup|\pi_\Gamma'|)\,|r| + |\pi_\Gamma(0)| + \|f_\Gamma\|_{L^\infty(\Gamma)} \tag{5.3}$$
$$\beta_\varepsilon(r)\,(r - m_0) \ge \delta_0|\beta_\varepsilon(r)| - c \quad \text{and} \quad \beta_{\Gamma,\varepsilon}(r)\,(r - m_0) \ge \delta_0|\beta_{\Gamma,\varepsilon}(r)| - c. \tag{5.4}$$

Precisely, (5.2) and (5.4) hold for every $r \in \mathbb{R}$, the latter with a suitable $\delta_0 > 0$ and the notation $m_0 := (\chi_0)_\Omega$, while (5.3) is true for every $r \in \mathbb{R} \setminus (r_-^*, r_+^*)$ and ε small enough, for suitable $r_\pm^* \in D(\beta)$. Finally, we have

$$\|\partial_t\chi_\varepsilon(0)\|_*^2 + \tau_\varepsilon\|\partial_t\chi_\varepsilon(0)\|_{0,\Omega}^2 + \|\partial_t\chi_\varepsilon(0)\|_{0,\Gamma}^2 \le c. \tag{5.5}$$

At this point, we can start estimating.

First a priori estimate. We test (4.12) by ϑ_ε. At the same time, we test (4.13) by $\mathcal{N}\partial_t\chi_\varepsilon$ and (4.14) by $-\partial_t\chi_\varepsilon$, noting that $\partial_t\chi_\varepsilon$ has zero mean value by (4.25). Then, the terms involving λ_ε and w_ε cancel, the former obviously, the latter by (2.41). Hence, by integrating over $(0,t)$, owing to (2.43), and adding the same quantity for convenience, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\vartheta_\varepsilon(t)|^2 + \int_{Q_t} |\nabla\vartheta_\varepsilon|^2 + \kappa\int_{\Sigma_t} |\vartheta_\varepsilon|^2 \\
&\quad + \int_0^t \|\partial_t\chi_\varepsilon(s)\|_*^2\,ds + \tau_\varepsilon\int_{Q_t} |\partial_t\chi_\varepsilon|^2 + \int_{\Sigma_t} |\partial_t\chi_\varepsilon|^2 \\
&\quad + \frac{1}{2}\int_\Omega |\nabla\chi_\varepsilon(t)|^2 + \frac{\nu}{2}\int_\Gamma |\nabla_\Gamma\chi_\varepsilon(t)|^2 + \int_\Omega \widehat{\beta}_\varepsilon(\chi_\varepsilon(t)) + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(\chi_\varepsilon(t)) \\
&\quad + \|\chi_\varepsilon(t)\|_{0,\Omega}^2 + \|\chi_\varepsilon(t)\|_{0,\Gamma}^2 \\
&= \int_0^t \langle F(s), \vartheta_\varepsilon(s)\rangle\,ds - \int_\Omega \widehat{\pi}(\chi_\varepsilon(t)) - \int_\Gamma \widehat{\pi_\Gamma}(\chi_\varepsilon(t)) + \int_{\Sigma_t} f_\Gamma\,\partial_t\chi_\varepsilon \\
&\quad + \frac{1}{2}\int_\Omega |\vartheta_0|^2 + \frac{1}{2}\int_\Omega |\nabla\chi_0|^2 + \frac{\nu}{2}\int_\Gamma |\nabla_\Gamma\chi_0|^2 \\
&\quad + \int_\Omega \widehat{\beta}_\varepsilon(\chi_0) + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(\chi_0) + \int_\Omega \widehat{\pi}(\chi_0) + \int_\Gamma \widehat{\pi_\Gamma}(\chi_0) \\
&\quad + \|\chi_\varepsilon(t)\|_{0,\Omega}^2 + \|\chi_\varepsilon(t)\|_{0,\Gamma}^2
\end{aligned} \tag{5.6}$$

where we have set (see also (4.6))

$$\widehat{\pi}(r) := \int_0^r \pi(s)\,ds \quad \text{and} \quad \widehat{\pi_\Gamma}(r) := \int_0^r \pi_\Gamma(s)\,ds \quad \text{for } r \in \mathbb{R}. \tag{5.7}$$

We treat the first term on the right-hand side this way

$$\int_0^t \langle F(s), \vartheta_\varepsilon(s)\rangle \, ds \le \delta \int_0^t \|\vartheta_\varepsilon(s)\|_{1,\Omega}^2 \, ds + c_\delta \int_0^t \|F(s)\|_*^2 \, ds$$

and notice that the first of such contributions can be controlled by the left-hand side of (5.6) for small δ due to the second of (2.39). Next, the integral containing f_Γ can be estimated in a trivial way and no trouble arises from all the terms involving the initial values but those related to $\widehat{\beta}_\varepsilon$ and $\widehat{\beta}_{\Gamma,\varepsilon}$. However, for such terms, we can apply (5.1) and (2.30) and derive that

$$\int_\Omega \widehat{\beta}_\varepsilon(\chi_0) + \int_\Gamma \widehat{\beta}_{\Gamma,\varepsilon}(\chi_0) \le \int_\Omega \widehat{\beta}(\chi_0) + \int_\Gamma \bigl(\widehat{\beta}_\Gamma(\chi_0) + (2C_\Gamma/\alpha)|\chi_0|\bigr) = c.$$

Finally, as the functions $\widehat{\pi}$ and $\widehat{\pi_\Gamma}$ have at most a quadratic growth since π and π_Γ are Lipschitz continuous by (2.5), we just need to estimate the two last norms of (5.6). To this aim, we apply (2.47)–(2.48) and have

$$\begin{aligned}
&\|\chi_\varepsilon(t)\|_{0,\Omega}^2 + \|\chi_\varepsilon(t)\|_{0,\Gamma}^2 \\
&\le \|\chi_0\|_{0,\Omega}^2 + \delta \int_0^t \|\partial_t \chi_\varepsilon(s)\|_*^2 \, ds + c_\delta \int_0^t \|\chi_\varepsilon(s)\|_{1,\Omega}^2 \, ds \\
&\quad + \|\chi_0\|_{0,\Gamma}^2 + \delta \int_0^t \|\partial_t \chi_\varepsilon(s)\|_{0,\Gamma}^2 \, ds + c_\delta \int_0^t \|\chi_\varepsilon(s)\|_{0,\Gamma}^2 \, ds.
\end{aligned}$$

At this point, we can easily conclude that

$$\begin{aligned}
&\|\vartheta_\varepsilon\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \|\chi_\varepsilon\|_{L^\infty(0,T;V)\cap H^1(0,T;V^*)} + \|\chi_\varepsilon\|_{L^\infty(0,T;V_\Gamma)\cap H^1(0,T;H_\Gamma)} \\
&\quad + \tau_\varepsilon^{1/2} \|\partial_t \chi_\varepsilon\|_{L^2(0,T;H)} + \|\widehat{\beta}_\varepsilon(\chi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} + \|\widehat{\beta}_{\Gamma,\varepsilon}(\chi_\varepsilon)\|_{L^\infty(0,T;L^1(\Gamma))} \le c
\end{aligned} \tag{5.8}$$

just by choosing δ small enough and applying the Gronwall lemma.

Second a priori estimate. We want both to estimate $\partial_t \vartheta_\varepsilon$ in $L^2(0,T;H)$ and to improve (5.8). To this aim, we split F as $F_1 + F_2$ with $F_1 \in L^2(0,T;H)$ and $F_2 \in H^1(0,T;V^*)$ (and let the values of c below to depend on such a decomposition, for simplicity). Next, we test (2.22) by $\partial_t \vartheta_\varepsilon$. At the same time, we differentiate equations (4.13) and (4.14) with respect to time and test the equalities we obtain by $\mathcal{N}\partial_t \chi_\varepsilon$ and $-\partial_t \chi_\varepsilon$, respectively. Finally, we integrate over $(0,t)$ and take the sum. As before, we use the properties of $\mathcal{N}$. We get

$$\begin{aligned}
&\int_{Q_t} |\partial_t \vartheta_\varepsilon|^2 + \frac{1}{2} \int_\Omega |\nabla \vartheta_\varepsilon(t)|^2 + \frac{\kappa}{2} \int_\Gamma |\vartheta_\varepsilon(t)|^2 \\
&\quad + \frac{1}{2} \|\partial_t \chi_\varepsilon(t)\|_*^2 + \frac{\tau_\varepsilon}{2} \int_\Omega |\partial_t \chi_\varepsilon(t)|^2 + \frac{1}{2} \int_\Gamma |\partial_t \chi_\varepsilon(t)|^2 \\
&\quad + \int_{Q_t} |\partial_t \nabla \chi_\varepsilon|^2 + \nu \int_{\Sigma_t} |\partial_t \nabla_\Gamma \chi_\varepsilon|^2 + \int_{Q_t} \beta_\varepsilon'(\chi_\varepsilon) \, |\partial_t \chi_\varepsilon|^2 + \int_{\Sigma_t} \beta_{\Gamma,\varepsilon}'(\chi_\varepsilon) \, |\partial_t \chi_\varepsilon|^2 \\
&= \int_{Q_t} \bigl(F_1 - \lambda_\varepsilon'(\chi_\varepsilon) \, \partial_t \chi_\varepsilon\bigr) \, \partial_t \vartheta_\varepsilon + \int_0^t \langle F_2(s), \partial_t \vartheta_\varepsilon(s)\rangle \, ds + \frac{1}{2} \int_\Omega |\nabla \vartheta_0|^2 + \frac{\kappa}{2} \int_\Gamma |\vartheta_0|^2 \\
&\quad + \frac{1}{2} \|\partial_t \chi_\varepsilon(0)\|_*^2 + \frac{\tau_\varepsilon}{2} \int_\Omega |\partial_t \chi_\varepsilon(0)|^2 + \frac{1}{2} \int_\Gamma |\partial_t \chi_\varepsilon(0)|^2 \\
&\quad - \int_{Q_t} \pi'(\chi_\varepsilon(t)) \, |\partial_t \chi_\varepsilon|^2 - \int_{\Sigma_t} \pi_\Gamma'(\chi_\varepsilon(t)) \, |\partial_t \chi_\varepsilon|^2 \\
&\quad + \int_{Q_t} \bigl(\lambda_\varepsilon''(\chi_\varepsilon) \, \partial_t \chi_\varepsilon \, \vartheta_\varepsilon + \lambda_\varepsilon'(\chi_\varepsilon) \, \partial_t \vartheta_\varepsilon\bigr) \, \partial_t \chi_\varepsilon + \int_{\Sigma_t} \partial_t f_\Gamma \, \partial_t \chi_\varepsilon \, .
\end{aligned} \tag{5.9}$$

All the terms on the left-hand side are nonnegative. More precisely, the left-hand side controls $\|\vartheta_\varepsilon(t)\|_{1,\Omega}^2$ thanks to the second of (2.39). As far as those on right-hand side are concerned, we note that the terms involving $\lambda'_\varepsilon(\chi_\varepsilon)$ cancel. Moreover, the integral related to F_1 can be dealt with in a trivial way. Next, an integration by parts yields

$$\int_0^t \langle F_2(s), \partial_t \vartheta_\varepsilon(s)\rangle \, ds = \langle F_2(t), \vartheta_\varepsilon(t)\rangle - \langle F_2(0), \vartheta_0\rangle - \int_0^t \langle \partial_t F_2(s), \vartheta_\varepsilon(s)\rangle \, ds$$
$$\leq \delta \|\vartheta_\varepsilon(t)\|_{1,\Omega}^2 + c_\delta \|F_2(t)\|_*^2 + c + \int_0^t \|\vartheta_\varepsilon(s)\|_{1,\Omega}^2 \, ds.$$

The terms related to the initial values are estimated by (5.5) and the integrals involving π, π_Γ, and f_Γ are estimated by (5.8) since π' and π'_Γ are bounded and $\partial_t f_\Gamma \in L^2(0,T;H)$. Finally, we deal with the only non-trivial term. We owe to (4.1) and to the Hölder inequality and use the continuous embedding $V \subset L^4(\Omega)$ and inequality (2.49). Moreover, we take (5.8) into account and term C_1 the value of its right-hand side. Then, we have

$$\int_{Q_t} \lambda''_\varepsilon(\chi_\varepsilon) \, |\partial_t \chi_\varepsilon|^2 \, \vartheta_\varepsilon \leq C_\lambda \int_0^t \|\partial_t \chi_\varepsilon(s)\|_{L^4(\Omega)}^2 \|\vartheta_\varepsilon(s)\|_{L^2(\Omega)} \, ds$$
$$\leq C_\lambda C_1 \int_0^t \|\partial_t \chi_\varepsilon(s)\|_{L^4(\Omega)}^2 \, ds \leq C_\lambda C_1 \int_0^t \Big(\delta \|\partial_t \nabla \chi_\varepsilon(s)\|_{0,\Omega}^2 + c_\delta \|\partial_t \chi_\varepsilon(s)\|_*^2 \Big) \, ds$$
$$\leq \delta C_\lambda C_1 \int_{Q_t} |\partial_t \nabla \chi_\varepsilon|^2 + c_\delta C_\lambda C_1^3 = \delta C_\lambda C_1 \int_{Q_t} |\partial_t \nabla \chi_\varepsilon|^2 + c_\delta \, .$$

Therefore, we conclude that

$$\|\partial_t \vartheta_\varepsilon\|_{L^2(0,T;H)} + \|\vartheta_\varepsilon\|_{L^\infty(0,T;V)} + \|\partial_t \chi_\varepsilon\|_{L^\infty(0,T;V^*) \cap L^2(0,T;V)}$$
$$+ \|\partial_t \chi_\varepsilon\|_{L^\infty(0,T;H_\Gamma) \cap L^2(0,T;V_\Gamma)} + \tau_\varepsilon^{1/2} \|\partial_t \chi_\varepsilon\|_{L^\infty(0,T;H)} \leq c \tag{5.10}$$

by choosing δ small enough and applying the Gronwall lemma.

Consequences. We observe that (4.1) implies that $|\lambda_\varepsilon(r)| \leq c(1+r^2)$ for all $r \in \mathbb{R}$. Therefore, by owing to the Hölder inequality and to the continuous embedding $V \subset L^6(\Omega)$, we have

$$\|\lambda_\varepsilon(\chi_\varepsilon)\|_{L^\infty(0,T;L^3(\Omega))} \leq c\big(1 + \|\chi_\varepsilon\|_{L^\infty(0,T;L^6(\Omega))}^2\big) \leq c\big(1 + \|\chi_\varepsilon\|_{L^\infty(0,T;V)}^2\big) \leq c \tag{5.11}$$

thanks to (5.8). By using (5.10) as well, we can prove that

$$\|\partial_t \lambda_\varepsilon(\chi_\varepsilon)\|_{L^2(0,T;L^3(\Omega))} \leq c \quad \text{and} \quad \|\lambda'_\varepsilon(\chi_\varepsilon)\vartheta_\varepsilon\|_{L^\infty(0,T;L^3(\Omega))} \leq c \tag{5.12}$$

since $\partial_t \lambda_\varepsilon(\chi_\varepsilon) = \lambda'_\varepsilon(\chi_\varepsilon) \, \partial_t \chi_\varepsilon$ and $|\lambda'_\varepsilon(r)| \leq c(1 + |r|)$ for all $r \in \mathbb{R}$.

Further estimates. At this point, no significant changes have to be made to the argument of [5, Sect. 5] in order to obtain the estimates that are needed to conclude. Indeed, in the quoted paper, the appoximating problem differs from ours just in two points. Namely, equation (4.12) does not appear and there is a given function f in place of $\lambda'_\varepsilon(\chi_\varepsilon) \, \vartheta_\varepsilon$ in equation (4.14). On the other hand, just the assumption $f \in L^\infty(0,T;H)$ is necessary from now on, and estimate (5.12) has already been established. Furthermore, all the assumptions that are used in [5] to solve the technical issues are listed in our Theorem 2.5 as well. Therefore, we can conclude that

$$\|\xi_\varepsilon\|_{L^2(0,T;H)} + \|\xi_{\Gamma,\varepsilon}\|_{L^2(0,T;H_\Gamma)} + \|w_\varepsilon\|_{L^\infty(0,T;V)} \leq c \tag{5.13}$$

just by arguing exactly as in [5].

Conclusion. By collecting all the above estimates, we can easily infer existence for problem (2.22)–(2.24). Even though the argument is very similar to the one of [5], we prefer to proceed with some detail. By using standard compactness results, we see that limit functions exist such that the following convergence

$$\begin{aligned}
\vartheta_\varepsilon \to \vartheta \quad & \text{weakly star in } L^\infty(0,T;V) \cap H^1(0,T;H) && (5.14)\\
\chi_\varepsilon \to \chi \quad & \text{weakly star in } H^1(0,T;V) \cap W^{1,\infty}(0,T;V^*) && (5.15)\\
\partial_t \lambda_\varepsilon(\chi_\varepsilon) \to \zeta \quad & \text{weakly in } L^2(0,T;L^3(\Omega)) && (5.16)\\
\chi_\varepsilon|_\Gamma \to \chi|_\Gamma \quad & \text{weakly star in } H^1(0,T;V_\Gamma) \cap W^{1,\infty}(0,T;H_\Gamma) && (5.17)\\
\tau_\varepsilon \partial_t \chi_\varepsilon \to \tau \partial_t \chi \quad & \text{weakly star in } L^\infty(0,T;H) && (5.18)\\
w_\varepsilon \to w \quad & \text{weakly star in } L^\infty(0,T;V) && (5.19)\\
\beta_\varepsilon(\chi_\varepsilon) \to \xi \quad & \text{weakly in } L^2(0,T;H) && (5.20)\\
\beta_{\Gamma,\varepsilon}(\chi_\varepsilon|_\Gamma) \to \xi_\Gamma \quad & \text{weakly in } L^2(0,T;H_\Gamma) && (5.21)
\end{aligned}$$

holds at least for a subsequence, and now we prove that $(\vartheta, \chi, w, \xi, \xi_\Gamma)$ is a solution to our problem. First of all, the regularity requirements contained in (2.14)–(2.20) are fulfilled. Moreover, as the embedding $V \subset H$ and $V_\Gamma \subset H_\Gamma$ are compact, we can apply [13, Sect. 8, Cor. 4] and derive the strong convergence

$$\vartheta_\varepsilon \to \vartheta \quad \text{and} \quad \chi_\varepsilon \to \chi \quad \text{in } C^0([0,T];H) \quad \text{and} \quad \chi_\varepsilon|_\Gamma \to \chi|_\Gamma \quad \text{in } C^0([0,T];H_\Gamma) \tag{5.22}$$

and the corresponding convergence almost everywhere. In particular, the initial conditions (2.21) are fulfilled as well. Furthermore, we deduce that $\pi(\chi_\varepsilon)$ and $\pi_\Gamma(\chi_\varepsilon|_\Gamma)$ converge to $\pi(\chi)$ and to $\pi_\Gamma(\chi|_\Gamma)$ strongly in $C^0([0,T];H)$ and in $C^0([0,T];H_\Gamma)$, respectively, just by Lipschitz continuity, and that $\lambda_\varepsilon(\chi_\varepsilon)$ and $\lambda'_\varepsilon(\chi_\varepsilon)\,\vartheta_\varepsilon$ converge to $\lambda(\chi)$ and to $\lambda'(\chi)\,\vartheta$, respectively, at least weakly star in $L^\infty(0,T;L^3(\Omega))$ if we account for (5.11)–(5.12) as well. In particular, $\zeta = \partial_t \lambda(\chi)$. Finally, we have $\xi \in \beta(\chi)$ a.e. in Q and $\xi_\Gamma \in \beta_\Gamma(\chi|_\Gamma)$ a.e. on Σ with the same proof as in [5]. Therefore, we see that $(\vartheta, \chi, w, \xi, \xi_\Gamma)$ satisfies an integrated version of (2.22)–(2.24) which is equivalent to (2.22)–(2.24) itself. Hence, the proof of Theorem 2.5 is complete.

Remark 5.1. The solution constructed in the above proof is more regular than required in (2.14)–(2.20), as (5.14)–(5.21) clearly show. Moreover, if the functional F is given by (2.12) with $f \in L^2(0,T;H)$ and $\vartheta_\Gamma \in L^2(0,T;H_\Gamma)$, the solution $(\vartheta, \chi, w, \xi, \xi_\Gamma)$ satisfies all equation and boundary condition (1.11)–(1.15) (where we have to read ξ and ξ_Γ in place of $\beta(\chi)$ and $\beta_\Gamma(\chi)$, respectively), besides the variational formulation (2.22)–(2.24). Namely, all the ingredients of such equalities are functions (rather than functionals) and the equations and the boundary conditions hold a.e. in Q and a.e. on Σ, respectively. Let us start from the equation for temperature. By writing (1.11) in the sense of distributions and accounting for the regularity implied by (5.16), we derive that $\Delta\vartheta$ belongs at least to $L^2(0,T;H)$. Hence, we are allowed to perform integration by parts in space and can conclude that (1.11) and the Robin boundary condition are satisfied a.e. in Q and a.e. on Σ, respectively. The same we can do for equation (1.12) and the Neumann boundary condition $\partial_n w = 0$, since we have at least $\partial_t \chi \in L^2(0,T;H)$ (see (5.15)), whence also $\Delta w \in L^2(0,T;H)$. Finally, as far as equation (1.13) and the dynamic boundary condition (1.15) are concerned, we can simply quote [5, Rem. 5.4], where the analogue is proved for the case of a constant temperature just as consequence of the minimal regularity requirements and of the variational equations (2.23) and (2.24), where the term $\lambda'(\chi)\,\vartheta$ is replaced by a given function. As just the regularity $L^2(0,T;H)$ of the latter is used in that remark, the result applies to the present case, since $\lambda'(\chi)\,\vartheta \in L^2(0,T;L^3(\Omega))$, as we have observed in concluding the above proof.

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