THE CONSERVED PHASE FIELD SYSTEM WITH MEMORY

GIANNI GILARDI

Abstract. We consider the conserved phase field model for phase change and introduce thermal memory in it by replacing the Fourier law for the heat flux with the Gurtin–Pipkin law. We study the resulting system from the viewpoint of mathematical analysis and state existence, uniqueness, regularity, and long time behavior results.

1. Introduction

We start with the simplest conserved phase field system without memory, which is based on the dynamics for the order parameter given by the Cahn–Hilliard equation. The standard model reads

$$\partial_t(\vartheta + \ell\chi) - \Delta\vartheta = g \tag{1.1}$$
$$\partial_t\chi - \Delta w = 0 \tag{1.2}$$
$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta. \tag{1.3}$$

In these equations, ϑ represents the relative temperature, χ is an order or phase parameter, w is the so–called chemical potential, g is a given source term, and ℓ is the latent heat. The equations have to hold in $\Omega \times (0,T)$, where Ω is an open set in $\mathbb{R}^3$, which we assume to be bounded, connected, and smooth. This system is then complemented with initial and Neumann boundary conditions. In particular, homogeneous Neumann boundary conditions are given for w. This implies that

$$\int_\Omega \chi(t) = \int_\Omega \chi(0) \qquad \forall\, t \in [0,T]$$

i.e., the total mass of χ is conserved in time.

The above equation (1.1) comes from the energy balance

$$\partial_t(\vartheta + \ell\chi) + \operatorname{div}\mathbf{q} = g$$

provided that the Fourier law $\mathbf{q} = -\nabla\vartheta$ for the heat flux $\mathbf{q}$ is used. Instead, we consider the Gurtin–Pipkin law, i.e.,

$$\begin{aligned}
\mathbf{q}(x,t) &= -\int_{-\infty}^{t} k(t-s)\nabla\vartheta(x,s)\,ds \\
&= -\int_{-\infty}^{0} k(t-s)\nabla\vartheta(x,s)\,ds - \int_{0}^{t} k(t-s)\nabla\vartheta(x,s)\,ds \\
&= \mathbf{q}_0(x,t) - (k * \nabla\vartheta)(x,t)
\end{aligned}$$

where k is a given memory kernel depending only on time and $\mathbf{q}_0$ is known whenever the past history of ϑ is known. Hence, the energy balance becomes

$$\partial_t(\vartheta + \ell\chi) - \Delta(k * \vartheta) = f \tag{1.4}$$

where the right hand side f accounts for both g and $\mathbf{q}_0$.

On the other hand, we generalize equation (1.3) with a differential inclusion which might account for constraints on the phase parameter χ, and replace (1.3) by

$$w \in -\Delta\chi + \beta(\chi) + \sigma'(\chi) - \ell\vartheta \tag{1.5}$$

where β is a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ and σ is a smooth funcion. We use the notation σ' since the sum $\beta + \sigma'$ stands for the derivative (in a suitable sense) of the double–well part of a Ginzburg–Landau free energy potential $\mathcal{F}$ (see, e.g., [3]). Indeed, the expression of $\mathcal{F}$ involves the sum $j + \sigma$, where j is the convex function whose subdifferential is β.

Finally, we can let the latent heat ℓ depend on χ. This corresponds to replace ℓ by a given smooth function of χ in (1.4–5). If we term this function $\lambda'(\chi)$ and rewrite (1.4) as

$$\partial_t\vartheta + \ell\partial_t\chi - \Delta(k * \vartheta) = f$$

the above term $\ell\,\partial_t\chi$ and the term $\ell\vartheta$ in (1.5) have to be replaced by $\partial_t(\lambda(\chi))$ and $\lambda'(\chi)\,\vartheta$, respectively.

Hence, we study the following initial–boundary value problem

$$\partial_t\big(\vartheta + \lambda(\chi)\big) - \Delta(k * \vartheta) = g \tag{1.6}$$

$$\partial_t\chi - \Delta w = 0 \tag{1.7}$$

$$w = -\Delta\chi + \xi + \sigma'(\chi) - \lambda'(\chi)\,\vartheta \tag{1.8}$$

$$\xi \in \beta(\chi) \tag{1.9}$$

$$\partial_n(k * \vartheta) = \partial_n\chi = \partial_n w = 0 \tag{1.10}$$

$$\vartheta(0) = \vartheta_0 \qquad \text{and} \qquad \chi(0) = \chi_0 \tag{1.11}$$

where ∂_n is the normal derivative on $\partial\Omega$ and ϑ_0, χ_0 are given initial data.

The literature on phase field models without memory and on the Cahn–Hilliard equation is very wide and we confine ourselves to quoting the papers [1–3, 11–13], which are closer to the aim of this work, and the survey paper [15]. In particular, [13] deals with constraints like in (1.8–9).

A model with memory like (1.6–11) has been considered first in [14], where existence and uniqueness results are presented and a regularizing effect in time is shown in the simpler case of a cubic nonlinearity and of a latent heat independent of the phase parameter.

Instead, the results we present here are contained in [6-8]. The reader can see also the introduction of these papers for further references.

Finally, we want to quote some recent asymptotic analyses for the phase field system with memory in the framework of [14]. In [10], taking a perturbation of the Cattaneo kernel

$$k_\varepsilon(t) = \frac{1}{\varepsilon}\exp(-t/\varepsilon) \qquad (\varepsilon > 0) \tag{1.12}$$

as k, one proves that the obtained model tends to the model without memory as ε tends to zero. In [16] the term $\partial_t\chi$ in (1.7) is replaced by $\mu\,\partial_t\chi$ and the analysis as μ tends to zero is performed. Under some restriction on ℓ, one proves that such a model tends to the one obtained taking $\mu = 0$ formally and forgetting about the initial condition for χ.

As we have already said, we are going to state some of the results of [6–8], and we refer directly to them for the proofs. While the general structure of the above system is allowed for some of our theorems, a part of them deals with the simpler situation corresponding to the standard Cahn–Hilliard equation. Moreover, the two sets of results we give are based on different assumptions on the memory kernel. Indeed, beside $k \in L^1(0,T)$, typical assumptions on k are the following. Either k is smoother and $k(0) > 0$ or k is a kernel of positive type (see below). A prototype for both kinds of assumptions is the Cattaneo kernel (1.12).

2. General assumptions and abstract setting

In order to state problem (1.6–11) in a more precise form, we list our assumptions on the structure of the system and introduce a notation. As far as the graph β and the functions λ and σ are concerned, we always assume that

$$j : \mathbb{R} \to [0, +\infty) \quad \text{is convex, proper, and lower semicontinuous} \tag{2.1}$$

$$j(0) = 0, \quad \beta = \partial j, \quad \text{and} \quad \beta(0) \ni 0 \tag{2.2}$$

$$\lambda, \sigma \in C^1(\mathbb{R}) \qquad \text{and} \qquad \lambda',\, \sigma' \text{ are Lipschitz continuous} \tag{2.3}$$

and denote by $D(\beta)$ the effective domains of β. The same symbols β and $D(\beta)$ are used for the related maximal monotone operators and for their domains in the Hilbert spaces $L^2(\Omega)$ and $L^2(\Omega \times (0,T))$.

However, for some of our results we need the further assumption

$$\lambda \text{ is Lipschitz continuous} \tag{2.4}$$

or even we consider the simplest case

$$\beta(r) = r^3, \quad \sigma'(r) = -r, \quad \text{and} \quad \lambda'(r) = \ell r \quad (\ell > 0) \qquad \forall\, r \in \mathbb{R}. \tag{2.5}$$

Note that (2.3) allows λ to grow quadratically, while (2.4) does not.

As far as the memory kernel is concerned, we always assume that $k \in L^1(0,T)$ and that either of the following conditions is fulfilled

$$k \in W^{2,1}(0,T) \qquad \text{and} \qquad k(0) > 0 \tag{2.6}$$

$$\int_0^t (k * v)(s)\, v(s)\, ds \geq 0 \qquad \forall\, v \in L^2(0,T), \quad \forall\, t \in [0,T]. \tag{2.7}$$

Assumption (2.7) says that k is a kernel of positive type.

Next, we define the spaces

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad W := \{v \in H^2(\Omega) : \partial_n v = 0\}$$

so that the dense and compact embeddings

$$W \subset V \subset H \subset V' \subset W'$$

hold. Moreover, we define the operator $A : W \to H$ setting $A = -\Delta$, and extend it to the two operators $V \to V'$ and $H \to W'$, still termed A, by means of the formulas

$$\langle Au, v \rangle = \int_\Omega \nabla u \cdot \nabla v \qquad \forall\, u, v \in V$$

$$\langle Au, v \rangle = \int_\Omega u(-\Delta v) \qquad \forall\, u \in H, \quad \forall\, v \in W.$$

Then problem (1.6–11) can be written as follows

$$\big(\vartheta + \lambda(\chi)\big)' + A(k * \vartheta) = f \tag{2.8}$$

$$\chi' + Aw = 0 \tag{2.9}$$

$$w = A\chi + \xi + \sigma'(\chi) - \lambda'(\chi)\,\vartheta \tag{2.10}$$

$$\xi \in \beta(\chi) \tag{2.11}$$

$$\vartheta(0) = \vartheta_0 \qquad \text{and} \qquad \chi(0) = \chi_0 \tag{2.12}$$

where the abstract equations have to be read in either H, or V', or W', according to the regularity of the functions involved. Moreover, note that they include the Neumann boundary conditions of the original problem.

3. Results for a kernel of positive type

In this section we assume (2.5) and (2.7). Note that, in this case, (2.11) and (2.10) become definitions of ξ and w, respectively. Moreover, we suppose that the data fulfil the following regularity conditions

$$f \in L^1(0,T;H), \quad \vartheta_0 \in V, \quad \chi_0 \in H \tag{3.1}$$

and look for a solution to (2.8–12) satisfying

$$\vartheta \in H^1(0,T;W') \cap L^\infty(0,T;H) \tag{3.2}$$

$$\chi \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;W) \tag{3.3}$$

$$\vartheta + \ell\chi \in C^0([0,T];H) \tag{3.4}$$

$$w \in L^2(0,T;V). \tag{3.5}$$

Theorem 3.1. *Assume* (2.5), (2.7), *and* (3.1). *Then problem* (2.8–12) *has a unique solution satisfying the regularity requirements* (3.2–5).

Part of this result is already contained in [14], and the paper [8] essentially improves uniqueness, which was shown in [14] under some restrictions. Moreover, in the same work [8], a continuous dependence estimate in appropriate norms is proved.

The next result we want to present deals with the long time behavior of the solution. Therefore, we suppose that the above assumptions on the structure of our system and on the data hold for any $T > 0$. In addition, we assume

$$k \in L^1(0,+\infty) \qquad \text{and} \qquad \int_0^\infty k(s)\,ds \neq 0. \tag{3.6}$$

$$f \in L^1(0,\infty;H) \tag{3.7}$$

and consider the $\omega-$limit ω corresponding to a give solution defined as usual. The only point to be remarked is the choice of the topology. This comes essentially from (3.2–3), which ensure that (ϑ,χ) is continuous provided that it is considered as a $(V' \times H)-$valued function. Therefore, we define ω this way.

Definition 3.2. *A pair* $(\vartheta_\infty,\chi_\infty) \in V' \times H$ *belongs to* ω *if and only if there exists an increasing sequence* $t_n \nearrow +\infty$ *such that*

$$\vartheta(t_n) \to \vartheta_\infty \quad \text{in } V' \qquad \text{and} \qquad \chi(t_n) \to \chi_\infty \quad \text{in } H\,.$$

Then the following result holds. Its proof is given in [8]. It uses some ideas developed in [9] for the non conserved case and relies on global a priori estimates ensuring compactness and on the study of the translated trajectories.

Theorem 3.3. *Assume* (2.5), (2.7), *and* (3.1) *for any* $T > 0$. *In addition, assume* (3.6–7) *and let* (ϑ,χ) *and* ω *be the solution to* (2.8–12) *and the corresponding* $\omega-$*limit. Then* ω *is a nonempty, compact, and connected subset of* $V' \times H$. *Moreover, if* $(\vartheta_\infty,\chi_\infty) \in \omega$, *then* ϑ_∞ *assumes the constant value*

$$\vartheta_\infty = \frac{1}{|\Omega|}\left(\int_\Omega \vartheta_0 + \int_0^\infty \int_\Omega f\right) \tag{3.8}$$

and $\chi_\infty \in V$ *solves the problem*

$$-\Delta\chi_\infty + \chi_\infty^3 - \chi_\infty - \ell\vartheta_\infty = w_\infty \tag{3.9}$$

$$\int_\Omega \chi_\infty = \int_\Omega \chi_0 \qquad \text{and} \qquad \partial_n\chi_\infty = 0 \tag{3.10}$$

where w_∞ *is given by the constant*

$$w_\infty = \frac{1}{|\Omega|} \int_\Omega (\chi_\infty^3 - \chi_\infty) - \ell\vartheta_\infty. \tag{3.11}$$

Additionally, the entire family $\{\vartheta(t)\}$ *converges in the sense that*

$$\vartheta(t) \to \vartheta_\infty \quad \text{weakly in } H \text{ and strongly in } V' \text{ as } t \to \infty\,. \tag{3.12}$$

4. Results for a smooth kernel

This section deals with system (2.8–12) its full generality and uses assumption (2.6) on the memory kernel. The proofs are due to [6–7] and follow the method developed in [4–5] for the nonconserved case and based on the choice of new unknown functions. Therefore we introduce

$$u := 1 * \vartheta \qquad \text{and} \qquad z := 1 * \bigl(\vartheta + \lambda(\chi)\bigr) = u + 1 * \lambda(\chi). \tag{4.1}$$

and obtain

$$\bigl(u' + \lambda(\chi)\bigr)' + k(0)Au = f - A(k' * u) \tag{4.2}$$
$$\chi' + Aw = 0 \tag{4.3}$$
$$w = A\chi + \xi + \sigma'(\chi) - \lambda'(\chi)u' \tag{4.4}$$
$$\chi \in D(\beta) \quad \text{and} \quad \xi \in \beta(\chi) \tag{4.5}$$
$$u(0) = 0, \quad u'(0) = \vartheta_0, \quad \text{and} \quad \chi(0) = \chi_0 \tag{4.6}$$

if we use u as unknown function. In terms of z, instead, we obtain

$$z'' + k(0)Az = f - A(k' * z) + A\bigl(k * \lambda(\chi)\bigr) \tag{4.7}$$
$$\chi' + Aw = 0 \tag{4.8}$$
$$w = A\chi + \xi + \sigma'(\chi) - \lambda'(\chi)\bigl(z' - \lambda(\chi)\bigr) \tag{4.9}$$
$$\chi \in D(\beta) \quad \text{and} \quad \xi \in \beta(\chi) \tag{4.10}$$
$$z(0) = 0, \quad z'(0) = \vartheta_0 + \lambda(\chi_0), \quad \text{and} \quad \chi(0) = \chi_0. \tag{4.11}$$

Note that the left hand side of (4.7) is essentially the wave operator applied to z, due to assumption (2.6). Moreover, the operator A that appears on the right hand side is combined with the regularizing effect of the convolution and the term $A(k' * z)$ can essentially be considered as a perturbation. Hence, one expects the regularity for z that is typical for the solutions to second order hyperbolic equations.

As far as the data are concerned, we assume

$$f \in L^1(0,T;H) + W^{1,1}(0,T;V') \tag{4.12}$$
$$\vartheta_0 \in H, \qquad \chi_0 \in V, \qquad \text{and} \qquad j(\chi_0) \in L^1(\Omega) \tag{4.13}$$

and note that the V'-valued part of f could account also for inhomogeneous Neumann data for $k * \vartheta$ in the original problem. We have the following result.

Theorem 4.1. *Assume* (2.1–4)*. Assume moreover that the mean value of χ_0 is an interior point of $D(\beta)$, i.e.*

$$\mu_0 := \frac{1}{|\Omega|} \int_\Omega \chi_0 \ \in \ \operatorname{int} D(\beta). \tag{4.14}$$

Then there exists a unique pair (u, χ) and some pair (w, ξ) such that

$$u \in C^0([0,T]; V) \cap C^1([0,T]; H) \tag{4.15}$$
$$\chi \in H^1(0,T; V') \cap L^\infty(0,T; V) \cap L^2(0,T; W) \tag{4.16}$$
$$j(\chi) \in L^\infty(0,T; L^1(\Omega)) \tag{4.17}$$
$$w \in L^2(0,T; V) \tag{4.18}$$
$$\xi \in L^2(0,T; H) \tag{4.19}$$

which solves problem (4.2–6).

This theorem is due to [6] and its existence part is proved by regularization and a priori estimates. For both existence and uniqueness a number of technical points is present. Indeed, one would like to construct appropriate test functions by inverting the operator A, which is not one-to-one.

Moreover, it is shown in the same paper that the problem might have no solutions if (4.14) does not hold and that uniqueness for (w, ξ) is false in our general framework. On the contrary, uniqueness surely holds even for (w, ξ) if β is single valued.

Finally, we have a similar statement in term of the unknown function z, where the regularity property (4.15) is replaced by

$$z \in C^0([0,T]; V) \cap C^1([0,T]; H). \tag{4.20}$$

This setting is used in the regularity results stated below and due to [6] too.

Remark 4.2. The existence part of Theorem 4.1 is improved in [7], where assumption (2.4) is removed. The argument is technical and consists essentially in proving similar a priori estimates for the approximate solutions without using (2.4). The key point relies on the choice of a suitable test function, combined with a boot–strap procedure.

Theorem 4.3. *In addition to the assumptions of Theorem 4.1, suppose that*

$$f \in W^{1,1}(0,T; H) + W^{2,1}(0,T; V'), \quad f(0) \in H, \quad \text{and} \quad \vartheta_0 \in V. \tag{4.21}$$

Then, we have

$$z \in W^{2,\infty}(0,T; H) \cap W^{1,\infty}(0,T; V). \tag{4.22}$$

Theorem 4.4. *In addition to the assumptions of Theorem 4.1, suppose that*

$$f \in L^2(0,T; H) + W^{1,1}(0,T; V'), \quad \chi_0 \in D(\beta) \tag{4.23}$$

and that there exists some function ξ_0 *satisfying the following conditions*

$$\xi_0 \in H, \qquad \xi_0 \in \beta(\chi_0) \tag{4.24}$$

$$w_0 := A\chi_0 + \xi_0 + \sigma'(\chi_0) - \lambda'(\chi_0)\,\vartheta_0 \in V. \tag{4.25}$$

Then, we have

$$\chi \in W^{1,\infty}(0,T;V') \cap H^1(0,T;V) \cap L^\infty(0,T;W). \tag{4.26}$$

We note that (4.26) yields $\chi \in L^\infty(\Omega \times (0,T))$, due to the continuous embedding $W \subset L^\infty(\Omega)$, which holds in our threedimensional setting.

References

1. H.W. Alt and I. Pawlow: *On the entropy principle of phase transition models with a conserved order parameter*, Adv. Math. Sci. Appl., **6** (1996), 291–376.
2. D. Brochet, D. Hilhorst, and A. Novick–Cohen: *Maximal attractor and inertial sets for a conserved phase field model*, Adv. Differential Equations, **1** (1996), 547–578.
3. G. Caginalp: *The dynamics of a conserved phase field system: Stefan-like, Hele-Shaw, and Cahn-Hilliard models as asymptotic limits*, IMA J. Appl. Math., **44** (1990), 77–94.
4. P. Colli, G. Gilardi, and M. Grasselli: *Global smooth solution to the standard phase–field model with memory*, Adv. Differential Equations, **2** (1997), 453-486.
5. P. Colli, G. Gilardi, and M. Grasselli: *Well–posedness of the weak formulation for the phase–field model with memory*, Adv. Differential Equations, **2** (1997), 487–508.
6. P. Colli, G. Gilardi, M. Grasselli, and G. Schimperna: *The conserved phase–field system with memory*, submitted.
7. P. Colli, G. Gilardi, M. Grasselli, and G. Schimperna: *Global existence for the conserved phase field model with memory and quadratic nonlinearity*, to appear.
8. P. Colli, G. Gilardi, Ph. Laurençot, A. Novick–Koen: *Uniqueness and long–time behaviour for the conserved phase–field system with memory*, Discrete Contin. Dynam. Systems, **5** (1999), 375–390.
9. P. Colli and Ph. Laurençot: *Existence and stabilization of solutions to the phase–field model with memory* , J. Integral Equations Appl., **10** (1998), 169–194.
10. V. Felli: graduation thesis, Pavia, 1999.
11. N. Kenmochi: *Systems of nonlinear PDEs arising from dynamical phase transitions*, in "Phase transitions and hysteresis", A. Visintin ed., pp. 39–86, Lecture Notes in Math., **1584**, Springer–Verlag, Berlin, 1994.

12. N. Kenmochi and M. Niezgódka: *Nonlinear system for non–isothermal diffusive phase separation*, J. Math. Anal. Appl., **188** (1994), 651–679.

13. N. Kenmochi, M. Niezgódka, and I. Pawlow: *Subdifferential operator approach to the Cahn–Hilliard equation with constraint*, J. Differential Equations, **117** (1995), 320–356.

14. A. Novick–Cohen: *Conserved phase–field equations with memory*, in "Curvature flows and related topics", A. Damlamian, J. Spruck, and A. Visintin eds., pp. 179–197, GAKUTO Internat. Ser. Math. Sci. Appl., **5**, Gakkōtosho, Tokyo, 1995.

15. A. Novick–Cohen: *The Cahn–Hilliard equation: mathematical and modeling perspectives*, Adv. Math. Sci. Appl., **8** (1998), 965–985.

16. E. Rocca: graduation thesis, Pavia, 1999.

Gianni Gilardi
Dipartimento di Matematica "F. Casorati"
Via Ferrata 1
27100 Pavia, Italy