Cahn-Hilliard and Thin film equations with nonlinear mobility as gradient flows in weighted Wasserstein metrics

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Jointly with S. Lisini, D. Matthes, R. McCann
Outline

1 Evolution PDE’s with a gradient flow structure
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2 The dynamical approach to weighted transport distances
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2. The dynamical approach to weighted transport distances

3. Basic tools for metric gradient flows: displacement convexity, variational approximation (JKO and WED schemes), flow interchange.
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2 The dynamical approach to weighted transport distances

3 Basic tools for metric gradient flows: displacement convexity, variational approximation (JKO and WED schemes), flow interchange.
A general class of evolutionary PDE’s

In many applications one is interested in nonnegative integrable solutions to evolution equations of the type

$$\partial_t u - \text{div} \left( m(u) D \frac{\delta \Phi}{\delta u} \right) = 0 \quad \text{in } \Omega \times (0, \infty), \quad \Omega \subset \mathbb{R}^d,$$

with Neumann-variational boundary conditions

$$\mathbf{n} \cdot D u = 0, \quad \mathbf{n} \cdot D \left( m(u) \frac{\delta \Phi}{\delta u} \right) = 0$$

and initial condition \( u_0 \in L^1(\Omega), \ u_0 \geq 0. \)
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and initial condition $$u_0 \in L^1(\Omega), \quad u_0 \geq 0.$$ The equation can be split

$$\begin{cases} \partial_t u + \text{div} w = 0 \\ \text{(Continuity equation)} \end{cases}$$
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$$\begin{cases} 
\partial_t u + \text{div} w = 0 \\
w = m(u) v = -m(u) D\psi 
\end{cases}$$

(Continuity equation)

(Flux structure)

$m : [0, +\infty) \to [0, +\infty)$ is a given mobility function associated to the equation.
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\psi = \frac{\delta \Phi}{\delta u} & \text{(Nonlinear variational condition)}
\end{cases}
\]

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Examples: 2nd order equations

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Drift-diffusion-interaction (\( m(u) = u \)):
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\partial_t u = \Delta u + \text{div}(u D V) + \text{div}(u D W * u) = \text{div} \left( u \, D(\log u + V + W * u) \right),
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Chemotaxis with overcrowding prevention [Hillen-Painter ’01]:

\[ \partial_t u = \text{div} \left( Du + m(u) D(W * u) \right) = \text{div} \left( m(u) D(F'(u) + W * u) \right) \]
Examples: 4th order equations

**Thin film** (typically $m(u) = u^\alpha$):

$$\partial_t u + \text{div} (m(u) D\Delta u) = 0,$$

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\Phi(u) := \frac{1}{2} \int \mathbb{R} d|Du|^2, 

\Phi(u) := \frac{1}{2\beta} \int \mathbb{R} d|Du|^{2\beta},$

Derrida-Lebowitz-Speer-Spohn '91 [Bleher-Lebowitz-Speer, Jüngel, Pinnau, Matthes, Gianazza-Toscani-S.]

Cahn-Hilliard: ($m(u) = u(1-u)$)

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[Elliott-Garcke '96]
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[Bernis-Friedman ’90, Bertsch-Dal Passo-Garcke-Gr"un; Becker-Gr"un, Carrillo-Toscani ’02, Carlen-Ulusoy ’07]
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[Elliott-Garcke ‘96]
The gradient flow structure: a formal motivation

\[ \partial_t u + \text{div} \ w = 0, \]

\[ \psi = \frac{\delta \Phi}{\delta u} \]

\[ \frac{d}{dt} \Phi(u_t) = \int_{\mathbb{R}^d} \partial_t u \frac{\delta \Phi}{\delta u} \ dx = -\int_{\mathbb{R}^d} (\text{div} \ w) \psi \ dx = \int_{\mathbb{R}^d} w \cdot D\psi \ dx \]
The gradient flow structure: a formal motivation

$$\partial_t u + \text{div} \ w = 0, \quad \ w = m(u)v \quad \psi = \frac{\delta \Phi}{\delta u}$$

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$$= \int_{\mathbb{R}^d} \nu \cdot \nabla \psi \ m(u) \ dx \geq -\left( \int_{\mathbb{R}^d} |\nabla \psi|^2 \ m(u) \ dx \right)^{1/2} \left( \int_{\mathbb{R}^d} |\nu|^2 \ m(u) \ dx \right)^{1/2}$$
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\textbf{Ansatz:} interpret

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\left( \int_{\mathbb{R}^d} |v|^2 \, m(u) \, dx \right)^{1/2}
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as the “velocity” of the moving family \( u \).
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If we want to decrease \( \Phi \) as fast as possible, we have to choose

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v = -D\psi
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3. Basic tools for metric gradient flows: displacement convexity, variational approximation (JKO and WED schemes), flow interchange.
Weighted transport distances: the dynamical approach

[Benamou-Brenier ’00]
We interpret \( u \) as the density of a (probability) measure \( \rho = u \, dx \) and we consider a time dependent family \( u_t, \ t \in [0, T] \), of densities satisfying the nonlinear continuity equation

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The scalar velocity at time $t$ is given by

$$\mathcal{V}_t[u_t] := \| v_t \|_{L^2(m(u_t); \mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |v_t(x)|^2 \, m(u_t) \, dx \right)^{1/2}.$$
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The length of the curve $u$ between $t_0$ and $t_1$

$$\mathcal{L}^{t_1}_{t_0}[u] := \int_{t_0}^{t_1} \mathcal{V}_t[u] \, dt = \int_{t_0}^{t_1} \left( \int_{\mathbb{R}^d} |v_t(x)|^2 \, m(u_t) \, dx \right)^{1/2} \, dt$$
**Weighted transport distances: the dynamical approach**

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$$

**Weighted transport distance $W_m$** between $u_0$ and $u_1$:

$$
W_m(u_0, u_1) := \min \left\{ \mathcal{L}_{t=0}^{1}[u] : u|_{t=0} = u_0, \ u|_{t=1} = u_1 \right\}.
$$
Limiting cases

\[ m(r) \equiv 1 \quad \leftrightarrow \quad \text{Homogeneous dual } W^{-1,2}(\mathbb{R}^d) \text{ distance.} \]

\[
W_m(u_0, u_1) = \sup \left\{ \int_{\mathbb{R}^d} (u_0 - u_1) \varphi \, dx : \int_{\mathbb{R}^d} |D\varphi|^2 \, dx \leq 1 \right\}
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Hilbert Theory [Benilan, Brezis, Crandall, Pazy, . . . ∼’70]
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\[ v = w, \quad \partial_t u + \text{div } w = 0. \quad W^2_m(u_0, u_1) = \min \left\{ \int |w|^2 : \text{div } w = u_1 - u_2 \right\} \]

Hilbert Theory [Benilan, Brezis, Crandall, Pazy, ... \sim ’70]

\[ m(r) = r \iff \text{Wasserstein distance, } W_m = W; \]
characterization in terms of optimal transport, linear transport equation

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Limiting cases

$m(r) \equiv 1 \iff \text{Homogeneous dual } \mathcal{W}^{-1,2}(\mathbb{R}^d) \text{ distance.}$

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[JORDAN-KINDERLEHRER-OTTO ’98, OTTO ’01]

Applications: optimal transport, existence and asymptotic behaviour of solutions, contraction properties, Logarithmic Sobolev Inequalities, approximation algorithms, curvature and metric measure spaces, stability,...

[AMBROSIO-GIGLI-S., AGUEH, BRENIER, CARRILLO, CARLEN, MCCANN, GANGBO, GIACOMELLI, GIANAZZA-TOSCANI-S., LISINI, OTTO, SLEPCEV, STURM, VILLANI, WESTDICKENBERG, ...]
The interest of the method and the main problems

ADVANTAGES

- **Non-negativity** is for free.
- A general approximation scheme, which is a variational formulation of the backward Euler method, is always available.
- **Decay of the generating functional** $\Phi$ along the (discrete/continuous) flow.
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- You lose the linear structure of the underlying space.
- The distance is not flat and the space behaves like the an infinitely dimensional, non-smooth, positively curved Riemannian manifold.
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▶ Are there interesting convexity properties of the integral functionals and related functional inequalities?
The variational problem

**Problem**

*Given nonnegative densities* $u_0, u_1 \in L^1_{\text{loc}}(\mathbb{R}^d)$ *find a minimizer of the action functional*

$$
\int_0^1 \int_{\mathbb{R}^d} |v_t|^2 m(u_t) \, dx \, dt \quad \text{s.t.} \quad \partial_t u + \text{div}(m(u_t)v_t) = 0, \quad u|_{t=0,1} = u_{0,1}.
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**Direct method of the calculus of variations:** fix the densities \( u_0, u_1 \) and take a minimizing sequence \((u^n_t, w^n_t, v^n_t)\) with \( w^n_t = m(u^n_t)v^n_t \), such that

\[
\partial_t u^n_t + \text{div}(m(u^n_t)v^n_t) = 0, \quad u^n\big|_{t=0,1} = u_{0,1}, \quad \int_0^1 \int_{\mathbb{R}^d} |v^n_t|^2 m(u^n_t) \, dx \, dt \to \inf
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**Problem:** sublevels of the minimizing functional are only *weakly*\(^*\) relatively compact: we get *weak*\(^*\) convergence of a suitable subsequence but the equation \( \partial_t u_t + \text{div}(m(u)v) = 0 \) is nonlinear in the couple \((u, v)\).
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Basic idea: write everything in terms of $(u, w)!$ Since $w = m(u)v$ we minimize

$$A(u, w) := \int_0^1 \int_{\mathbb{R}^d} A(u_t, w_t) \, dx \, dt \quad \text{s.t.} \quad \partial_t u + \text{div} w_t = 0, \quad u|_{t=0,1} = u_{0,1}. $$

where

$$A(u, w) := \frac{|w|^2}{m(u)}$$
Convexity (and l.s.c.) of the action requires a concave mobility

**Lemma**

The function

\[ A : (u, w) \in (0, +\infty) \times \mathbb{R}^d \to \frac{|w|^2}{m(u)} \in [0, +\infty] \]

is convex iff \( m : [0, +\infty) \to [0, \infty) \) is concave.
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Two cases:

A) \( m : [0, +\infty) \to [0, +\infty) \) is concave and nondecreasing.

Model example: \( m(u) = u^\alpha, \) \( 0 \leq \alpha \leq 1. \) In this case \( A(\lambda u, \lambda w) \) is superlinear as \( \lambda \uparrow +\infty, \) except when \( w = 0. \)
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**B)** \( m : [0, M] \rightarrow [0, +\infty) \) is concave with \( m(0) = m(M) = 0 \).

Model example: \( m(u) = u(M - u) \). In this case \( A(u, w) = +\infty \) if \( u > M \) and all the densities \( u \) are uniformly bounded.
A rigorous definition through convex functional of measures

To get weak* lower semicontinuity of $A$, we extend it to couples $(\rho, \nu)$ where $\rho \in \mathcal{M}_{1\text{oc}}(\mathbb{R}^d)$ is a nonnegative Radon measure and $\nu \in \mathcal{M}_{1\text{oc}}(\mathbb{R}^d; \mathbb{R}^d)$ is a Radon vector measure.
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Moreover, the function $A$ is no more 1–homogeneous in the couple $(\rho, \nu)$, so that the definition of $A$ also depends from a reference measure $\gamma \in M_{\text{loc}}(\mathbb{R}^d)$ (usually the Lebesgue measure, but not necessarily).
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**Definition (The case of a sublinear mobility)**

If $\rho \in M_{\text{loc}}(\mathbb{R}^d), \nu \in M_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ we set

$$A(\rho, \nu | \gamma) := \int_{\mathbb{R}^d} A\left(\frac{d\rho}{d\gamma}, \frac{d\nu}{d\gamma}\right) d\gamma$$

Given $\rho_0, \rho_1$ we have

$$W_{m, \gamma}^2(\rho_0, \rho_1) := \inf \left\{ \int_0^1 A(\rho_t, \nu_t) dt \mid \partial_t \rho + \text{div} \nu = 0, \quad \rho|_{t=0,1} = \rho_{0,1} \right\}$$

We call $M_{m, \gamma}[\sigma]$ the collection of all measures at finite distance from $\sigma$.

[Dolbeault-Nazaret-S. '09, Lisini-Marigonda '10]
The role of $\gamma$ and simple properties of $W_{m,\gamma}$

1. Typically $\gamma = \mathcal{L}^d$ (omitted in $W_m$).
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**Theorem (Dolbeault-Nazaret-S.)**

Suppose that $\gamma^n \rightharpoonup \gamma$, $\rho^n_i \rightharpoonup \rho_i$ in $\mathcal{M}_{loc}(\mathbb{R}^d)$ and $m^n \downarrow m$ pointwise in $[0, +\infty)$. Then

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**Simplest case:** bounded $\Omega$ with $m$ defined on a bounded interval. In this case $W_m$ induces the weak* topology on $L^\infty_+(\Omega)$. 
Outline

1. Evolution PDE's with a gradient flow structure

2. The dynamical approach to weighted transport distances

3. Basic tools for metric gradient flows: displacement convexity, variational approximation (JKO and WED schemes), flow interchange.
Displacement convexity for weighted transport distances

A functional $\Phi$ is displacement convex if for every $u_0, u_1$ there exists a geodesic $u_t$, $t \in [0, 1]$, w.r.t. $W_m$ connecting $u_0$ to $u_1$ such that

$$W_m(u_t, u_s) = |t - s|W_m(u_0, u_1), \quad \Phi(u_t) \leq (1 - t)\Phi(u_0) + t\Phi(u_1).$$

The functional generating the Heat equation is always displacement convex.
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**Theorem (Generalized McCann condition \[Carrillo-Lisini-S.-Slepcev '09\])**

The functional

$$\Phi(u) = \int F(u) \, dx$$

is displacement convex in $\mathcal{M}_m(\Omega)$ with respect to the distance $W_m$ if

$$r \mapsto \frac{H(r)}{m(r)^{1-1/d}} \text{ is nonnegative and non decreasing in } (0, +\infty),$$

where

$$H(r) := \int_0^r F''(z)m(z)m'(z) \, dz.$$
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\]

The functional generating the Heat equation is always displacement convex.
Given $u_0 \in D(\Phi) \subset L^\infty_+(\Omega)$ and a relaxation parameter $\varepsilon > 0$ consider the space-time minimization of the WED functional

$$
\mathcal{J}_\varepsilon(u_0) := \min \left\{ \int_0^\infty \frac{e^{-t/\varepsilon}}{\varepsilon} \left( \varepsilon \int_{\mathbb{R}^d} |v_t|^2 m(u_t) \, dx + \Phi(u_t) \right) \, dt : \partial_t u_t + \text{div} \left( m(u_t) v_t \right) = 0, \quad u(\cdot, 0) = u_0 \right\}
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Weighted Energy-Dissipation (WED) approximation

Given \( u_0 \in D(\Phi) \subset L^\infty_+(\Omega) \) and a relaxation parameter \( \varepsilon > 0 \) consider the space-time minimization of the WED functional

\[
\mathcal{J}_\varepsilon(u_0) := \min \left\{ \int_0^\infty \frac{e^{-t/\varepsilon}}{\varepsilon} \left( \varepsilon \int_{\mathbb{R}^d} |\nu_t|^2 m(u_t) \, dx + \Phi(u_t) \right) \, dt : \partial_t u_t + \text{div} \left( m(u_t) \nu_t \right) = 0, \quad u(\cdot, 0) = u_0 \right\}
\]

**Theorem (Rossi-S.-Segatti-Stefanelli)**

Assume that \( \Phi \) is displacement convex w.r.t. \( W_m \) and has compact sublevels. Then the family of minimizers \( \{u_\varepsilon\} \) of the WED functional is relatively compact and every limit point is a gradient flow of \( \Phi \).
The JKO-De Giorgi’s Minimizing movement scheme

- Choose a partition of \((0, +\infty)\) of **step size** \(\tau > 0\)
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\[
\begin{align*}
U^0_{\tau} \\
\tau \\
t_0 & \quad t_1 & \quad t_2 & \quad t_3 & \quad t_4 & \cdots & \quad t_n
\end{align*}
\]

- Starting from \(U^0_{\tau} := \rho_0\) **find recursively** \(U^n_{\tau}, \ n = 1, 2, \ldots,\)

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\frac{U^n_{\tau} - U^{n-1}_{\tau}}{\tau} + \nabla \Phi(U^n_{\tau}) = 0
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\frac{U^n_\tau - U^{n-1}_\tau}{\tau} + \nabla \Phi(U^n_\tau) = 0 \implies U^n_\tau \in \arg\min_V \frac{W^2_m(V, U^{n-1}_\tau)}{2\tau} + \Phi(V)
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- Starting from \(U^0_\tau := \rho_0\) find recursively \textbf{minimizers} \(U^n_\tau, \ n = 1, 2, \ldots\)

\[
\frac{U^n_\tau - U^{n-1}_\tau}{\tau} + \nabla \Phi(U^n_\tau) = 0 \implies U^n_\tau \in \underset{V}{\text{argmin}} \frac{W^2_m(V, U^{n-1}_\tau)}{2\tau} + \Phi(V)
\]

- \(U_\tau\) is the \textbf{piecewise constant} interpolant of \(\{U^n_\tau\}_n\).

We look for \textbf{convergence results} of \(U_\tau\) as \(\tau \downarrow 0\).
First variation along auxiliary flows

**MAIN IDEA:** take the first variation of the minimum problem

\[ U^n_\tau \in \arg\min_V W^2(V, U^{n-1}_\tau) + \Phi(V) \]

along the gradient flow \( S^\Psi \) generated by other “good” auxiliary functionals \( \Psi \).
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HEURISTICS: in an euclidean space \( S^\Phi, S^\Psi \) corresponds to

\[ u_t := S^\Phi(u_0) \text{ solves } \frac{d}{dt} u = -\nabla \Phi(u), \quad w_t := S^\Psi(w_0) \text{ solves } \frac{d}{dt} w = -\nabla \Psi(w) \]
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**HEURISTICS:** in an euclidean space $S^\Phi, S^\Psi$ corresponds to

$u_t := S^\Phi_t(u_0)$ solves $\frac{d}{dt} u = -\nabla \Phi(u)$, \quad $w_t := S^\Psi_t(w_0)$ solves $\frac{d}{dt} w = -\nabla \Psi(w)$

If $u_0 = w_0$ then we have the "commutation" identity

$$\frac{d}{d\varepsilon} \Phi(w_\varepsilon)\bigg|_{\varepsilon=0^+} = \frac{d}{d\varepsilon} \Psi(u_\varepsilon)\bigg|_{\varepsilon=0^+}$$
First variation along auxiliary flows

**MAIN IDEA:** take the first variation of the minimum problem

\[ U^n_n \in \arg\min_V W^2(V, U^{n-1}_\tau) + \Phi(V) \]

along the gradient flow \( S^\Psi \) generated by other "good" auxiliary functionals \( \Psi \).

**HEURISTICS:** in an euclidean space \( S^\Phi, S^\Psi \) corresponds to

\[
\begin{align*}
\u_t &:= S^\Phi_t(u_0) \text{ solves } \frac{d}{dt} u = -\nabla \Phi(u), & w_t &:= S^\Psi_t(w_0) \text{ solves } \frac{d}{dt} w = -\nabla \Psi(w)
\end{align*}
\]

If \( u_0 = w_0 \) then we have the "commutation" identity

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\frac{d}{d\epsilon} \Phi(w_\epsilon) \bigg|_{\epsilon=0^+} = \frac{d}{d\epsilon} \Psi(u_\epsilon) \bigg|_{\epsilon=0^+} \quad \left( = -\langle \nabla \Phi(w_0), \nabla \Psi(u_0) \rangle \right)
\]
**First variation along auxiliary flows**

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**RECIPE:** if the derivative of the (main) functional $\Phi$ along the (auxiliary) flow $S^\Psi$ is negative

then $\Psi$ is a Lyapunov functional for the main flow $S^\Phi$
First variation along auxiliary flows

MAIN IDEA: take the first variation of the minimum problem

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If \( u_0 = w_0 \) then we have the "commutation" identity

\[ \frac{d}{d\varepsilon} \Phi(w_\varepsilon) \bigg|_{\varepsilon=0^+} = \frac{d}{d\varepsilon} \Psi(u_\varepsilon) \bigg|_{\varepsilon=0^+} \quad \left( = -\langle \nabla \Phi(w_0), \nabla \Psi(u_0) \rangle \right) \]

RECIPE: if the derivative of the (main) functional \( \Phi \) along the (auxiliary) flow \( S^\Psi \) is negative

then \( \Psi \) is a Lyapunov functional for the main flow \( S^\Phi \)

Look for good flows \( S^\Psi \) having \( \Phi \) as Lyapunov functional
First variation along auxiliary flows

**MAIN IDEA:** take the first variation of the minimum problem

$$U^n_\tau \in \arg\min_V W^2(V, U^{n-1}_\tau) + \Phi(V)$$

along the gradient flow $S^\psi$ generated by other “good” auxiliary functionals $\Psi$.

**HEURISTICS:** in an euclidean space $S^\Phi, S^\psi$ corresponds to

$$u_t := S^\phi_t(u_0) \text{ solves } \frac{d}{dt} u = -\nabla \Phi(u), \quad w_t := S^\psi_t(w_0) \text{ solves } \frac{d}{dt} w = -\nabla \Psi(w)$$

If $u_0 = w_0$ then we have the “commutation” identity

$$\frac{d}{d\varepsilon} \Phi(w_\varepsilon) \bigg|_{\varepsilon = 0^+} = \frac{d}{d\varepsilon} \Psi(u_\varepsilon) \bigg|_{\varepsilon = 0^+} \quad \left( = -\langle \nabla \Phi(w_0), \nabla \Psi(u_0) \rangle \right)$$

**RECIPE:** if the derivative of the (main) functional $\Phi$ along the (auxiliary) flow $S^\psi$ is negative (up to lower order terms)

then $\Psi$ is a Lyapunov functional for the main flow $S^\phi$ (up to lower order terms).

Look for good flows $S^\psi$ having $\Phi$ as Lyapunov functional.
A Lyapunov-type estimate at the discrete level

Suppose that $\Psi$ generates a flow $w_t = S_t^\Psi(w)$ satisfying a suitable metric formulation.
A Lyapunov-type estimate at the discrete level

Suppose that $\Psi$ generates a flow $w_t = S_t^\Psi(w)$ satisfying a suitable metric formulation. We call $\mathcal{D}$ the dissipation of $\Phi$ along $S^\Psi$

$$\mathcal{D}(w) := \left. - \frac{d}{d\varepsilon} \Phi(S_{\varepsilon}^\Psi(w)) \right|_{\varepsilon=0^+} = \limsup_{\varepsilon \downarrow 0} \Phi(w) - \Phi(S_{\varepsilon}^\Psi(w))$$
A Lyapunov-type estimate at the discrete level

Suppose that $\psi$ generates a flow $w_t = S_t^\psi(w)$ satisfying a suitable metric formulation. We call $\mathcal{D}$ the dissipation of $\Phi$ along $S^\psi$

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$$\frac{d}{dt} \psi(u_t) = -\mathcal{D}(w) \implies \psi(u_t) + \int_0^t \mathcal{D}(u_s) \, ds \leq \psi(u_0)$$
A Lyapunov-type estimate at the discrete level

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**Theorem (Discrete flow-interchange estimate)**

If $U^n_\tau$ is a minimizer of $V \mapsto \frac{W^2(V, U^{n-1}_\tau)}{2\tau} + \Phi(V)$ then

$$\psi(U^n_\tau) + \tau \mathcal{D}(U^n_\tau) \leq \psi(U^{n-1}_\tau)$$
**Auxiliary flow for the Cahn-Hilliard equation**

A typical example in the case of the Cahn-Hilliard equation with mobility $m(u) = u(1 - u)$ is given by the (displacement convex) entropy functional

$$
\Psi(w) = \int_\Omega w \log w + (1 - w) \log(1 - w) \, dx
$$

$S^\Psi$ is the heat flow $\partial_t w - \Delta w = 0$
Auxiliary flow for the Cahn-Hilliard equation

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$$\Psi(w) = \int_{\Omega} w \log w + (1 - w) \log(1 - w) \, dx$$

$S^\Psi$ is the heat flow $\partial_t w - \Delta w = 0$

The functional

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^d} |Du|^2 \, dx$$

decays along the heat flow with

$$\mathcal{D}(w) = -\frac{d}{d\varepsilon} \Phi(S^\Psi(w))\big|_{\varepsilon=0} = \int_{\Omega} |\Delta w|^2 \, dx = \int_{\Omega} |D^2 w|^2 \, dx$$
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The discrete flow-interchange estimate shows that $\Psi$ is a Lyapunov functional and satisfies

$$\Psi(U^n_\tau) + \tau \int_\Omega |D^2 U^n_\tau|^2 \, dx \leq \Psi(U^{n-1}_\tau).$$
Auxiliary flow for the Cahn-Hilliard equation

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The discrete flow-interchange estimate shows that $\Psi$ is a Lyapunov functional and satisfies

$$\Psi(U^n_{\tau}) + \tau \int_{\Omega} |D^2 U^n_{\tau}|^2 \, dx \leq \Psi(U^{n-1}_{\tau}).$$

In term of $U_{\tau}$ it corresponds to

$$\int_0^T \int_{\Omega} |D^2 U_{\tau}|^2 \, dx \, dt \leq C.$$
An example of convergence result

Assume that

$$P'(r) = m(r) W''(r) \geq -C \quad \text{in } (0, 1),$$

and the initial condition $u_0$ satisfies

$$0 \leq u_0 \leq 1, \quad \Phi(u_0) = \int_{\Omega} \left( \frac{1}{2} |Du|^2 \, dx + W(u_0) \right) \, dx$$

There exists an infinitesimal subsequence of time steps $\tau_k \downarrow 0$ such that $U_{\tau_k} \to u$ pointwise in $L^2(\mathbb{R}^d)$ and in $L^2(0, T; W^{1,2}(\mathbb{R}^d))$ as $k \to \infty$.

$$u \in C^0_w([0, +\infty); W^{1,2}(\Omega)) \cap L^2_{\text{loc}}([0, +\infty); W^{2,2}(\mathbb{R}^d))$$
is a non-negative global solution of the weak formulation of the Cahn-Hilliard equation

$$\int \int \left( u \partial_t \zeta - \Delta u \, \text{div} \left( m(u) D\zeta \right) + P(u) \Delta \zeta \right) \, dx \, dt = 0,$$

for every test function $\zeta \in C^\infty_c(\overline{\Omega} \times (0, \infty))$ such that $D\zeta \cdot \mathbf{n} = 0$ on $\partial \Omega \times (0, \infty)$. 
An example of convergence result

Assume that

\[ P'(r) = m(r) W''(r) \geq -C \text{ in } (0, 1), \]

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**Theorem**

There exists an infinitesimal subsequence of time steps \( \tau_k \downarrow 0 \) such that

\[ U_{\tau_k} \rightarrow u \text{ pointwise in } L^2(\mathbb{R}^d) \text{ and in } L^2(0, T; W^{1,2}(\mathbb{R}^d)) \text{ as } k \uparrow \infty \]
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**Theorem**

There exists an infinitesimal subsequence of time steps \( \tau_k \downarrow 0 \) such that

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\( u \in C^0([0, +\infty); W^{1,2}(\Omega)) \cap L^2_{\text{loc}}([0, +\infty); W^{2,2}(\mathbb{R}^d)) \) is a non-negative global solution of the weak formulation of the Cahn-Hilliard equation.
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**Theorem**

*There exists an infinitesimal subsequence of time steps \( \tau_k \downarrow 0 \) such that*

\[ U_{\tau_k} \to u \quad \text{pointwise in} \ L^2(\mathbb{R}^d) \text{ and in} \ L^2(0, T; W^{1,2}(\mathbb{R}^d)) \quad \text{as} \ k \uparrow \infty \]

\( u \in C^0_w([0, +\infty); W^{1,2}(\Omega)) \cap L^2_{\text{loc}}([0, +\infty); W^{2,2}(\mathbb{R}^d)) \) is a non-negative global solution of the weak formulation of the Cahn-Hilliard equation

\[ \int\int \left( u \partial_t \zeta - \Delta u \ \text{div} \ (m(u)D\zeta) + P(u)\Delta \zeta \right) \, dx \, dt = 0, \]

for every test function \( \zeta \in C^\infty_0(\Omega \times (0, \infty)) \) such that \( D\zeta \cdot n = 0 \) on \( \partial\Omega \times (0, \infty) \).
Open problems

▶ More explicit characterizations of $W_m$ and of measures at finite $W_m$-distance.
▶ Develop a duality approach to the weighted distances and find a precise characterization of their geodesics. [Carliet-Nazaret-Cardaliaguet '12]. Curvature properties?
▶ Study the gradient flow of other integral functionals: potential and interaction energies do not behave well with respect to the weighted distances.
▶ What about non-concave mobilities?
▶ ......