Hamel basis and additive functions

Hamel basis

References: [Hei Section 4.1], [Ku Section 4.2, Chapter 11], [NS Kapitola 4.7], [A Section 6F][1]

Existence of Hamel basis

Definition 1. Let $V$ be a vector space over a field $K$. We say that $B$ is a Hamel basis in $V$ if $B$ is linearly independent and every vector $v \in V$ can be obtained as a linear combination of vectors from $B$.

This is equivalent to the condition that every $x \in V$ can be written in precisely one way as

$$\sum_{i \in F} c_i x_i$$

where $F$ is finite, $c_i \in K$ and $x_i \in B$ for each $i \in F$.

It is also easy to see that for any vector space $W$ and any map $g: B \to W$ there exists exactly one linear map $f: V \to W$ such that $f|_B = g$.

Theorem 1. Let $V$ be a vector space over a field $K$. Let $A$ be an linearly independent subset of $V$. Then there exist a Hamel basis $B$ of $V$ such that $A \subseteq B$. (Any linearly independent set is contained in a basis.)

Proof. Zorn’s lemma. □

Corollary 1. Every vector space has a Hamel basis.

Proof. For $V = \{0\}$ we have a basis $B = \emptyset$.

If $V \neq \{0\}$, we can take any non-zero element $x \in V$ and use Theorem 1 for $A = \{x\}$. □

In some cases we are able to write down a basis explicitly, for example in finitely-dimensional space or in the following example. However, the claim that a Hamel basis exists for each vector space over any field already implies AC (see [HR Form 1A]).

Example 1. Let $c_{00}$ be the space of all real sequences which have only finitely many non-zero terms. Then $\{e^{(i)}; i \in \mathbb{N}\}$, where the sequence $e^{(i)}$ is given by $e^{(i)}_n = \delta_{in}$, is a Hamel basis of this space.

1See also: [thales.doa.fmph.uniba.sk/sleziak/texty/rozne/AC/cont.pdf]
Cardinality of Hamel basis

Proposition 1. If $B_1$, $B_2$ are Hamel bases of a vector space $V$, then $\text{card } B_1 = \text{card } B_2$.

Because of the above result, it makes sense to define Hamel dimension of a vector space $V$ as the cardinality of any of its bases.

Hamel bases in linear normed spaces and Banach spaces

Cardinality. Recall that a subset $A$ of a topological space $X$ is called meagre in $X$ if it is a countable union of nowhere-dense sets. Baire category theorem: If $X$ is a complete metric space, then $X$ is not meagre in $X$; i.e., $X$ cannot be obtained as a countable union of nowhere-dense sets. (Similar claim is true for locally compact Hausdorff spaces.)

Theorem 2. Let $X$ be an infinite-dimensional Banach space.

a) If $S$ is a subspace of $X$ which has countable Hamel basis, then $X$ is meagre in $X$.

b) Any Hamel basis of $X$ is uncountable.

The proof uses Baire category theorem and the fact that every finitely-dimensional subspace of a Banach space is closed (see [FHH] Proposition 1.36]). The same argument can be used to show analogous result for completely metrizable topological vector spaces (see [AB] Corollary 5.23). The above result can be in fact improved: It can be shown that cardinality of infinite-dimensional Banach space is at least $\mathfrak{c}$. We will give here a proof from [L].

We first recall a few fact about almost disjoint families (see [BS] §III.1, [B] Theorem 5.35, [JW] Theorems 17.17, 17.18).

Definition 2. Let $\mathcal{A} = \{A_i; i \in I\}$ be a system of subset of $X$. We say that $\mathcal{A}$ is an almost disjoint family or AD family on $X$, if $\text{card } A_i = \text{card } X$ for each $i \in I$ and the intersection $A_i \cap A_j$ is finite for each $i, j \in I, i \neq j$.

Lemma 1. If $X$ is an infinite countable set then there is an AD family on $X$ of cardinality $\mathfrak{c}$.

Proof. We will work with $X = \mathbb{Q}$. (The obtained AD family can be transferred to any infinite countable set.) For every $r \in \mathbb{R}$ there is an injective sequence $f_r : \mathbb{N} \to \mathbb{Q}$ of rational numbers, which converges to $r$. Put $A_r = f_r[\mathbb{N}]$. It is easy to see that $\{A_r; r \in \mathbb{R}\}$ is an AD family.

I should mention that I’ve learned about some of these results (and their proofs) from discussions at http://math.stackexchange.com See http://math.stackexchange.com/questions/74101/, http://math.stackexchange.com/questions/33282/ and http://math.stackexchange.com/questions/79184/.
Theorem 3. If $X$ is an infinite-dimensional Banach space then Hamel dimension of $X$ is at least $\mathfrak{c}$.

Proof. We first construct inductively systems $\{x_i; i \in \mathbb{N}\} \subseteq X$ and $\{x_i^*; i \in \mathbb{N}\} \subseteq X^*$ such that $x_i^*(x_j) = \delta_{ij}$ and $\|x_i\| = 1$.

Let us describe the inductive step in detail. Suppose we have already constructed $x_1, \ldots, x_k$ and $x_1^*, \ldots, x_k^*$ fulfilling the above conditions. Then the space $X$ can be written as $X = [x_1, \ldots, x_k] \oplus X'$ and the space $X'$ is again infinite-dimensional. Then we can choose any $x_{k+1} \in X'$ such that $\|x_{k+1}\| = 1$.

The map $x_{k+1}^*: [x_1, \ldots, x_{k+1}] \to \mathbb{R}$ given by $x_{k+1}^*(x_i) = \delta_{ij}$ is linear map on a finitely-dimensional subspace, hence it is continuous. By Hahn-Banach theorem it can be extended to a linear continuous function from $X$ to $\mathbb{R}$.

The above conditions imply $x_k / \in \{x_j; j \in \mathbb{N}, j \neq k\}$, since $x_k / \notin (x_k^*)^{-1}(0)$ and the later set is a closed subspace of $X$ containing $\{x_j; j \in \mathbb{N}, j \neq k\}$.

Now let $A = \{A_i; i \in \mathbb{R}\}$ be an AD family on $\mathbb{N}$. For each $i \in \mathbb{R}$ we define

$$a_i = \sum_{j \in A_i} \frac{1}{2^j} x_j.$$  

(Not that $\|\sum_{j \in A_i} \frac{1}{2^j} x_j\| \leq 1$, which implies that the above series is Cauchy and thus convergent.)

We will show that $\{a_i; i \in \mathbb{R}\}$ is an independent set. By Theorem 1 this implies that Hamel dimension of $X$ is at least $\mathfrak{c}$.

Let us assume that $\sum_{i \in F} c_i a_i = 0$ for some finite set $F$, where all $c_i$'s are non-zero. Let

$$P := \bigcup_{i,j \in F; i \neq j} (A_i \cap A_j),$$

This set is finite, since $A$ is an AD family. The above finite sum can be rewritten as

$$\sum_{j=1}^{\infty} d_j x_j = 0,$$

where $d_j = \frac{c_j}{2^j}$ whenever $i \in F$ and $j \in A_i \setminus P$. Since each set $A_i \setminus P$ is infinite, we have infinitely many non-zero coefficients in this sum. Thus we can rewrite the last equation as

$$x_k = \sum_{i \neq k} f_i x_i$$

for some $k$ and $f_i \in \mathbb{R}$, which contradicts the assumption that $x_k / \notin \{x_j; j \neq k\}$.

Existence of unbounded linear functionals.

Proposition 2. If $X$ is an infinite-dimensional linear normed space, then there exist non-continuous linear function $f: X \to \mathbb{R}$.

3Here we used the fact that if $f \in X^*$ and $f(x) \neq 0$, then $X = \text{Ker } f \oplus \{x\}$. 

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Proof. Choose an infinite independent set \( \{x_n; n \in \mathbb{N}\} \) such that \( \|x_n\| = 1 \) for each \( n \in \mathbb{N} \) and a function \( f: X \to \mathbb{R} \) such that \( f(x_n) = n \).

**Continuity of coordinate functionals.** If \( b \) is a Hamel basis of a vector space \( X \) over \( \mathbb{R} \), and we define \( f_b: x \to \mathbb{R} \) which assigns to \( x \) its \( b \)-th coordinate, i.e., \( x = \sum_{b \in B} f_b(x)b \) for each \( x \in X \), then \( f_b \) is a linear function from \( X \) to \( \mathbb{R} \).

Suppose that \( X \) is, moreover, a Banach space. We would like to know whether the functions \( f_b \) are continuous. We will show that at most finitely many of them can be continuous.

**Proposition 3.** Let \( B \) be a Hamel basis of a Banach space \( X \). Let \( f_b, b \in B \), be the coordinate functionals. Then there is only finitely many \( b \)'s such that \( f_b \) is continuous.

**Proof.** Suppose that \( \{b_i; i \in \mathbb{N}\} \) is an infinite subset of \( B \) such that each \( f_{b_i} \) is continuous. W.l.o.g. we may assume that \( \|b_i\| = 1 \).

Let

\[
x := \sum_{i=1}^{\infty} \frac{1}{2^i} b_i.
\]

(Since \( X \) is complete, the above sum converges.)

We also denote \( x_n := \sum_{i=1}^{n} \frac{1}{2^i} b_i \). Since \( x_n \) converges to \( x \), we have \( f_{b_k}(x_n) = \lim_{n \to \infty} f_{b_k}(x_n) = \frac{1}{2^k} \) for each \( k \in \mathbb{N} \). Thus the point \( x \) has infinitely many non-zero coordinates, which contradicts the definition of Hamel basis.

We can give another proof based on Banach-Steinhaus theorem (uniform boundedness principle). We show first the following:

**Lemma 2.** Let \( B \) be a Hamel basis of a Banach space \( X \). Let \( f_b, b \in B \), be the coordinate functionals. Let \( C = \{b \in B; f_b \text{ is continuous}\} \). Then \( \sup\{\|f_b\|; b \in C\} < \infty \).

**Proof.** For any \( x \in X \) there is at most finitely many \( b \)'s in \( C \) such that \( f_b(x) \neq 0 \). This implies that \( \sup_{b \in C} |f_b(x)| \) is finite. Banach-Steinhaus theorem this implies \( \sup\{\|f_b\|; b \in C\} < \infty \).

**Proof of Proposition 3.** Let \( B \) be any Hamel basis for \( X \). For any choice of constants \( c_b, b \in B \), is the set \( \{c_b f_b; b \in B\} \) a Hamel basis as well. The coordinate functionals for this new basis are \( g_b = \frac{1}{c_b} f_b \). If the set \( C = \{b \in B; f_b \text{ is continuous}\} \) is infinite, then by an appropriate choice of constant \( c_b \) we can obtain \( \sup\{\|f_b\|; b \in C\} = \infty \), which contradicts the above lemma.

It is easy to show that finitely many of coordinate functionals can be continuous. If \( X \) is a Banach space with a basis \( B \) and \( x_1, \ldots, x_n \notin X \), then \( [x_1, \ldots, x_n] \oplus X \) is a Banach space with a basis \( \{x_1, \ldots, x_n\} \cup B \) and there are at least \( n \) continuous coordinate functionals.

Also in the space \( c_{00} \) from Example 1 with sup-norm all coordinate functionals are continuous. The space \( c_{00} \) is, of course, not complete.
Cauchy functional equation

References: [Ku, Section 5.2, Chapter 12], [S, Section 2.1], [Ka, Chapter 1], [Kh, Chapter 7], [Her, Section 5.1], [A, Appendix to Chapter 6]

Let us study the functions $f: \mathbb{R} \to \mathbb{R}$ fulfilling

$$f(x + y) = f(x) + f(y).$$  (1)

The equation (1) is called Cauchy equation and functions fulfilling (1) are called additive functions.

It is easy to show that

**Lemma 3.** If a function $f: \mathbb{R} \to \mathbb{R}$ fulfills (1), then

$$f(qx) = qf(x)$$

holds for every $q \in \mathbb{Q}$, $x \in \mathbb{R}$.

This shows, that the additive functions are precisely the linear maps if we consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$.

**Lemma 3** implies that

**Theorem 4.** Every continuous solution (1) is of the form $f(x) = ax$ for some $a \in \mathbb{R}$.

Non-linear solutions

Using the existence of Hamel basis in $\mathbb{R}$ (as a vector space over $\mathbb{Q}$) we can show that

**Theorem 5.** There exist non-linear solution of (1), i.e. functions $f: \mathbb{R} \to \mathbb{R}$ that fulfill (1) but are not of the form $f(x) = ax$.

**Theorem 6.** If $f$ is a non-linear solution of (1), then the graph of this function

$$G(f) = \{(x, f(x)); x \in \mathbb{R}\}$$

is dense in $\mathbb{R}^2$.

The proof can be found e.g. in [Her, Theorem 5.4].

Theorems 4 and 5 suggest that well-behaved solutions of (1) are linear and that non-linear solutions have to be, in some sense, pathological. Let us mention a one more result in this direction.

**Theorem 7.** Every measurable solution of (1) is linear.

An elegant proof is given in [Her, Theorem 5.5].

This last result means that by showing the existence of non-continuous solutions of (1) we have also obtained the existence of non-measurable sets.
References


