Hamel basis and additive functions

Hamel basis

References: [Hei, Section 4.1], [Ku, Section 4.2, Chapter 11], [NS, Kapitola 4.7], [A, Section 6F] [1]

Existence of Hamel basis

Definition 1. Let V be a vector space over a field K. We say that B is a *Hamel basis* in V if B is linearly independent and every vector $v \in V$ can be obtained as a linear combination of vectors from B.

This is equivalent to the condition that every $x \in V$ can be written in precisely one way as

$$\sum_{i\in F} c_i x_i$$

where F si finite, $c_i \in K$ and $x_i \in B$ for each $i \in F$.

It is also easy to see that for any vector space W and any map $g\colon B \to W$ there exists exactly one linear map $f\colon V \to W$ such that $f|_B = g$.

Theorem 1. *Let V be a vector space over K. Let A be an linearly independent subset of V. Then there exist a Hamel basis B of V such that $A \subseteq B$. (Any linearly independent set is contained in a basis.)*

Proof. Zorn's lemma. □

Corollary 1. *Every vector space has a Hamel basis.*

Proof. For $V = \{0\}$ we have a basis $B = \emptyset$.

If $V \neq \{0\}$, we can take any non-zero element $x \in V$ and use Theorem 1 for $A = \{x\}$. □

In some cases we are able to write down a basis explicitly, for example in finitely-dimensional space or in the following example. However, the claim that a Hamel basis exists for each vector space over any field already implies AC (see [HR, Form 1A]).

Example 1. Let c_{00} be the space of all real sequences which have only finitely many non-zero terms. Then $\{e^{(i)}; i \in \mathbb{N}\}$, where the sequence $e^{(i)}$ is given by $e^{(i)}_n = \delta_{in}$, is a Hamel basis of this space.

[1] See also: `thales.doa.fmph.uniba.sk/sleziak/texty/rozne/AC/cont.pdf`

Cardinality of Hamel basis

Proposition 1. *If B_1, B_2 are Hamel bases of a vector space V, then* $\operatorname{card} B_1 = \operatorname{card} B_2$.

Because of the above result, it makes sense to define *Hamel dimension* of a vector space V as the cardinality of any of its bases.

Hamel bases in linear normed spaces and Banach spaces

[2]

Cardinality. Recall that a subset A of a topological space X is called *meagre* in X if it is a countable union of nowhere-dense sets. Baire category theorem: If X is a complete metric space, then X is not meagre in X; i.e., X cannot be obtained as a countable union of nowhere-dense sets. (Similar claim is true for locally compact Hausdorff spaces.)

Theorem 2. *Let X be an infinite-dimensional Banach space.*

a) *If S is a subspace of X which has countable Hamel basis, then X is meagre in X.*

b) *Any Hamel basis of X is uncountable.*

The proof uses Baire category theorem and the fact that every finitely-dimensional subspace of a Banach space is closed (see [FHH$^+$, Proposition 1.36]). The same argument can be used to show analogous result for completely metrizable topological vector spaces (see [AB, Corollary 5.23]).

The above result can be in fact improved: It can be shown that cardinality of infinite-dimensional Banach space is at least $\mathfrak{c}$. We will give here a proof from [L].

We first recall a few fact about almost disjoint families (see [BŠ, §III.1], [B, Theorem 5.35], [JW, Theorems 17.17, 17.18]).

Definition 2. Let $\mathcal{A} = \{A_i; i \in I\}$ be a system of subset of X. We say that $\mathcal{A}$ is an *almost disjoint family* or *AD family* on X, if $\operatorname{card} A_i = \operatorname{card} X$ for each $i \in I$ and the intersection $A_i \cap A_j$ is finite for each $i, j \in I$, $i \neq j$.

Lemma 1. *If X is an infinite countable set then there is an AD family on X of cardinality $\mathfrak{c}$.*

Proof. We will work with $X = \mathbb{Q}$. (The obtained AD family can be transferred to any infinite countable set.)

For every $r \in \mathbb{R}$ there is an injective sequence $f_r \colon \mathbb{N} \to \mathbb{Q}$ of rational numbers, which converges to r. Put $A_r = f_r[\mathbb{N}]$. It is easy to see that $\{A_r; r \in \mathbb{R}\}$ is an AD family. □

[2] I should mention that I've learned about some of these results (and their proofs) from discussions at `http://math.stackexchange.com`. See `http://math.stackexchange.com/questions/74101/`, `http://math.stackexchange.com/questions/33282/` and `http://math.stackexchange.com/questions/79184/`.

Theorem 3. *If X is an infinite-dimensional Banach space then Hamel dimension of X is at least $\mathfrak{c}$.*

Proof. We first construct inductively systems $\{x_i; i \in \mathbb{N}\} \subseteq X$ and $\{x_i^*; i \in \mathbb{N}\} \subseteq X^*$ such that $x_i^*(x_j) = \delta_{ij}$ and $\|x_i\| = 1$.

Let us describe the inductive step in detail. Suppose we have already constructed $x_1, \dots, x_k$ and $x_1^*, \dots, x_k^*$ fulfilling the above conditions. Then the space X can be written as $X = [x_1, \dots, x_k] \oplus X'$ and the space X' is again infinite-dimensional.[3] Then we can choose any $x_{k+1} \in X'$ such that $\|x_{k+1}\| = 1$. The map $x_{k+1}^* \colon [x_1, \dots, x_{k+1}] \to \mathbb{R}$ given by $x_{k+1}^*(x_i) = \delta_{ij}$ is linear map on a finitely-dimensional subspace, hence it is continuous. By Hahn-Banach theorem it can be extended to a linear continuous function from X to $\mathbb{R}$.

The above conditions imply $x_k \notin \overline{[\{x_j; j \in \mathbb{N}, j \neq k\}]}$, since $x_k \notin (x_k^*)^{-1}(0)$ and the later set is a closed subspace of X containing $\{x_j; j \in \mathbb{N}, j \neq k\}$.

Now let $\mathcal{A} = \{A_i; i \in \mathbb{R}\}$ be an AD family on $\mathbb{N}$. For each $i \in \mathbb{R}$ we define

$$a_i = \sum_{j \in A_i} \frac{1}{2^j} x_j.$$

(Not that $\|\frac{1}{2^j} x_j\| \leq \frac{1}{2^j}$, which implies that the above series is Cauchy and thus convergent.)

We will show that $\{a_i; i \in \mathbb{R}\}$ is an independent set. By Theorem 1 this implies that Hamel dimension of X is at least $\mathfrak{c}$.

Let us assume that $\sum_{i \in F} c_i a_i = 0$ for some finite set F, where all c_i's are non-zero. Let

$$P := \bigcup_{\substack{i,j \in F \\ i \neq j}} (A_i \cap A_j).$$

This set is finite, since $\mathcal{A}$ is an AD family. The above finite sum can be rewritten as

$$\sum_{j=1}^{\infty} d_j x_j = 0,$$

where $d_j = \frac{c_i}{2^j}$ whenever $i \in F$ and $j \in A_i \setminus P$. Since each set $A_i \setminus P$ is infinite, we have infinitely many non-zero coefficients in this sum. Thus we can rewrite the last equation as

$$x_k = \sum_{i \neq k} f_i x_i$$

for some k and $f_i \in \mathbb{R}$, which contradicts the assumption that $x_k \notin \overline{\{x_j; j \neq k\}}$. ∎

Existence of unbounded linear functionals.

Proposition 2. *If X is an infinite-dimensional linear normed space, then there exist non-continuous linear function $f \colon X \to \mathbb{R}$.*

[3] Here we used the fact that if $f \in X^*$ and $f(x) \neq 0$, then $X = \operatorname{Ker} f \oplus [x]$.

Proof. Choose an infinite independent set $\{x_n; n \in \mathbb{N}\}$ such that $\|x_n\| = 1$ for each $n \in \mathbb{N}$ and a function $f\colon X \to \mathbb{R}$ such that $f(x_n) = n$. □

Continuity of coordinate functionals. If b is a Hamel basis of a vector space X over $\mathbb{R}$, and we define $f_b\colon x \to \mathbb{R}$ which assigns to x its b-th coordinate, i.e., $x = \sum_{b\in B} f_b(x)b$ for each $x \in X$, then f_b is a linear function from X to $\mathbb{R}$.

Suppose that X is, moreover, a Banach space. We would like to know whether the functions f_b are continuous. We will show that at most finitely many of them can be continuous.

Proposition 3. *Let B be a Hamel basis of a Banach space X. Let f_b, $b \in B$, be the coordinate functionals. Then there is only finitely many b's such that f_b is continuous.*

Proof. Suppose that $\{b_i; i \in \mathbb{N}\}$ is an infinite subset of B such that each f_{b_i} is continuous. W.l.o.g. we may assume that $\|b_i\| = 1$.

Let
$$x := \sum_{i=1}^{\infty} \frac{1}{2^i} b_i.$$
(Since X is complete, the above sum converges.)

We also denote $x_n := \sum_{i=1}^{n} \frac{1}{2^i} b_i$. Since x_n converges to x, we have $f_{b_k}(x) = \lim\limits_{n\to\infty} f_{b_k}(x_n) = \frac{1}{2^k}$ for each $k \in \mathbb{N}$. Thus the point x has infinitely many non-zero coordinates, which contradicts the definition of Hamel basis. □

We can give another proof based on Banach-Steinhaus theorem (uniform boundedness principle). We show first the following:

Lemma 2. *Let B be a Hamel basis of a Banach space X. Let f_b, $b \in B$, be the coordinate functionals. Let $C = \{b \in B; f_b$ is continuous$\}$. Then* $\sup\{\|f_b\|; b \in C\} < \infty$.

Proof. For any $x \in X$ there is at most finitely many b's in C such that $f_b(x) \neq 0$. This implies that $\sup_{b\in C}|f_b(x)|$ is finite. Banach-Steinhaus theorem this implies $\sup\{\|f_b\|; b \in C\} < \infty$. □

Proof of Proposition 3. Let B be any Hamel basis for X. For any choice of constants c_b, $b \in B$, is the set $\{c_b f_b; b \in B\}$ a Hamel basis as well. The coordinate functionals for this new basis are $g_b = \frac{1}{c_b} f_b$. If the set $C = \{b \in B; f_b$ is continuous$\}$ is infinite, then by an appropriate choice of constant c_b we can obtain $\sup\{\|f_b\|; b \in C\} = \infty$, which contradicts the above lemma. □

It is easy to show that finitely many of coordinate functionals can be continuous. If X is a Banach space with a basis B and $x_1, \dots, x_n \notin X$, then $[x_1, \dots, x_n] \oplus X$ is a Banach space with a basis $\{x_1, \dots, x_n\} \cup B$ and there are at least n continuous coordinate functionals.

Also in the space c_{00} from Example 1 with sup-norm all coordinate functionals are continuous. The space c_{00} is, of course, not complete.

Cauchy functional equation

References: [Ku, Section 5.2, Chapter 12], [S, Section 2.1], [Ka, Chapter 1], [Kh, Chapter 7], [Her, Section 5.1], [A, Appendix to Chapter 6]

Let us study the functions $f\colon \mathbb{R} \to \mathbb{R}$ fulfilling

$$f(x+y) = f(x) + f(y). \tag{1}$$

The equation (1) is called *Cauchy equation* and functions fulfilling (1) are called *additive functions.*

It is easy to show that

Lemma 3. *If a function $f\colon \mathbb{R} \to \mathbb{R}$ fulfills (1), then*

$$f(qx) = qf(x)$$

holds for every $q \in \mathbb{Q}$, $x \in \mathbb{R}$.

This shows, that the additive functions are precisely the linear maps if we consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$.

Lemma 3 implies that

Theorem 4. *Every continuous solution (1) is of the form $f(x) = ax$ for some $a \in \mathbb{R}$.*

Non-linear solutions

Using the existence of Hamel basis in $\mathbb{R}$ (as a vector space over $\mathbb{Q}$) we can show that

Theorem 5. *There exist non-linear solution of (1), i.e. functions $f\colon \mathbb{R} \to \mathbb{R}$ that fulfill (1) but are not of the form $f(x) = ax$.*

Theorem 6. *If f is a non-linear solution of (1), then the graph of this function*

$$G(f) = \{(x, f(x)); x \in \mathbb{R}\}$$

is dense in $\mathbb{R}^2$.

The proof can be found e.g. in [Her, Theorem 5.4].

Theorems 4 and 6 suggest that well-behaved solutions of (1) are linear and that non-linear solutions have to be, in some sense, pathological. Let us mention a one more result in this direction.

Theorem 7. *Every measurable solution of (1) is linear.*

An elegant proof is given in [Her, Theorem 5.5].

This last result means that by showing the existence of non-continuous solutions of (1) we have also obtained the existence of non-measurable sets.

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