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Exponential attractors for a nonlinear reaction-diffusion system in $\mathbb{R}^3$

Messoud EFENDIEV [a], Alain MIRANVILLE [b], Sergey ZELIK [c]

[a] Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, 10117 Berlin, Germany
E-mail: efendiev@wias-berlin.de

[b] Mathématiques, SP2MI, Université de Poitiers, téléport 2, avenue Marie-et-Pierre-Curie, 86962 Futuroscope Chasseneuil cedex, France
E-mail: miranv@wallis.sp2mi.univ-poitiers.fr

[c] Institute for Information and Transmission Problems, Russian Academy of Sciences, Bolshoi Karetnii 19, Moscow 101447, GSP-4, Russia
E-mail: zelik@iitp.ru

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Abstract. We give in this Note a construction of exponential attractors for a class of operators in Banach spaces (and not in Hilbert spaces only as it is the case for the classical constructions). We then apply this result to a reaction-diffusion system in $\mathbb{R}^3$. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Attracteurs exponentiels pour un système de réaction-diffusion dans $\mathbb{R}^3$

Résumé. *Nous donnons dans cette Note une construction d'attracteurs exponentiels pour une classe d'opérateurs dans des espaces de Banach* (*et non dans des espaces de Hilbert uniquement comme c'est le cas pour les constructions usuelles*). *Puis, nous appliquons ce résultat à un système de réaction-diffusion dans* $\mathbb{R}^3$. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Nous nous intéressons dans cette Note à l'existence d'attracteurs exponentiels (*voir* la définition ci-après) pour des équations de réaction-diffusion dans $\mathbb{R}^3$ de la forme (1).

La difficulté essentielle est qu'ici nous ne pouvons pas utiliser les constructions classiques (*voir* [1] et [3]) ; la raison étant que les espaces dans lesquels nous travaillons (définis dans la section 1) n'ont pas de structure hilbertienne. En effet, les constructions usuelles font appel de manière essentielle à des projecteurs orthogonaux de rang fini.

Afin de contourner cette difficulté, nous donnons dans cette Note (*voir* proposition 1 ci-dessous) une construction d'attracteurs exponentiels, valable dans des espaces de Banach, qui généralise celle de [3]

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pour des opérateurs s'écrivant comme somme d'une contraction et d'un opérateur compact (dans un sens précisé ci-dessous). On en déduit alors l'existence d'un attracteur exponentiel pour (1).

Introduction

Our aim in this Note is to prove the existence of exponential attractors for reaction-diffusion equations in $\mathbb{R}^3$ of the form (1) below.

In [4], the authors obtained the existence of finite-dimensional attractors for (1) under very restrictive conditions on the nonlinear term. Then, in [6], the authors were able to weaken these assumptions. However, they obtained the finite-dimensionality of the attractors in $\mathrm{L}^2(\mathbb{R}^3)$, whereas the associated semigroup was constructed on functions that are bounded as $|x| \to \infty$ (*see* below for the exact definition of the phase space). Here, by constructing an exponential attractor, we are able to prove the finite-dimensionality of the global attractor in the phase space. Such spaces of bounded functions play a crucial role for problem (1). Indeed, we have the existence of the finite-dimensional global attractor in these spaces without any decay assumption on the initial data. Of course, we need a decay condition on the forcing term, but this decay can be arbitrarily slow (*see* Section 1 below), and, without this decay assumption, the attractor has generally infinite dimension (*see* [6] and [7]; *see* also Remark 3 below). Thus, these spaces give here the sharp border between the finite and the infinite-dimensionality of the global attractor.

Now, the study of exponential attractors has also an interest on its own. Indeed, compared to an exponential attractor, the global attractor presents two defaults for practical purposes. Indeed, it is very sensitive to perturbations and the rate of attraction of the trajectories may be small. An exponential attractor however, as its name indicates, attracts exponentially the trajectories and will thus be more stable. Furthermore, in some situations, the global attractor can be very simple (say, reduced to one point) and thus fails to capture interesting transcient behaviors. Again, in such situations, an exponential attractor could be a more suitable object.

In [1], the authors proposed a construction of exponential attractors for equations in unbounded domains (*see* also [4]). However, as it is the case for the usual construction of [3], it is only valid in Hilbert spaces; indeed, it makes an essential use of orthogonal projectors with finite rank. This construction will thus not apply to (1) (we shall see below that the phase space for our problem is not a Hilbert space).

We propose, for maps that can be decomposed into the sum of a contraction and of a compact (in a sense precised below) map, a construction that is not based on projectors and that is therefore valid in Banach spaces. As an application, we obtain the existence of an exponential attractor for (1) in the phase space.

We only give in this Note the proof of our main result (Proposition 1 below). Indications on the proofs of some results may be found in [6]. Furthermore, all the details will appear in [5].

1. Setting of the problem

This Note is devoted to the study of the long time behavior of the solutions of the following problem:

$$\begin{cases} \partial_t u = \Delta_x u - f(u, \nabla_x u) - \lambda_0 u + g(t), \quad x \in \mathbb{R}^3, \\ u|_{t=\tau} = u_\tau. \end{cases} \tag{1}$$

Here, $u = (u^1, \ldots, u^k)$ is an unknown vector valued function, $f = (f^1, \ldots, f^k)$ and $g(t) = g(t,x) = (g^1(t,x), \ldots, g^k(t,x))$ are given functions, Δ_x is the Laplacian with respect to the variables $x = (x_1, x_2, x_3)$ and λ_0 is a fixed strictly positive number.

We assume that the nonlinear term f satisfies the conditions:

$$\begin{cases} 1. \quad f \in \mathrm{C}^1(\mathbb{R}^k \times \mathbb{R}^{3k}, \mathbb{R}^k); \\ 2. \quad f(v,p) \cdot v \geqslant 0; \\ 3. \quad |f(v,p)| \leqslant |v| Q(|v|)(1 + |p|^q), \ q < 2; \end{cases} \tag{2}$$

for every $v \in \mathbb{R}^k$, $p \in \mathbb{R}^{3k}$ and for some monotonous function Q. (Here and below, we denote by $u \cdot v$ the inner product in $\mathbb{R}^k$.)

In order to introduce the phase space for our problem and to impose the assumptions on the right-hand side g, we give the following definition (*see* [6]):

DEFINITION 1. – Let $\mathrm{B}^R_{x_0}$ be an open R-ball in $\mathbb{R}^3$ centered at x_0 and let as usual $\mathrm{W}^{\ell,p}(\mathrm{B}^R_{x_0})$ denote the Sobolev space of the functions on $\mathrm{B}^R_{x_0}$ whose derivatives up to the order ℓ belong to $\mathrm{L}^p(\mathrm{B}^R_{x_0})$ ($\|u, \mathrm{B}^R_{x_0}\|_{\ell,p} \equiv \|u\|_{\mathrm{W}^{\ell,p}(\mathrm{B}^R_{x_0})}$). For every $\ell \geqslant 0$ and $1 \leqslant p \leqslant \infty$, we define the space

$$\mathrm{W}^{\ell,p}_{\mathrm{b}}(\mathbb{R}^3) \equiv \Big\{ u \in D'\big(\mathbb{R}^3\big) \ : \ \|u\|_{\mathrm{b},\ell,p} = \sup_{x_0 \in \mathbb{R}^3} \big\|u, \mathrm{B}^1_{x_0}\big\|_{\ell,p} < \infty \Big\}$$

(roughly speaking, the space $\mathrm{W}^{\ell,p}_{\mathrm{b}}(\mathbb{R}^3)$ consists of functions whose derivatives up to the order ℓ are bounded as $|x| \to \infty$) and the space

$$\mathrm{W}^{\ell,p}_{\mathrm{b},0}(\mathbb{R}^3) \equiv \Big\{ u \in \mathrm{W}^{\ell,p}_{\mathrm{b}}(\mathbb{R}^3) \ : \ \lim_{|x_0| \to \infty} \big\|u, \mathrm{B}^1_{x_0}\big\|_{\ell,p} = 0 \Big\}.$$

In other words, the functions in $\mathrm{W}^{\ell,p}_{\mathrm{b},0}(\mathbb{R}^3)$ decay as $|x| \to \infty$. We define similarly the spaces $\mathrm{L}^p_{\mathrm{b}}(\mathbb{R}^3)$ and $\mathrm{L}^p_{\mathrm{b},0}(\mathbb{R}^3)$ (corresponding to $\ell = 0$).

We assume that the right-hand side $g \in \mathrm{C}^1_{\mathrm{b}}(\mathbb{R}, \mathrm{L}^2_{\mathrm{b},0}(\mathbb{R}^3))$ and is quasiperiodic with respect to t with ℓ independent frequencies, i.e., there exist a function

$$G \in \mathrm{C}^1\big(\mathbb{T}^\ell, \mathrm{L}^2_{\mathrm{b},0}(\mathbb{R}^3)\big),$$

$\mathbb{T}^\ell$ being the ℓ-dimensional torus, rationally independent frequencies $\alpha = (\alpha^1, \ldots, \alpha^\ell)$ and the initial phase $\phi_0 = (\phi^1_0, \ldots, \phi^\ell_0) \in \mathbb{T}^\ell$ such that

$$g(t,x) = G(\phi_0 + \alpha t, x).$$

The phase space for problem (1) will be the space $\Phi = \mathrm{W}^{2-\delta,2}_{\mathrm{b}}(\mathbb{R}^3)$, where $\delta > 0$ is chosen such that $\delta < \min\{1/2, 1/q - 1/2\}$ and the exponent q is the same as in (2).

Remark 1. – Since the exponent δ in the definition of the phase space is small enough, one can easily verify, using the third assumption of (2) and the Sobolev embedding theorems, that $f(v, \nabla_x v) \in \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^3)$ if $v \in \Phi$ and consequently equation (1) can be understood in the sense of distributions.

Proceeding as in [6], we have the following result:

THEOREM 1. – *Let the above assumptions hold. Then, problem* (1) *has a unique solution* $u(t) \in \Phi$, *for every* $u_\tau \in \Phi$. *Moreover, the following estimate holds*:

$$\big\|u(t)\big\|_\Phi \leqslant Q_1\big(\|u_\tau\|_\Phi\big)\, \mathrm{e}^{-\varepsilon(t-\tau)} + Q_1\big(\|G\|_{\mathrm{C}^1(\mathbb{T}^\ell, \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^3))}\big),$$

where $\varepsilon > 0$ *and* Q_1 *is some monotonous function depending only on the equation.*

COROLLARY 1. – *Theorem* 1 *implies that the* (*family of*) *operators* (*called the process associated with the equation*):

$$U_g(t,\tau)\,:\,\Phi \longrightarrow \Phi, \quad u(t) = U_g(t,\tau)u_\tau,$$

are well defined and are bounded as $t-\tau \to \infty$.

2. Existence of an exponential attractor

In order to study the long time behavior of the nonautonomous equation (1), we actually consider, following [2], the family of equations

$$\begin{cases} \partial_t u = \Delta_x u - f(u,\nabla_x u) - \lambda_0 u + \xi(t), \\ u|_{t=\tau} = u_\tau, \end{cases} \tag{3}$$

for all $\xi \in \mathcal{H}(g)$, where the hull $\mathcal{H}(g)$ is defined as follows:

$$\mathcal{H}(g) = \{G(\phi + \alpha t, x) :\ \phi \in \mathbb{T}^\ell\}.$$

Since the functions $\xi \in \mathcal{H}(g)$ can be parametrized by the points ϕ of the ℓ-dimensional torus, we shall denote by $U_\phi(t,\tau)$ the family of processes associated with (3) (instead of $U_\xi(t,\tau)$, with $\xi(t) = G(\phi+\alpha t)$).

It is known (*see* for instance [2]) that the family of processes $\{U_\phi(t,\tau),\ \phi \in \mathbb{T}^\ell\}$ can be extended to a semigroup $\mathbb{S}_t$ acting on $\Phi \times \mathbb{T}^\ell$ by formula

$$\mathbb{S}_t(v,\phi) \equiv \big(U_\phi(t,0)v, T_t\phi\big), \quad T_t\phi \equiv \phi + \alpha t. \tag{4}$$

Thus, instead of studying the long time behavior of the single equation (1), we shall actually study the long time behavior of the trajectories of the semigroup $\mathbb{S}_t : \Phi \times \mathbb{T}^\ell \to \Phi \times \mathbb{T}^\ell$.

We recall that the compact set $\mathcal{A} \subset \Phi \times \mathbb{T}^\ell$ is called the global attractor for the semigroup $\mathbb{S}_t$ on $\Phi \times \mathbb{T}^\ell$ if it is invariant by $\mathbb{S}_t$, i.e.,

$$\mathbb{S}_t\mathcal{A} = \mathcal{A}, \quad \text{for } t \geqslant 0, \tag{5}$$

and it attracts the bounded subsets of $\Phi \times \mathbb{T}^\ell$ as $t \to \infty$, i.e., for every $B \subset \Phi \times \mathbb{T}^\ell$ bounded

$$\lim_{t\to\infty} \operatorname{dist}_{\Phi\times\mathbb{T}^\ell}\{\mathbb{S}_t B,\ \mathcal{A}\} = 0,$$

where dist_V denotes the Hausdorff semi-distance in V.

THEOREM 2. – *Let the above assumptions hold. Then, the semigroup* $\mathbb{S}_t$ *defined by* (4) *possesses the global attractor* $\mathcal{A}$ *on* $\Phi \times \mathbb{T}^\ell$. *Moreover, this attractor has finite fractal dimension in* $\Phi \times \mathbb{T}^\ell$:

$$\dim_{\mathrm{F}}\big(\mathcal{A}, \Phi \times \mathbb{T}^\ell\big) < \infty.$$

The existence of the global attractor is obtained in [6]. Now, the fact that is has finite fractal dimension in the topology of $\Phi \times \mathbb{T}^\ell$ will be a consequence of the existence of an exponential attractor below (we note however that this last property can be proved directly, *see* [5] and [7]).

Remark 2. – We recall that we require that $g(t) \in \mathrm{L}^2_{\mathrm{b},0}(\mathbb{R}^3)$, for every $t \geqslant 0$. It is worth emphasizing here that $\mathrm{L}^2(\mathbb{R}^3) \subset \mathrm{L}^2_{\mathrm{b},0}(\mathbb{R}^3)$. Consequently, right-hand sides g belonging to $\mathrm{L}^2(\mathbb{R}^3)$ are also admissible. We also note that $\mathrm{L}^2_{\mathrm{b},0}(\mathbb{R}^3)$ is larger than $\mathrm{L}^2(\mathbb{R}^3)$, since *arbitrary* decay rates as $|x| \to \infty$ are allowed. For example, the function $g(x) = 1/\ln(|x|^2+2)$ belongs to $\mathrm{L}^2_{\mathrm{b},0}(\mathbb{R}^3)$, but evidently $g \notin \mathrm{L}^2(\mathbb{R}^3)$.

Remark 3. – As noted in the introduction, such decay rates of the right-hand side g as $|x| \to \infty$ ($g(t) \in \mathrm{L}^2_{\mathrm{b},0}(\mathbb{R}^3)$) is $essential$ to prove the finite-dimensionality of the global attractor. Indeed, even in the autonomous case $g(t)=g$, the global attractor may have infinite fractal dimension if $g\in \mathrm{L}^2_{\mathrm{b}}(\mathbb{R}^3)$ but $g\notin \mathrm{L}^2_{\mathrm{b},0}(\mathbb{R}^3)$ (*see* for instance [6] or [7]).

We first recall that a compact set $\mathcal{M}\subset \Phi\times\mathbb{T}^\ell$ is called an exponential attractor for the semigroup $\mathbb{S}_t$ on $\Phi\times\mathbb{T}^\ell$ if it is semi-invariant by $\mathbb{S}_t$, i.e.,

$$\mathbb{S}_t\mathcal{M}\subset\mathcal{M},\quad \text{for } t\geqslant 0, \tag{6}$$

it attracts *exponentially* the bounded subsets of $\Phi\times\mathbb{T}^\ell$, i.e., there exists a constant $\mu>0$ such that for every $B\subset\Phi\times\mathbb{T}^\ell$ bounded

$$\mathrm{dist}_{\Phi\times\mathbb{T}^\ell}\{S_tB,\ \mathcal{M}\}\leqslant C\left(\|B\|_{\Phi\times\mathbb{T}^\ell}\right)\mathrm{e}^{-\mu t},$$

and it has finite fractal dimension in $\Phi\times\mathbb{T}^\ell$, i.e., $\dim_{\mathrm{F}}(\mathcal{M},\Phi\times\mathbb{T}^\ell)<\infty$.

Remark 4. – We note that since we lose the invariance (assumption (6) instead of (5)), then, contrarily to the global attractor, an exponential attractor is not necessarily unique. However, we always have $\mathcal{A}\subset\mathcal{M}$.

THEOREM 3. – *Let the above assumptions hold. Then, the semigroup $\mathbb{S}_t$ defined by* (4) *possesses an exponential attractor $\mathcal{M}$ on $\Phi\times\mathbb{T}^\ell$.*

The proof of this theorem is based on the following sufficient conditions for the existence of an exponential attractor for maps in *Banach* spaces which generalize those given in [1] and [3] that are valid in Hilbert spaces only:

PROPOSITION 1. – *Let H and H_1 be two Banach spaces such that H_1 is compactly embedded in H. Let also X be a bounded subset of H. We consider a nonlinear map $L:X\to X$ such that L can be decomposed into a sum of two maps*

$$L=L_0+K,\quad L_0:X\longrightarrow H,\quad K:X\longrightarrow H,$$

where L_0 is a contraction, i.e.,

$$\big\|L_0(x_1)-L_0(x_2)\big\|_H\leqslant\alpha\|x_1-x_2\|_H,\quad \forall x_1,x_2\in X,\ \textit{with } \alpha<1/2, \tag{7}$$

and K satisfies the condition

$$\big\|K(x_1)-K(x_2)\big\|_{H_1}\leqslant C\|x_1-x_2\|_H,\quad \forall x_1,x_2\in X. \tag{8}$$

Then, the map $L:X\to X$ possesses an exponential attractor.

Sketch of the proof. – Let us fix $\theta>0$ such that $2(\alpha+\theta)<1$. Since X is bounded, there exists a ball $\mathrm{B}(R,x_0,H)$ of radius R centered at $x_0\in X$ in H which contains X. We set $E_0=V_0=\{x_0\}$. It follows from (8) that the H_1-ball $\mathrm{B}(CR,K(x_0),H_1)$ covers the image $K(X)$. We now cover this ball by a finite number of θR balls in H with centers y_i (it is possible to do so because the embedding $H_1\subset H$ is compact). Moreover, the minimal number of balls in this covering can be estimated as follows:

$$N_{\theta R}\big(\mathrm{B}(CR,K(x_0),H_1),H\big)=N_{\theta R}\big(\mathrm{B}(CR,0,H_1),H\big)=N_{\theta/C}\big(\mathrm{B}(1,0,H_1),H\big)\equiv N(\theta).$$

(It is essential for us that this number be independent of R.) Thus, we have constructed a θR-covering of $K(X)$. It follows from assumption (7) that the family of balls with centers $y_i+L_0(x_0)$ and with radius $(\alpha+\theta)R$ covers $L(X)$. Now, the centers of the balls in this covering may go out of $L(X)$ and even out of X. To avoid this problem, we increase the radius twice and construct the new $2(\alpha+\theta)$-covering $\{\mathrm{B}(2(\alpha+\theta)R,x_1^i,H)\}$, $i=1,\dots,N(\theta)$, of $L(X)$ so that $x_1^i\in L(X)$. We then set $V_1=\{x_1^i,$ $i=1,\dots,N(\theta)\}$.

Applying the above procedure to every ball in this new covering, we obtain the $(2(\alpha+\theta))^2 R$-covering of $\mathrm{L}^2(X)$ with $N(\theta)^2$ balls. We denote by V_2 the set of their centers. Repeating this procedure, we finally construct a sequence of sets $V_k \subset \mathrm{L}^k(X)$ such that

$$\mathrm{dist}_H\big(\mathrm{L}^k(X), V_k\big) \leqslant R\big(2(\alpha+\theta)\big)^k \quad \text{and} \quad \#V_k \leqslant N(\theta)^k. \tag{9}$$

To obtain the invariance, we now introduce the sequence of sets $E_k = L(E_{k-1}) \cup V_k$ and we set

$$E_\infty = \bigcup_{k=1}^{\infty} E_k; \qquad \mathcal{M} = [E_\infty]_H,$$

where $[\cdot]_H$ denotes the closure in H. Let us verify that $\mathcal{M}$ is an exponential attractor for L on X. Indeed, the invariance follows immediately from our construction. Since $V_k \subset \mathcal{M}$ and $2(\alpha+\theta) < 1$, the exponential attraction property is a consequence of (9). Thus, there remains to estimate the dimension of $\mathcal{M}$ or, equivalently, that of E_∞.

We note that $L(X) \subset X$ and that

$$\bigcup_{k\geqslant n} E_k \subset \mathrm{L}^n(X) \subset \bigcup_{v\in V_n} \mathrm{B}\big(v, R\big(2(\theta+\alpha)\big)^n, H\big).$$

We fix $\varepsilon > 0$ and we choose the smallest integer n such that $R(2(\alpha+\theta))^n \leqslant \varepsilon$. Then

$$N_\varepsilon(E_\infty, H) \leqslant N_\varepsilon\Big(\bigcup_{k\leqslant n} E_k\Big) + N_\varepsilon\Big(\bigcup_{k>n} E_k\Big) \leqslant \sum_{k\leqslant n} \#E_k + \#V_{n+1} \leqslant C_2 N(\theta)^n.$$

Here, we have used the fact that $\#E_k \leqslant C_1 N(\theta)^n$, which can be easily deduced from the recurrent formula $\#E_n \leqslant \#E_{n-1} + N(\theta)^n$. Thus, $\dim_{\mathrm{F}}(X, H) \leqslant \log_2 N(\theta)/\log_2(1/(2(\theta+\alpha)))$, and Proposition 1 is proved. $\square$

In order to have compactness when proving Theorem 3, we obtain, after having introduced the proper decomposition (similar in spirit to that considered in [4]), estimates in weighted Sobolev spaces. In particular, we use the compactness of the injection $\mathrm{W}^{2-\delta/2,2}(\mathbb{R}^3, \varphi\,\mathrm{d}x) \subset \Phi$ for proper weights φ. The details will appear in [5].

Remark 5. – Analogous sufficient conditions are given in [7] for the existence of the global attractor.

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