Appunti del Corso di

Equazioni alle derivate parziali II
Attrattori

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1. Semigroups of operators and dynamical systems

A dynamical system (DS) is a set of parameters, state variables which evolves with respect to time. To be more precise, a DS $(X, S(t))$ is determined by a phase space X, X being a real Banach space which consists of all possible values of the parameters describing the state of the system (*phase space*), and an evolution map $S(t) : X \to X$, which allows to find the state of the system at time $t > 0$ if the initial state at $t = 0$ is known. Some authors (e.g., Robinson) call it semi-dinamical system (being defined for nonnegative times only).

Very often, in mechanics or in physics, the evolution of the phenomena under consideration is governed by systems of differential equations. If the system consists of ordinary differential equations (ODE) then it can be formulated as

$$u'(t) = F(t, u(t))$$

for some nonlinear function $F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$. If F does not depends explicitly on time the equation is called *autonomous*. From now on we consider only this situation, that is

$$u'(t) = F(u(t))$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$. Assume that for any initial datum $u_0 \in \mathbb{R}^n$, our equation has a unique solution $u \in C([0, \infty), \mathbb{R}^n)$ satisfying the initial condition $u(0) = u_0$. Also, we suppose that the solution continuously depends on the initial data. Then we have a so-called *finite-dimensional* DS since here the phase space is $X = \mathbb{R}^n$. In the case of a system whose possible initial states are described by functions $u_0 = u_0(x)$ depending on the *spatial* variable x, the evolution is usually governed by partial differential equations (PDE). Hence, the corresponding phase space X is some infinite-dimensional function space (e.g. $X = L^2(\Omega)$, or $X = H_0^1(\Omega)$). Such DS are usually called *infinite-dimensional*.

To refer to concrete examples, we can think to

$$u'(t) = \Delta u(t) + f(u(t))$$

or

$$\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} v(t) \\ \Delta u(t) + f(u(t)) \end{pmatrix}$$

where Δ is the Laplacian with homogeneous Dirichlet boundary condition. In the former case the *phase space* X is $L^2(\Omega)$, whereas in the latter is $H_0^1(\Omega) \times L^2(\Omega)$. Observe that we have considered autonomous equations, i.e., without an explicit dependence on t. Again, assume that for any initial datum $u_0 \in X$, our equations have a unique solution $u \in C([0,\infty), X)$ satisfying the initial condition $u(0) = u_0$. Here by "solution" we mean a solution in some weak sense. Also, we suppose that the solution continuously depends on the initial data.

In all the quoted examples (autonomous equations) we denote by

$$S(t)u_0 = u(t), \quad t \geq 0$$

the solution at time $t \geq 0$ of the equation with datum $u_0 \in X$ given at the initial time $t = 0$. More in details, we built a one-parameter family $\{S(t)\}_{t \geq 0}$ of maps from X into X. Obviously, this family has to reflect the results we know about the equation. Therefore, existence and uniqueness of global solutions for all initial data translates into requiring that $S(t) : X \to X$ is well-defined for every $t \geq 0$. Since the system is autonomous, the solution at time $t + \tau$ with initial datum u_0 is the same of the solution at time t with initial datum $u(\tau)$, where $u(\tau)$ is the solution at time τ with initial datum u_0. In terms of $S(t)$ we have the equality $S(t+\tau) = S(t)S(\tau)$, where $S(t)S(\tau)$ means $S(t) \circ S(\tau)$. Since $u(0) = u_0$ this means that $S(0) = \mathbb{I}$. For every u_0, $u(t)$ is continuous in X, that is, $S(\cdot)u_0 \in C([0,\infty), X)$ for all $u_0 \in X$. Finally, if $u_0 \to \bar{u}_0$ then $u(t) \to \bar{u}(t)$. This reads $S(t) \in C(X, X)$, for all $t \geq 0$. Summarizing all these considerations we have

1.1 Definition. Let X be a real Banach space. A *dynamical system* (or a *strongly continuous semigroup of operators*, or a *C_0-semigroup of operators*) on X is a one-parameter family of functions $S(t) : X \to X$ $(t \geq 0)$ satisfying the properties:

S.1 $S(0) = \mathbb{I}$;

S.2 $S(t+\tau) = S(t)S(\tau)$, for all $t, \tau \geq 0$;

S.3 $t \mapsto S(t)x \in C([0,\infty), X)$, for all $x \in X$;

S.4 $S(t) \in C(X, X)$, for all $t \geq 0$.

Notice that (S.2) implies that $S(t)$ and $S(\tau)$ commute for all $t, \tau \geq 0$.

Remark. The limitation X Banach space is superfluous. At this stage, the definition makes sense for X topological space (no linear structure is needed).

Remark. Notice that we do not require the joint continuity

$$(t, x) \mapsto S(t)x \in C([0,\infty) \times X, X)$$

even if in most practical situations originating from PDE this is the case.

1.2 [Exercise] Let $\mathcal{K} \subset X$ be a compact set. If the joint continuity holds, prove that $\forall\, \varepsilon > 0$ and $\forall\, T > 0$ there exists $\delta = \delta(\varepsilon, T, \mathcal{K})$ such that, for $u \in X$ and $k \in \mathcal{K}$ then it holds

$$\|u - k\| \leq \delta \quad \Longrightarrow \quad \|S(t)u - S(t)k\| \leq \varepsilon, \qquad \forall\, t \in [0, T].$$

2. Introduction to global attractors

"It is impossible to study the properties of a single mathematical trajectory. The physicist knows only bundles of trajectories, corresponding to slightly different initial conditions."

Léon Brillouin

The qualitative study of DS of finite dimension goes back from the beginning of the 20th century with the pioneering works of Poincaré on the N-body problem. One of the most surprising and significant facts discovered at first was that even relatively simple equations can generate very complicated chaotic behaviors. Moreover these types of systems are extremely sensitive to initial conditions, that is, the trajectories (solutions) with close but different initial data diverge exponentially.

Let us examine a simple example that should clarify Brillouin's words. Consider the dynamical system generated by the ODE in $X = \mathbb{R}$

$$u' = u - u^3.$$

If the initial datum u_0 is positive, then $u(t) \to 1$ as $t \to \infty$. Conversely, if u_0 is negative, $u(t) \to -1$ for $t \to \infty$. If $u_0 = 0$, then $u(t) \equiv 0$. Therefore a very small change of the initial datum can produce a big difference as $t \to \infty$.

Moreover, we mention the famous example of the Lorenz system which is defined by the system of ODE in $X = \mathbb{R}^3$

$$\begin{cases} x' = \sigma(y - x) \\ y' = -xz + rx - y \\ z' = xy - bz \end{cases}$$

Here σ, r and b are three positive numbers representing the Prandtl and Rayleigh numbers and the aspect ratio. These equations are obtained by truncation of the Navier-Stokes equations and give an approximate description of the horizontal fluid layer heated from below. The warmer fluid formed at the bottom tends to rise, creating convection currents. This is similar to what happens in the earth's atmosphere. For a sufficiently intense heating, the time evolution has a sensitive dependence on the initial conditions, thus representing a very irregular and chaotic convection. This fact was used by Lorenz to justify the so-called "butterfly effect", a metaphor for the imprecision of weather forecast.

These examples show that if we want to perform an asymptotic analysis which is relevant from the physical viewpoint, we have to change our perspective, and find a way to get information on the evolution of a set of initial data, rather then a single initial datum.

The theory of DS in finite dimensions has been extensively developed by many mathematicians during the 20th century. In particular, it is known that, very often, the trajectories (solutions) of a chaotic system are localized, up to a transient process, in some subset of the phase space having a very complicated fractal geometric structure which accumulates the nontrivial dynamics of the system.

We now turn to infinite-dimensional DS generated by PDE. A first important difficulty which arises here is that the analytic structure of a PDE is essentially more complicated than that of an ODE and, in particular, we do not have in general the unique solvability theorem as for ODE. So that, even finding the proper phase space and the rigorous construction of the associated DS can be a highly nontrivial problem. It suffices to recall the example of the three-dimensional Navier-Stokes system (one of the most important equation of mathematical physics) for which the required associate DS has not been constructed yet! Nevertheless, there exists a large number of equations for which the problem of the global existence and uniqueness of a solution has been solved. Thus, the question of extending (in these cases) the highly developed finite-dimensional DS theory to infinite dimension arises naturally. Observe that in this context the evolution system generated by PDE is treated as an ODE whose solutions (trajectories) are viewed as curves in a suitable phase space of infinite dimension.

One of the most important class of such equations consists of the so-called *dissipative* PDE. This mathematical notion originates from the fact that in many natural phenomena various kinds of dissipation are present, like, e.g., viscosity, friction, heat loss. In the study of the evolution of these phenomena it can be observed the decaying of some form of *energy* (to be suitably defined). From the mathematical viewpoint, a dynamical system may be called dissipative if there exists a bounded absorbing set; that is, a subset in the phase space which *attracts*, in some sense, any trajectory without being necessarily compact or of finite fractal dimension. We emphasize once more that the phase space X is infinite-dimensional. Nevertheless, it was observed in some experiments that, up to a transient process, the trajectories of the DS under study are localized inside *very thin* subsets of the phase space. These subsets present a complicated geometric structure which attracts all the non trivial dynamics of the system. It was conjectured a little later that these sets are, in some proper sense, finite-dimensional and that the dynamics restricted to them (they are *invariant* in time) can be effectively described by a finite number of parameters. Thus, (when this conjecture turns out to be true), in spite of the infinite-dimensional initial phase space, the effective dynamics (reduced to this invariant set) is finite-dimensional and can be studied by using the algorithms and concepts of the classical finite-dimensional DS theory. In particular, this means that the infinite-dimensionality plays here only the role of (possibly essential) technical difficulties which cannot, however, produce any new dynamical phenomena which are not observed in the finite-dimensional theory. This fact can be of some help, for instance, for possible numerical approximations. Hence, in this context, a further nice mathematical feature of a dissipative dynamical system is the existence of the global attractor, that is, the minimal compact set (so, very thin!) that attracts uniformly any bounded set in the phase space. In some cases it can be proved that the global attractor possesses final *fractal* dimension.

Finally, we would like to mention the study of attractors for nonautonomous systems, that is, systems of PDE in which time appears explicitly. This situation is much more delicate and not still completely exploited.

3. The global attractor

Here and in the sequel, let X be a real Banach space, and let $S(t)$ be a dynamical system on X. In fact all the results that will be given hold for X complete metric space in which balls are connected, upon replacing the norm, whenever it occurs, with the distance.

Limit sets

Let us begin with some definitions.

3.1 Definition. Let $x \in X$, the *orbit* of x (or *trajectory* starting at x) is the set

$$\Gamma(x) = \bigcup_{t \geq 0} S(t)x \subset X.$$

3.2 Definition. A function $y : \mathbb{R} \to X$ is a *complete bounded trajectory* of $S(t)$ if

$$\sup_{t \in \mathbb{R}} \|y(t)\| < \infty$$

and

$$y(t + \tau) = S(t)y(\tau), \qquad \forall\, t \geq 0, \forall\, \tau \in \mathbb{R}.$$

3.3 [**Exercise**] Show that $y \in C(\mathbb{R}, X)$.

3.4 Definition. An *equilibrium point* of $S(t)$ is a point $u_0 \in X$ such that

$$S(t)u_0 = u_0, \qquad \forall\, t \geq 0.$$

If we assume that $S(t)$ is the family of operators associated to the equation

$$\frac{d}{dt}\, u(t) = A(u)(t),$$

then u_0 is an equilibrium point if and only if $A(u_0) = 0$ (in some weak sense).

3.5 Definition. Denote by $S(t)\mathcal{B} = \bigcup_{x\in\mathcal{B}} S(t)x$.
A nonempty set $\mathcal{B} \subset X$ is (positively) *invariant* for $S(t)$ if

$$S(t)\mathcal{B} \subset \mathcal{B}, \qquad \forall\, t \geq 0.$$

A nonempty set $\mathcal{B} \subset X$ is *fully invariant* for $S(t)$ if

$$S(t)\mathcal{B} = \mathcal{B}, \qquad \forall\, t \geq 0.$$

Remark. If $\mathcal{B}$ is fully invariant, then the whole set is important in the dynamics since no part of $\mathcal{B}$ disappears as we run the dynamics on $\mathcal{B}$ forward in time.

Remark. If $\mathcal{B} \subset X$ is an invariant set for $S(t)$, then the restriction of $S(t)$ on $\mathcal{B}$ is a dynamical system on $\mathcal{B}$. If in addition $\mathcal{B}$ is fully invariant and $S(t)$ is injective on $\mathcal{B}$, then $S(-t) = S(t)^{-1}$ is well defined. It is then easy to check that (S.2)-(S.3) hold for the restriction of $S(t)$ on $\mathcal{B}$, replacing $[0,\infty)$ with $\mathbb{R}$. In terms of the differential equation associated to $S(t)$, this means that, for data on $\mathcal{B}$, we have *backwards uniqueness*. Finally, if $\mathcal{B}$ is compact, then (S.4) holds for all $t \in \mathbb{R}$.

3.6 [Exercise] *Let $B \subset X$ be a compact invariant set for $S(t)$. Show that $S(t)$ is a strongly continuous group of operators on $\mathcal{B}$.*

3.7 Definition. The *ω-limit* set of a nonempty set $\mathcal{B} \subset X$ is given by

$$\omega(\mathcal{B}) = \big\{x \in X : \exists\, t_n \to \infty, x_n \in \mathcal{B} \text{ with } S(t_n)x_n \to x\big\}.$$

In particular the ω-limit set of a single point $x_0 \in X$ is

$$\omega(x_0) = \big\{x \in X : \exists\, t_n \to \infty \text{ with } S(t_n)x_0 \to x\big\}.$$

Remark. Notice that $\omega(\mathcal{B})$ might be empty. If x is an equilibrium point, then $\omega(x) = \{x\}$.

Now we want to describe better the properties of an ω-limit set. For the sake of brevity we denote by $\overline{B}$ the closure of $B \subset X$ with respect to the metric induced by the norm in X.

3.8 Proposition. *Let $B \subset X$ and $\omega(B)$ be a nonempty set. Then we have*

- $\omega(\mathcal{B}) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} S(\tau)\mathcal{B}}$
- $\omega(S(t)\mathcal{B}) = \omega(\mathcal{B}), \quad \forall t \geq 0$
- *$S(t)\omega(\mathcal{B}) \subset \omega(\mathcal{B})$, that is $\omega(B)$ is invariant*

PROOF (1) Let

$$x \in \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} S(\tau)\mathcal{B}}.$$

Then

$$x \in \overline{\bigcup_{\tau\geq t} S(\tau)\mathcal{B}}, \qquad \forall\, t \geq 0.$$

Setting $t = n \in \mathbb{N}$, we find $\tau_n \geq n$ and $x_n \in B$ such that

$$\|S(\tau_n)x_n - x\| \leq \frac{1}{n}$$

from which we deduce $S(\tau_n)x_n \to x$, as $n \to +\infty$, and then $x \in \omega(B)$.

Conversely, if $x \in \omega(\mathcal{B})$, then there exist $\tau_n \to +\infty$ and $x_n \in B$ such that $S(\tau_n)x_n \to x$, as $n \to +\infty$. Choosing $t \geq 0$ we can take $\tau_n \geq t$. Then we get $x \in \overline{\bigcup_{\tau\geq t} S(\tau)\mathcal{B}}$ for all $t \geq 0$. Finally, $x \in \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} S(\tau)\mathcal{B}}$.

(2) Fix $t \geq 0$. Consider $x \in \omega(S(t)\mathcal{B})$. Then there exist $\tau_n \to +\infty$ and $x_n \in \mathcal{B}$ such that $S(\tau_n)(S(t)x_n) = S(\tau_n + t)x_n \to x$, as $n \to +\infty$. Setting $\tau_n + t = t_n$ we easily obtain $x \in \omega(\mathcal{B})$.

Conversely, let $x \in \omega(\mathcal{B})$. Then there exist $t_n \to +\infty$ and $x_n \in \mathcal{B}$ such that $S(t)x_n \to x$, as $n \to +\infty$. Fix $t \geq 0$, we can take $t_n \geq t$. Hence we have

$$S(t_n)x_n = S(t_n - t)(S(t)x_n) = S(\tau_n)(S(t)x_n) \to x$$

as $n \to +\infty$. We conclude $x \in \omega(S(t)\mathcal{B})$.

(3) Let $x \in \omega(\mathcal{B})$. Then $S(t_n)x_n \to x$, as $n \to +\infty$, for some $x_n \in \mathcal{B}$ and $t_n \to \infty$. Fix $t \geq 0$. Since $S(t) \in C(X, X)$, then

$$S(t + t_n)x_n = S(t)S(t_n)x_n \to S(t)x$$

as $n \to +\infty$. That is, $S(t)x \in \omega(\mathcal{B})$. ◇

Dissipative systems

3.9 Definition. A nonempty set $\mathcal{B}_0 \subset X$ is an *absorbing set* for $(X, S(t))$ if for every bounded set $\mathcal{B} \subset X$ there exists $t_{\mathcal{B}} \geq 0$ such that

$$S(t)\mathcal{B} \subset \mathcal{B}_0, \qquad \forall\, t \geq t_{\mathcal{B}}.$$

3.10 Definition. $(X, S(t))$ is *dissipative* if it has a bounded absorbing set.

3.11 [Exercise] If $\mathcal{B}_0 \subset X$ is an absorbing set, then

$$\mathcal{B}_1 = \bigcup_{t \geq t_{\mathcal{B}_0}} S(t)\mathcal{B}$$

is a bounded, absorbing, invariant set.

Since in a dissipative system all the trajectories eventually enter in $\mathcal{B}_0$, one might expect that the set

$$\bigcup_{x \in \mathcal{B}_0} \omega(x)$$

would capture the asymptotic dynamics. However, as it will be clear in the sequel, this set turns out to be too small to give the necessary information.

Unfortunately, in infinite-dimensional Banach spaces, balls are not so nice sets. For instance, they are not compact. So the knowledge of the existence of a bounded absorbing set gives little information on the longterm dynamics. Then one might try to find a compact absorbing set. In most situations, this is hopeless. The idea is then to look for compact sets that, though not absorbing, "attract" all the orbits departing from bounded sets. This "attraction" can be measured in terms of the Hausdorff semidistance.

3.12 Definition. If $\mathcal{A}$ and $\mathcal{B}$ are nonempty subsets of X, the *Hausdorff semidistance* between $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$\mathrm{dist}_{\mathcal{H}}(\mathcal{A}, \mathcal{B}) = \sup_{a \in \mathcal{A}} \|a - \mathcal{B}\| = \sup_{a \in \mathcal{A}} \inf_{b \in \mathcal{B}} \|a - b\|.$$

Notice that the Hausdorff semidistance is not symmetric (so it is not a distance). To obtain a metric on subsets of X, we need to use the *symmetric Hausdorff distance*

$$\mathrm{dist}_{sym}(\mathcal{A}, \mathcal{B}) = \max(\mathrm{dist}_{\mathcal{H}}(\mathcal{A}, \mathcal{B}), \mathrm{dist}_{\mathcal{H}}(\mathcal{B}, \mathcal{A})).$$

3.13 [Exercise] $\mathrm{dist}_{\mathcal{H}}(\mathcal{A}, \mathcal{B}) = 0$ if and only if $\mathcal{A} \subset \overline{\mathcal{B}}$

3.14 Definition. A nonempty set $\mathcal{K} \subset X$ is an *attracting set* for $(X, S(t))$ if for every bounded set $\mathcal{B} \subset X$

$$\lim_{t \to \infty} \mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}, \mathcal{K}) = 0.$$

3.15 Definition. $(X, S(t))$ is *asymptotically compact* if it has a compact attracting set.

3.16 Proposition. *If $(X, S(t))$ is asymptotically compact then it possesses a bounded absorbing set (that is $(X, S(t))$ is dissipative).*

PROOF If $\mathcal{K} \subset X$ is a compact attracting set then, for any bounded $\mathcal{B} \subset X$,

$$\lim_{t\to\infty} \left(\sup_{x\in\mathcal{B}} \inf_{y\in\mathcal{K}} \|S(t)x - y\| \right) = 0$$

that is, for any $\varepsilon > 0$ there exists $t_{\mathcal{B}} > 0$ such that

$$\sup_{x\in\mathcal{B}} \inf_{y\in\mathcal{K}} \|S(t)x - y\| < \varepsilon, \quad \forall\, t > t_{\mathcal{B}}.$$

Hence, for any $x \in X$, we have

$$\inf_{y\in\mathcal{K}} \|S(t)x - y\| < \varepsilon, \quad \forall\, t > t_{\mathcal{B}}.$$

We can construct a sequence $y_n \in \mathcal{K}$ such that

$$\|S(t)x - y_n\| < \varepsilon, \quad \forall\, t > t_{\mathcal{B}}, \quad \forall\, n \in \mathbb{N}.$$

Since $\mathcal{K}$ is compact, we can extract a subsequence y_{n_k} converging to $y_0 \in \mathcal{K}$ and then

$$\|S(t)x - y_0\| \leq \varepsilon, \quad \forall\, t > t_{\mathcal{B}}.$$

If we define

$$\mathcal{K}_\varepsilon = \{z \in X \,:\, \exists\, y \in \mathcal{K} \,:\, \|z - y\| \leq \varepsilon\}$$

then $\mathcal{K}_\varepsilon$ is bounded ($\mathcal{K}$ is compact) and we have

$$S(t)x \in \mathcal{K}_\varepsilon, \quad \forall\, x \in \mathcal{B}, \quad \forall\, t > t_{\mathcal{B}}.$$

So $\mathcal{K}_\varepsilon$ is an absorbing set. ◇

If a dynamical systems is asymptotically compact, we can possibly think of finding the smallest compact attracting set. This set should describe properly the asymptotic dynamics.

The global attractor

3.17 Definition. A nonempty set $\mathcal{A} \subset X$ is said to be the *global attractor* of $(X, S(t))$ if it enjoys the following properties:

A.1 $\mathcal{A}$ is fully invariant for $S(t)$;

A.2 $\mathcal{A}$ is an attracting set;

A.3 $\mathcal{A}$ is compact.

We speak of *the* global attractor rather than *a* global attractor since, if $\mathcal{A}$ exists, it is necessarily unique. Indeed, it holds

3.18 Proposition. *Assume that $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are two global attractors of $(X, S(t))$. Then $\mathcal{A} = \tilde{\mathcal{A}}$.*

PROOF Using (A.2),

$$\lim_{t\to\infty} \mathrm{dist}_{\mathcal{H}}(S(t)\tilde{\mathcal{A}}, \mathcal{A}) = 0.$$

But on account of (A.1), $S(t)\tilde{\mathcal{A}} = \tilde{\mathcal{A}}$. Hence $\mathrm{dist}(\tilde{\mathcal{A}}, \mathcal{A}) = 0$ which implies $\tilde{\mathcal{A}} \subset \overline{\mathcal{A}} = \mathcal{A}$ (since $\mathcal{A}$ is closed due to (A.3)). Interchanging the role of $\mathcal{A}$ and $\tilde{\mathcal{A}}$, we get the reverse inclusion. ◇

Needless to say that the global attractor might not exist. Just consider the trivial example $S(t) = \mathbb{I}$ for all $t \geq 0$.

The global attractor satisfies some peculiar maximal and minimal properties.

3.19 Proposition. *Let $\mathcal{A}$ be the global attractor of $(X, S(t))$.*

(i) *Let $\tilde{\mathcal{A}}$ be a bounded set satisfying (A.1). Then $\mathcal{A} \supset \tilde{\mathcal{A}}$.*

(ii) *Let $\tilde{\mathcal{A}}$ be a closed set satisfying (A.2). Then $\mathcal{A} \subset \tilde{\mathcal{A}}$.*

PROOF **(i)** Since $\tilde{\mathcal{A}}$ is bounded, then

$$\lim_{t\to\infty} \mathrm{dist}_{\mathcal{H}}(S(t)\tilde{\mathcal{A}}, \mathcal{A}) = 0.$$

But on account of (A.1), $S(t)\tilde{\mathcal{A}} = \tilde{\mathcal{A}}$. Hence $\mathrm{dist}_{\mathcal{H}}(\tilde{\mathcal{A}}, \mathcal{A}) = 0$ which implies $\tilde{\mathcal{A}} \subset \overline{\mathcal{A}} = \mathcal{A}$ (since $\mathcal{A}$ is closed due to (A.3)).

(ii) Since $\tilde{\mathcal{A}}$ is an attracting set and A is bounded, then

$$\lim_{t\to\infty} \mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{A}, \tilde{\mathcal{A}}) = 0.$$

But $\mathcal{A}$ is also invariant, hence $\mathrm{dist}_{\mathcal{H}}(\mathcal{A}, \tilde{\mathcal{A}}) = 0$ which implies $\mathcal{A} \subset \overline{\tilde{\mathcal{A}}} = \tilde{\mathcal{A}}$ (since $\tilde{\mathcal{A}}$ is closed). ◇

On account of the previous result, we can say that the global attractor (when it exists) is the bigger compact invariant set (with reference to (i)) and the smallest compact attracting set (with reference to (ii)). This is the reason why in literature $\mathcal{A}$ is sometimes called the *maximal attractor* or the *minimal attractor.*

3.20 [Exercise] Prove that if x is an equilibrium point of $S(t)$ and $\mathcal{A}$ is the global attractor then $x \in \mathcal{A}$.

PROOF If $S(t)x = x$, $\forall\, t \geq 0$, then $\tilde{\mathcal{A}} = \{x\}$ is fully invariant and bounded. Hence (by 3.11 (i)), $\tilde{\mathcal{A}} = \{x\} \subset \mathcal{A}$. ◇

3.21 [Exercise] Prove that if $S(t)x$ is a periodic orbit and $\mathcal{A}$ is the global attractor then $S(t)x \in A$, $\forall t \geq 0$.

In the next result we show a characterization of the global attractor

3.22 Theorem. *If $\mathcal{A}$ is the global attractor of $(X, S(t))$ then it holds*

$$\mathcal{A} = \Big\{ y(0) : \textit{ y is a complete bounded trajectory of } S(t) \Big\}$$

that is, $\mathcal{A}$ is the section at time $t = 0$ (or at any time $t = t^*$) of all the complete bounded trajectories.

PROOF Let $\mathcal{A}$ be the global attractor of $S(t)$, and set

$$\tilde{\mathcal{A}} = \Big\{ y(0) : y \text{ is a complete bounded trajectory of } S(t) \Big\}.$$

If y is a complete bounded trajectory of $S(t)$, then the set $\mathcal{B} = y(\mathbb{R})$ is bounded. Moreover, $S(t)\mathcal{B} = \mathcal{B}$ for all $t \geq 0$. Hence, by Proposition 3.19 (i), $\mathcal{B} \subset \mathcal{A}$ which gives $\tilde{\mathcal{A}} \subset \mathcal{A}$.

To show the converse, let $x \in \mathcal{A}$. Using the fully invariance of $\mathcal{A}$ we build in a recursive (backward) way a sequence $x_n \in \mathcal{A}$ (with $x_0 = x$) such that

$$S(1)x_{n+1} = x_n$$

from which we deduce

$$S(k)x_{n+k} = x_n, \quad \forall\, k \in \mathbb{N}.$$

Define then

$$y(t) = S(t+n)x_n, \qquad \text{for } t \geq -n.$$

Notice that this definition is coherent; indeed, if $t \geq -m$ and we assume, for instance, $n \geq m$, then $(t \geq -m > -n)$

$$S(t+n)x_n = S(t+m+n-m)x_n = S(t+m)S(n-m)x_n = S(t+m)x_m.$$

Observe that $y(t) \in \mathcal{A}$ for every $t \in \mathbb{R}$. So, in particular, y is bounded. Moreover, if $\tau \in \mathbb{R}$ and $t \geq 0$,

$$y(t+\tau) = S(t+\tau+n)x_n = S(t)S(\tau+n)x_n = S(t)y(\tau).$$

Hence, we have proved that y is a complete bounded trajectory of $S(t)$. Since $y(0) = S(0)x_0 = x_0 = x$, we obtain the inclusion $\tilde{\mathcal{A}} \supset \mathcal{A}$. ◇

Remark. If $S(t)$ is injective on $\mathcal{A}$, then for every $x \in \mathcal{A}$ there is just one complete bounded trajectory of $S(t)$ passing through x. Hence $S(t)$ is a strongly continuous group of operators on $\mathcal{A}$.

3.23 Theorem. *Let $\mathcal{A}$ be the global attractor of $(X, S(t))$, with X connected. Then $\mathcal{A}$ is connected.*

PROOF Let $\mathcal{B}$ be a ball containing $\mathcal{A}$. Then $\mathcal{B}$ is clearly (path) connected. Assume by contradiction $\mathcal{A}$ be not connected. Then there are two disjoint open sets U_1 and U_2 such that $\mathcal{A} \cap U_j \neq \emptyset$ and $\mathcal{A} \subset U_1 \cup U_2$. Since $S(t) \in C(X, X)$ for every fixed $t \geq 0$, we get that $S(n)\mathcal{B}$ is connected for every $n \in \mathbb{N}$. Moreover,

$$S(n)\mathcal{B} \supset S(n)\mathcal{A} = \mathcal{A}, \qquad \forall\, n \in \mathbb{N}$$

hence

$$U_j \cap S(n)\mathcal{B} \neq \emptyset, \qquad \forall\, n \in \mathbb{N}.$$

Then for every $n \in \mathbb{N}$ there is $x_n \in S(n)\mathcal{B} \setminus (U_1 \cup U_2)$, due to the connectedness of $S(n)\mathcal{B}$. We know that

$$\lim_{n\to\infty} \mathrm{dist}_{\mathcal{H}}(S(n)\mathcal{B}, \mathcal{A}) = 0$$

so in particular

$$\lim_{n\to\infty} \Big[\inf_{a\in\mathcal{A}} \|x_n - a\| \Big] = 0.$$

Thus we can find a sequence $a_n \in \mathcal{A}$ such that

$$\lim_{n\to\infty} \|x_n - a_n\| = 0.$$

Using the compactness of $\mathcal{A}$, we reckon that (up to a subsequence) $a_n \to a \in \mathcal{A}$. But then we have also $x_n \to a$, which implies that $a \notin U_1 \cup U_2$ and therefore $a \notin \mathcal{A}$. Contradiction. $\diamond$

3.24 [Exercise] Using the definition and the related properties, find the global attractor of the dynamical system generated by the ODE

$$x' = x - x^3.$$

Up to now, we investigated some properties of the global attractor. We now show that indeed there are cases in which this object exists.

The main existence theorem

Let us show to begin a "conditional" existence result.

3.25 Theorem. *Let $\mathcal{B} \subset X$ be a bounded nonempty set. Assume that $\omega(\mathcal{B})$ is nonempty, compact and attracting for $(X, S(t))$. Then $\omega(\mathcal{B})$ is the global attractor of $(X, S(t))$.*

PROOF We only have to prove that $\omega(\mathcal{B})$ is fully invariant. In fact, we need to show that

$$S(t)\omega(\mathcal{B}) \supset \omega(\mathcal{B}), \qquad \forall\, t \geq 0$$

the reverse inclusion being always true (we already know that $\omega(\mathcal{B})$ is invariant). Fix $t \geq 0$, and let $z \in \omega(\mathcal{B})$. Then there exist $t_n \to \infty$ and $z_n \in \mathcal{B}$ such that

$$S(t_n)z_n \to z.$$

We may suppose $t_n \geq t$ for all $n \in \mathbb{N}$. Since $\omega(\mathcal{B})$ is attracting, we get in particular

$$\lim_{n\to\infty} \mathrm{dist}_{\mathcal{H}}(S(t_n - t)\mathcal{B}, \omega(\mathcal{B})) = 0$$

that is

$$\lim_{n\to\infty} \left(\sup_{z\in\mathcal{B}} \inf_{x\in\omega(\mathcal{B})} \|S(t_n - t)z - x\| \right) = 0.$$

Hence, we deduce

$$\lim_{n\to\infty} \left[\inf_{x\in\omega(\mathcal{B})} \|S(t_n - t)z_n - x\| \right] = 0.$$

So there is a sequence $x_n \in \omega(\mathcal{B})$ such that

$$\lim_{n\to\infty} \|S(t_n - t)z_n - x_n\| = 0.$$

But $\omega(\mathcal{B})$ is compact, thus, up to a subsequence, $x_n \to x \in \omega(\mathcal{B})$ as $n \to \infty$, which yields at once

$$S(t_n - t)z_n \to x, \quad n \to \infty.$$

Using the continuity of $S(t)$,

$$S(t)S(t_n - t)z_n \to S(t)x, \quad n \to \infty.$$

On the other hand,

$$S(t)S(t_n - t)z_n = S(t_n)z_n \to z, \quad n \to \infty.$$

We conclude that $z = S(t)x$, i.e., $z \in S(t)\omega(\mathcal{B})$. $\diamond$

To carry out successfully our analysis, we need to introduce some technical tools. Recall the following well-known fact.

3.26 Theorem. *If $\mathcal{B}$ is a closed subset of X then $\mathcal{B}$ is compact if and only if $\mathcal{B}$ is totally bounded, that is, $\mathcal{B}$ can be covered by finitely many balls of radius ε, for every $\varepsilon > 0$.*

The above result suggests a way to measure how far a set is from being compact.

3.27 Definition. Given a bounded set $\mathcal{B} \subset X$, the *Kuratowski measure of noncompactness* $\alpha(\mathcal{B})$ is defined by

$$\alpha(\mathcal{B}) = \inf \big\{ d : \mathcal{B} \text{ has a finite cover of balls of } X \text{ of diameter less than } d \big\}.$$

Here are some basic properties.

3.28 Proposition.

K.1 $\alpha(\mathcal{B}) = 0$ *if and only if* $\overline{\mathcal{B}}$ *is compact.*

K.2 $\mathcal{B}_1 \subset \mathcal{B}_2$ *implies* $\alpha(\mathcal{B}_1) \leq \alpha(\mathcal{B}_2)$*; and* $\alpha(\mathcal{B}_1 \cup \mathcal{B}_2) = \max\{\alpha(\mathcal{B}_1), \alpha(B_2)\}$.

K.3 $\alpha(\mathcal{B}) = \alpha(\overline{\mathcal{B}})$.

K.4 *Let* $\{\mathcal{B}_t\}_{t\geq 0}$ *be a family of nonempty bounded closed sets such that* $\mathcal{B}_{t_1} \supset \mathcal{B}_{t_2}$ *for* $t_1 < t_2$*, and* $\lim_{t\to\infty} \alpha(\mathcal{B}_t) = 0$*. Denote* $\mathcal{B} = \bigcap_{t\geq 0} \mathcal{B}_t$*. Then*

- **(i)** $\mathcal{B}$ *is nonempty;*
- **(ii)** $\mathcal{B}$ *is compact;*
- **(iii)** *if the sets* $\mathcal{B}_t$ *are connected for all* t*, then* $\mathcal{B}$ *is connected.*

3.29 [Exercise] *Prove (K.1)-(K.3) and (K.4ii).*

In order to prove (K.4i), we state and prove a stronger result.

3.30 Proposition. *Let* $\{\mathcal{B}_t\}_{t\geq 0}$ *be as in* (K.4)*. Let* $t_n \uparrow \infty$*, and let* $\zeta_n \in \mathcal{B}_{t_n}$*. Then there exist* $\zeta \in \bigcap_{t\geq 0} \mathcal{B}_t$ *and a subsequence* $\zeta_{n_k} \to \zeta$*.*

PROOF Select $m \in \mathbb{N}$. Then there is n_m such that

$$\alpha(\mathcal{B}_{t_n}) \leq \frac{1}{m}, \qquad \forall\, n \geq n_m.$$

In particular, there exists a ball of diameter less than or equal to $1/m$ which contains infinitely many terms of the sequence ζ_n. Proceeding with a classical diagonalization method, we find a Cauchy subsequence ζ_{n_k}, that converges to some ζ, for X is complete. On the other hand, for any fixed $t \geq 0$, $\zeta_{n_k} \in \mathcal{B}_t$ for all $k \geq k_0(t)$. So $\zeta \in \mathcal{B}_t$ (since $\mathcal{B}_t$ is closed) and therefore $\zeta \in \bigcap_{t\geq 0} \mathcal{B}_t$. ◇

3.31 [Exercise] Prove (K.4iii). [**Hint:** Use the arguments of the above proposition and of the proof of Theorem 3.23].

3.32 [Exercise] Show that if $\mathcal{K}$ is relatively compact then

$$\operatorname{dist}_{\mathcal{H}}(\mathcal{B}, \mathcal{K}) \leq c \qquad \Longrightarrow \qquad \alpha(\mathcal{B}) \leq 2c.$$

We are now ready to state the main existence result.

3.33 Theorem. *Let $(X, S(t))$ be a dynamical system. Assume the following hypotheses:*

(i) *there exists a bounded absorbing set $\mathcal{B}_0 \subset X$;*

(ii) *there exists a sequence $t_n \geq 0$ such that $\lim_{n\to\infty} \alpha(S(t_n)\mathcal{B}_0) = 0$;*

Then $\omega(\mathcal{B}_0)$ is the global attractor of $(X, S(t))$.

PROOF In light of Theorem 3.25, we have to show that $\omega(\mathcal{B}_0)$ is nonempty, compact and attracting. The result will be obtained in three steps.

Step 1. *We have $\lim_{t\to\infty} \alpha(S(t)\mathcal{B}_0) = 0$.*

On account of (i), there is $t_0 \geq 0$ such that

$$S(t)\mathcal{B}_0 \subset \mathcal{B}_0, \qquad \forall\, t \geq t_0.$$

Let us fix $\varepsilon > 0$. From (ii) there is t_{n_0} such that $\alpha(S(t_{n_0})\mathcal{B}_0) < \varepsilon$.

Then, taking $t \geq t_0 + t_{n_0}$,

$$S(t)\mathcal{B}_0 = S(t_{n_0})S(t - t_{n_0})\mathcal{B}_0 \subset S(t_{n_0})\mathcal{B}_0$$

which yields

$$\alpha(S(t)\mathcal{B}_0) \leq \alpha(S(t_{n_0})\mathcal{B}_0) < \varepsilon, \quad \forall\, t \geq t_0 + t_{n_0}$$

Hence, $\lim_{t\to\infty} \alpha(S(t)\mathcal{B}_0) = 0$. ◇

Step 2. *$\omega(\mathcal{B}_0)$ is nonempty and compact.*

Let $t_0 \geq 0$ be as above. If $t \geq t_0$ there holds

$$\bigcup_{\tau\geq t} S(\tau)\mathcal{B}_0 = \bigcup_{\sigma\geq 0} S(t+\sigma)\mathcal{B}_0 = \bigcup_{\sigma\geq 0} S(t-t_0)S(\sigma+t_0)\mathcal{B}_0$$

$$\subset \bigcup_{\sigma\geq 0} S(t-t_0)\mathcal{B}_0 = S(t-t_0)\mathcal{B}_0.$$

Observe that $S(t-t_0)\mathcal{B}_0$ is definitely bounded. Hence, on account of Step 1,

$$\lim_{t\to\infty} \alpha\Big(\bigcup_{\tau\geq t} S(\tau)\mathcal{B}_0\Big) = 0.$$

Set then for $t \geq 0$

$$\mathcal{A}_t = \overline{\bigcup_{\tau\geq t} S(\tau)\mathcal{B}_0}.$$

The sets $\mathcal{A}_t$ are closed, bounded (definitely) and nested, and

$$\lim_{t\to\infty} \alpha(\mathcal{A}_t) = 0.$$

Therefore, by (K.4) and Proposition 3.8

$$\omega(\mathcal{B}_0) = \bigcap_{t\geq 0} \mathcal{A}_t$$

is nonempty and compact. ◇

Step 3. $\omega(\mathcal{B}_0)$ *is an attracting set.*

Assume not. Then there exist a bounded set $\mathcal{B}$, $z_n \in \mathcal{B}$, $\delta > 0$, and a sequence $\tau_n \uparrow \infty$ such that

$$\inf_{x\in\omega(\mathcal{B}_0)} \|S(\tau_n)z_n - x\| \geq \delta.$$

Choose $t_* \geq 0$ such that

$$S(t_*)\mathcal{B} \subset \mathcal{B}_0$$

and let n be large enough such that $\tau_n \geq t_*$. Clearly,

$$\zeta_n = S(\tau_n)z_n \in \mathcal{A}_{\tau_n - t_*}.$$

Since $\mathcal{A}_t$ satisfy the assumptions of Proposition 3.30, then there is a subsequence

$$\zeta_{n_k} \to \zeta \in \bigcap_{t\geq 0} \mathcal{A}_t = \omega(\mathcal{B}_0).$$

This means that the sequence $S(\tau_n)z_n$ has a cluster point in $\omega(\mathcal{B}_0)$, which is a contradiction. ◇

We highlight some situations in which the previous theorem applies at once, so yielding the existence of the global attractor.

3.34 Corollary. *If $(X\, S(t))$ has a compact absorbing set, then there exists the global attractor.*

PROOF Let $\mathcal{B}_0$ be a compact absorbing set. Then assumption i) of Theorem 3.33 is satisfied.

Since $\mathcal{B}_0$ is a compact absorbing set, then there exists $t_0 \geq 0$ such that

$$S(t)\mathcal{B}_0 \subset \mathcal{B}_0,\ \forall t \geq t_0.$$

Hence, $\alpha(S(t)\mathcal{B}_0) \subset \mathcal{B}_0 \leq \alpha(\mathcal{B}_0) = 0$, $\forall t \geq t_0$.

Finally, $\alpha(S(t)\mathcal{B}_0) \to 0$ as $t \to \infty$, and also ii) of Theorem 3.33 holds. ◇

3.35 Corollary. *Let $(X\, S(t))$ be asymptotically compact. Then there exists the global attractor.*

PROOF $(X\,S(t))$ admits a compact attracting set K. On account of Proposition 3.16, there exists a bounded absorbing set $\mathcal{B}_0$. Hence, i) of Theorem 3.33 holds.

Since K is attracting and $\mathcal{B}_0$ is bounded, then we have

$$\lim_{t\to\infty} \mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}_0, K) = \lim_{t\to\infty} \sup_{z\in\mathcal{B}_0} \inf_{y\in K} \|S(t)z - y\| = 0$$

Hence, there exists a sequence t_n such that

$$\sup_{z\in\mathcal{B}_0} \inf_{y\in K} \|S(t_n)z - y\| \leq \frac{1}{n}$$

from which

$$\inf_{y\in K} \|S(t_n)z - y\| \leq \frac{1}{n}, \quad \forall\, z \in \mathcal{B}_0.$$

We can find a sequence $y_j \in K$ such that $\|S(t_n)z - y_j\| \leq \frac{1}{n}$, $\forall\, z \in \mathcal{B}_0$, $\forall j \in \mathbb{N}$. Since K is compact there exist a subsequence $y_{j_k} \in K$ and $y_0 \in K$ such that $y_{j_k} \to y_0$. This implies $\|S(t_n)z - y_0\| \leq \frac{1}{n}$, $\forall\, z \in \mathcal{B}_0$. Consider now the set $K_{\frac{1}{n}} = \bigcup_{y\in K} B_y(\frac{1}{n})$ where $B_w(\rho)$ denotes a ball centered in w with radius $\rho > 0$. Since $K_{\frac{1}{n}}$ is a cover of K compact, then we can extract a finite subcover of balls centered in $\gamma, \ldots, \gamma_k \in K$ such that $K \subset \bigcup_{i=1,\ldots,k} B_{\gamma_i}(\frac{1}{n})$.

Observe that, for any $z \in \mathcal{B}_0$, there exists $y_0 \in K$ such that $\|S(t_n)z - y_0\| \leq \frac{1}{n}$. Moreover, for any $y_0 \in K$, there exists $\gamma_i \in \{\gamma_1, \ldots, \gamma_k\} \subset K$ such that $\|y_0 - y_i\| \leq \frac{1}{n}$. Hence we deduce $\|S(t_n)z - \gamma_i\| \leq \|S(t_n)z - y_0\| + \|y_0 - \gamma_i\| \leq \frac{2}{n}$, from which $S(t_n)\mathcal{B}_0 \subset \bigcup_{i=1,\ldots,k} B_{\gamma_i}(\frac{2}{n})$. Finally, we have $\alpha(S(t_n)\mathcal{B}_0) \leq \frac{4}{n}$ and then $\alpha(S(t_n)\mathcal{B}_0) \to 0$ as $n \to \infty$. So ii) of Theorem 3.33 holds. ⋄

3.36 Corollary. *Let $(X, S(t))$ be dissipative. In addition, assume*

$$S(t) = S_1(t) + S_2(t)$$

(where $S_j(t) : X \to X$ are not necessarily semigroups), with

$$\sup_{x\in\mathcal{B}_0} \|S_1(t)x\| \to 0 \quad \text{as } t \to \infty$$

and

$$S_2(t)\mathcal{B}_0 \subset K \ \ \text{compact}, \quad \forall\, t \geq 0.$$

Then there exists the global attractor $\mathcal{A} \subset K$.

PROOF Since $(X, S(t))$ is dissipative, it admits a bounded absorbing set $\mathcal{B}_0$.

Moreover, there exists a sequence t_n such that $\sup_{x\in\mathcal{B}_0} \|S_1(t_n)x\| \leq \frac{1}{n}$.

Consider $K_{\frac{1}{n}} = \bigcup_{y\in K} B_y(\frac{1}{n})$ that is a cover of the compact K. Then we can extract a finite subcover of balls centered in $\gamma_1, \ldots, \gamma_k \in K$ such that $K \subset \bigcup_{i=1,\ldots,k} B_{\gamma_i}(\frac{1}{n})$.

Let us fix now $x \in \mathcal{B}_0$. Then we have $S(t_n)x = S_1(t_n)x + S_2(t_n)x$. Since $S_2(t_n)x \in K$ then there exists $\gamma_i \in \{\gamma_1, \dots, \gamma_k\}$ such that $\|S_2(t_n)x - \gamma_i\| \le \frac{1}{n}$. Hence we have $\|S(t_n)x - \gamma_i\| = \|S_1(t_n)x + S_2(t_n)x - \gamma_i\| \le \frac{2}{n}$ from which $\alpha(S(t_n)\mathcal{B}_0) \le \frac{4}{n}$. Finally, one deduces $\alpha(S(t_n)\mathcal{B}_0) \to 0$, as $n \to \infty$. This implies the existence the global attractor $\mathcal{A}$.

To show that $\mathcal{A} \subset K$ it is enough to prove that the compact K is an attracting set. Let us consider a bounded set $\mathcal{B}$. Since $\mathcal{B}_0$ is an absorbing set, then there exists $t_0 \ge 0$ such that $S(t)\mathcal{B} \subset \mathcal{B}_0$, $\forall\, t \ge t_0$. Hence we can always reduce the analysis to the case $\mathcal{B} = \mathcal{B}_0$, taking $t \ge t_0$ ($S(t)\mathcal{B} = S(t-t_0)S(t_0)\mathcal{B} = S(t-t_0)\mathcal{B}_0$).

Recalling the properties of $S_1(t)$ and $S_2(t)$, then we have

$$\begin{aligned}\mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}_0, K) &= \sup_{z\in\mathcal{B}_0} \inf_{y\in K} \|S(t)z - y\| = \sup_{z\in\mathcal{B}_0} \inf_{y\in K} \|S_1(t)z + S_2(t)z - y\| \\ &\le \sup_{z\in\mathcal{B}_0} \inf_{y\in K} (\|S_1(t)z\| + \|S_2(t)z - y\|) \\ &\le \sup_{z\in\mathcal{B}_0} \|S_1(t)z\| + \sup_{z\in\mathcal{B}_0} \inf_{y\in K} \|S_2(t)z - y\| = \sup_{z\in\mathcal{B}_0} \|S_1(t)z\| \to 0, \text{ as } t \to \infty.\end{aligned}$$

From this we deduce that K is a compact attracting set and we conclude that $\mathcal{A} \subset K$. ◇

3.37 Corollary. *Let $(X, S(t))$ be dissipative. In addition, assume*

$$S(t) = S_1(t) + S_2(t)$$

(where $S_j(t) : X \to X$ are not necessarily semigroups), with

$$\sup_{x\in\mathcal{B}_0} \|S_1(t)x\| \to 0 \quad \text{as } t \to \infty$$

and

$$S_2(t)\mathcal{B}_0 \subset K(t) \quad \text{compact}, \quad \forall\, t \ge 0.$$

Then there exists the global attractor $\mathcal{A}$.

PROOF **[Exercise]** ◇

3.38 [Exercise] Prove that the dynamical system generated by the Lorenz system admits the global attractor.

Dynamics on the attractor

We explain in which sense the global attractor determines the longterm dynamics of the system. We discuss the particular (but sufficiently general) case when the semigroup is jointly continuous. Loosely speaking, what happens is that if we consider a trajectory departing from a point $x \in X$, after a sufficient time it will "look like" some trajectory on the attractor for a long time.

3.39 Theorem. *Assume $S(t)$ is jointly continuous, that is,*

$$(t,x) \mapsto S(t)x \in C([0,\infty) \times X, X)$$

and let $\mathcal{A}$ be the global attractor of $(X, S(t))$. Then, for every $x_0 \in X$, every $\varepsilon > 0$, and every $T > 0$ there exist $\tau = \tau(\varepsilon, T)$ and $a_0 \in \mathcal{A}$ such that

$$\|S(t+\tau)x_0 - S(t)a_0\| \leq \varepsilon, \qquad \forall\, t \in [0,T].$$

PROOF By force of Exercise 1.2, there exists $\delta = \delta(\varepsilon, T) > 0$ such that, taking $x_0 \in X$ and $a \in \mathcal{A}$, then

$$\|x_0 - a\| \leq \delta \quad \Longrightarrow \quad \|S(t)x - S(t)a\| \leq \varepsilon, \quad \forall t \in [0,T].$$

Since $\mathcal{A}$ is the global attractor, then

$$\lim_{t\to\infty} \mathrm{dist}_{\mathcal{H}}(S(\tau)x_0, \mathcal{A}) = 0.$$

Then there exists a time $\tau \geq 0$ such that $\mathrm{dist}_{\mathcal{H}}(S(\tau)x_0, \mathcal{A}) \leq \delta$ and then $\inf_{a\in\mathcal{A}} \|S(\tau)x_0 - a\| \leq \delta$.

Since $\mathcal{A}$ is compact, then there exists $a_0 \in \mathcal{A}$ such that $\mathrm{dist}_{\mathcal{H}}(S(\tau)x_0, \mathcal{A}) \leq \delta = \|S(\tau)x_0 - a_0\| \leq \delta$ Considering the two trajectories having initial data at time $t = 0$ the values $S(\tau)x_0$ and a_0, then we get

$$\|S(t+\tau)x_0 - S(t)a_0\| \leq \varepsilon, \qquad \forall\, t \in [0,T].$$

◇

Gradient systems and equilibrium points

We now want to examine in more detail the set $\mathcal{E}$ of equilibrium points of $S(t)$. Due to the continuity of $S(t)$, $\mathcal{E}$ is a closed set. Also, if there is the global attractor $\mathcal{A}$, then $\mathcal{E} \subset \mathcal{A}$. We already mentioned that the knowledge of $\mathcal{E}$ is not sufficient to understand the whole asymptotic dynamics. Here we consider a special class of dynamical systems, the so-called gradient systems.

The following definitions and results hold if we replace the whole space X with a closed set $X_0 \subset X$ that is invariant for $S(t)$. Clearly, in order to have information on the longterm dynamics, the set X_0 should not be too small. Certainly it should contain the set $\mathcal{E}$ and, if it exists, the global attractor $\mathcal{A}$.

3.40 Definition. A function $\Phi \in C(X, \mathbb{R})$ is said a *Lyapunov function* for $S(t)$ if

(i) $\Phi(S(t)x) \leq \Phi(x)$ for all $x \in X$ and $t \geq 0$;

(ii) $\Phi(S(t)x) = \Phi(x)$ for all $t > 0$ implies that $x \in \mathcal{E}$.

If $(X, S(t))$ has a Lyapunov function, then $(X, S(t))$ is called a *gradient system*.

3.41 [Exercise] Let Φ satisfy (i). Then the function $t \mapsto \Phi(S(t)x)$ is decreasing for all $x \in X$.

Notice that condition (ii) above implies that a gradient system cannot have periodic trajectories.

3.42 Theorem. *Let $(X, S(t))$ be a gradient system. Let $x \in X$ be such that*

$$\lim_{t\to\infty} \alpha\Big(\bigcup_{\tau\geq t} S(\tau)x\Big) = 0$$

(this is clearly true for all $x \in X$ whenever the global attractor $\mathcal{A}$ exists). Then $\mathcal{E} \neq \emptyset$ and $\omega(x) \subset \mathcal{E}$. Moreover, if $\mathcal{E}$ is discrete (i.e., with no cluster points) then $\omega(x) \in \mathcal{E}$.

Before going to the proof, let us notice that if $S(t)$ is a gradient system with a global attractor $\mathcal{A}$, then

$$\mathcal{E} = \bigcup_{x\in X} \omega(x).$$

This should tell the difference between taking the union of the ω-limits of all the points of a bounded set in place of the ω-limit of the whole set.

PROOF The sets

$$\mathcal{B}_t = \overline{\bigcup_{\tau\geq t} S(\tau)x}$$

are nonempty, closed, connected, nested, and

$$\lim_{t\to\infty} \alpha(\mathcal{B}_t) = 0.$$

So by (K.4) the set

$$\omega(x) = \bigcap_{t\geq 0} \mathcal{B}_t$$

is nonempty, compact and connected. Let now $z \in \omega(x)$. Then there exists $t_n \to \infty$ such that $S(t_n)x \to z$. Exploiting the Lyapunov function Φ, we get

$$L = \lim_{t\to\infty} \Phi(S(t)x) = \lim_{n\to\infty} \Phi(S(t_n)x) = \Phi(z).$$

In particular we conclude that $\Phi(w) = L$ for all $w \in \omega(x)$. The invariance of $\omega(x)$ then implies that

$$\Phi(S(t)z) = L, \qquad \forall\, t \geq 0$$

that is, $z \in \mathcal{E}$. Finally, if $\mathcal{E}$ is discrete, then $\omega(x)$, which is connected, can meet only a connected component of $\mathcal{E}$, thus $\omega(x) \in \mathcal{E}$. ◇

3.43 [Exercise] Let $S(t)$ be a gradient system on $\mathbb{R}$ that possesses a global attractor $\mathcal{A}$. Setting $m = \min \mathcal{E}$ and $M = \max \mathcal{E}$, prove that $\mathcal{A} = [m, M]$.

3.44 Example. Let $G \in C^2(\mathbb{R})$ and consider the dynamical system on $\mathbb{R}$ generated by

$$x'(t) = G'(x(t)).$$

We show that this is a gradient system. Set

$$\Phi(x) = -G(x), \qquad x \in \mathbb{R}.$$

Notice that, if $x(t)$ is the solution,

$$\frac{d}{dt}\Phi(x(t)) = -G'(x(t))x'(t) = -\big[G'(x(t))\big]^2.$$

Therefore $\Phi(x(t))$ is decreasing in t. Moreover, if

$$\Phi(x(t)) = \Phi(x(0)), \qquad \forall\, t \geq 0$$

this entails

$$x'(t) = G'(x(t)) = 0, \qquad \forall\, t \geq 0$$

i.e., x is an equilibrium point.

Apparently, the set $\mathcal{E}$ gives few information on the attractor. However, there is a way to recover the attractor from $\mathcal{E}$. Indeed, for gradient systems, all and only the trajectories which extend to complete trajectories "departing" (at $-\infty$) from $\mathcal{E}$ locate the whole attractor.

3.45 Definition. The *unstable manifold* of $\mathcal{E}$ is defined by

$$W^U(\mathcal{E}) = \left\{ \begin{array}{l} y(0) :\ y \text{ is a complete trajectory} \\ \text{of } S(t) \text{ and } \lim_{t\to\infty} \|y(-t) - \mathcal{E}\| = 0 \end{array} \right\}.$$

In an analogous manner, the unstable manifold of a point $x \in \mathcal{E}$ is

$$W^U(x) = \left\{ \begin{array}{l} y(0) :\ y \text{ is a complete trajectory} \\ \text{of } S(t) \text{ and } \lim_{t\to\infty} \|y(-t) - x\| = 0 \end{array} \right\}.$$

If $S(t)$ is not injective on X, then there might be more complete trajectories arriving at a given point. However notice that $W^U(\mathcal{E})$ is not empty if $\mathcal{E}$ is not empty. Indeed, if $x_0 \in \mathcal{E}$, then the constant function $y(t) = x_0$ is certainly a complete (bounded) trajectory.

3.46 Theorem. *Let $(X, S(t))$ be a gradient system with admits the global attractor $\mathcal{A}$ and a Lyapunov function Φ. Then*

$$\mathcal{A} = W^U(\mathcal{E}).$$

Moreover if $\mathcal{E}$ is a finite set, then

$$\mathcal{A} = \bigcup_{x \in \mathcal{E}} W^U(x).$$

Notice that since $\mathcal{E}$ is contained in $\mathcal{A}$, then it is compact. Therefore saying that $\mathcal{E}$ is discrete or finite is the same thing.

PROOF The inclusion $\mathcal{A} \supset W^U(\mathcal{E})$ follows directly from the characterization of $\mathcal{A}$ given in Theorem 3.22 (due to the presence of the attractor, the elements of $W^U(\mathcal{E})$ are in fact complete *bounded* trajectories [**Exercise**]).

Let then $x \in \mathcal{A}$, and let y be a complete bounded trajectory of $S(t)$ such that $y(0) = x$. Consider the set

$$\gamma(x) = \bigcap_{t \geq 0} \mathcal{B}_t \qquad \text{where} \qquad \mathcal{B}_t = \overline{\bigcup_{\tau \geq t} y(-\tau)}.$$

Notice that $\mathcal{B}_t \subset \mathcal{A}$, hence $\mathcal{B}_t$ is compact. So $\gamma(x)$, a nested intersection of compact sets, is nonempty and compact. Let $z \in \gamma(x)$. Then there is $t_n \to \infty$ such that

$$y(-t_n) \to z, \qquad (n \to \infty).$$

This bears

$$\Phi(y(-t_n)) \to \Phi(z), \qquad (n \to \infty).$$

But $\Phi(y(-t))$ is increasing as $t \to \infty$. Indeed, if $\tau > 0$,

$$\Phi(y(-t + \tau)) = \Phi(S(\tau)y(-t)) \leq \Phi(y(-t)).$$

Hence

$$L = \lim_{t \to \infty} \Phi(y(-t)) = \Phi(z).$$

On the other hand, for any fixed $t \geq 0$, we have

$$S(t)y(-t_n) = y(-t_n + t) \to S(t)z, \qquad (n \to \infty)$$

and therefore

$$L = \lim_{n \to \infty} \Phi(y(-t_n + t)) = \Phi(S(t)z)$$

which yields $z \in \mathcal{E}$. We have proved that $\gamma(x) \subset \mathcal{E}$. We are left to show that

$$\lim_{t \to \infty} \|y(-t) - \mathcal{E}\| = 0.$$

Indeed, if not, there are $\varepsilon > 0$ and $t_n \to \infty$ such that

$$\|y(-t_n) - \mathcal{E}\| \geq \varepsilon.$$

Since $y(-t_n) \in \mathcal{A}$, due to compactness, we have that, up to a subsequence, $y(-t_n) \to w$. But this means that $w \in \gamma(x)$, and consequently $w \in \mathcal{E}$. Thus

$$\lim_{n\to\infty} \|y(-t_n) - \mathcal{E}\| = \|w - \mathcal{E}\| = 0$$

leading to a contradiction. Finally, if $\mathcal{E}$ is finite, it is immediate to see that

$$W^U(\mathcal{E}) = \bigcup_{x\in\mathcal{E}} W^U(x).$$

This concludes the proof. $\diamond$

3.47 [Exercise] Discuss again the ODE

$$x' = x - x^3$$

on account of the above result.

3.48 [Exercise] Let $(X, S(t))$ be a gradient system, with Lyapunov function Φ, that possesses the global attractor $\mathcal{A}$. Assume also that $\mathcal{E} = \{x_0\}$. Prove that $\mathcal{A} = \{x_0\}$. [**Hint:** Given any complete trajectory $y(t)$, observe that $\Phi(y(t))$ is a decreasing function].

Some additional topics

Further developments of the theory not treated here, such as Hausdorff and fractal dimension of global attractor, stability of the attractor, exponential attractors, inertial manifolds, and nonautonomous dynamical systems, may be found in the references.

Bibliography

[1] A.V. Babin, M.I. Vishik: *Attractors of evolution equations*, North-Holland, Amsterdam, 1992.

[2] V.V. Chepyzhov, M.I. Vishik: *Attractors for equations of mathematical physics*, A.M.S. Colloquium Pub. vol.49, Providence, 2001.

[3] A. Miranville, S. Zelik, *Attractors for dissipative partial differential equations in bounded and unbounded domains*, Evolutionary equations. Vol. IV, 103-200, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008

[4] J.C. Robinson: *Infinite-dimensional dynamical systems*, Cambridge Texts in App. Math., Cambridge, 2001.

[5] R. Temam: *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York, 1997.

[6] M.I. Vishik: *Asymptotic behaviour of solutions of evolutionary equations*, Cambridge University Press, Cambridge, 1992.

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