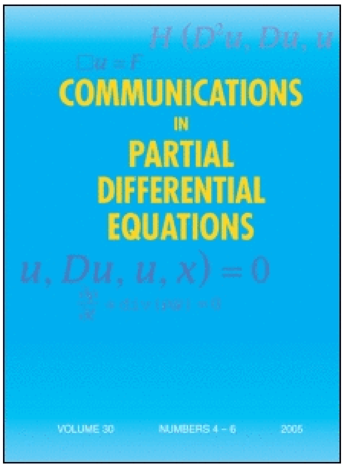

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On the 2D Cahn–Hilliard Equation with Inertial Term

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Galenko et al. proposed a modified Cahn–Hilliard equation to model rapid spinodal decomposition in non-equilibrium phase separation processes. This equation contains an inertial term which causes the loss of any regularizing effect on the solutions. Here we consider an initial and boundary value problem for this equation in a two-dimensional bounded domain. We prove a number of results related to well-posedness and large time behavior of solutions. In particular, we analyze the existence of bounded absorbing sets in two different phase spaces and, correspondingly, we establish the existence of the global attractor. We also demonstrate the existence of an exponential attractor.

Keywords Absorbing sets; Cahn–Hilliard equation; Exponential attractors; Global attractors; Logarithmic embedding inequality; Nonconvex potential.

Mathematics Subject Classification 35B40; 35B41; 35Q99; 82C26.

1. Introduction

The celebrated Cahn–Hilliard equation was proposed to describe phase separation phenomena in binary systems [9]. The standard version reads

$$u_t - \Delta(-\Delta u + f(u)) = 0, \tag{1.1}$$

where u represents the relative concentration of one species in a given domain $\Omega \subset \mathbb{R}^N$, $N \leq 3$, while f is the derivative of a non-convex potential accounting for the presence of two species (e.g., $f(u) = u(u^2 - 1)$). Many papers are devoted to the mathematical analysis of this equation and many different features have already been carefully analyzed. Here we recall, in particular, well-posedness for

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different boundary conditions and/or singular potentials, asymptotic behavior of solutions, existence of global and exponential attractors, analysis of the stationary states. We confine ourselves to quote only some contributions, namely, [4, 12–14, 29, 30, 32, 33, 35, 36, 38–41, 45–47, 51–55].

Among the phase transformations involved in phase separation, a peculiar one is named spinodal decomposition, which indicates a situation in which both the phases have an equivalent symmetry and differ only in composition (see [8], cf. also [23, 32]). It has been noted that, in certain materials like glasses, (1.1) needs to be modified in order to describe strongly non-equilibrium decomposition generated by deep supercooling into the spinodal region (cf. [18–20] and references therein). In this respect, Galenko et al., proposed a modification based on the relaxation of the diffusion flux (see, for instance, [16, 17]) which yields the following evolution equation

$$\epsilon u_{tt} + u_t - \Delta(-\Delta u + f(u)) = 0, \tag{1.2}$$

where $\epsilon > 0$ is a relaxation time. This equation shows a good agreement with experimental data extracted from light scattering on spinodally decomposed glasses (cf. [18]).

From the mathematical viewpoint, Equation (1.2) was first analyzed in [11] as a dissipative dynamical system. Those pioneering results were then improved and generalized in [57, 58] with some restrictions on ϵ. Then, in [21], the existence of a family of exponential attractors, robust as ϵ goes to 0, was established. All the quoted papers were devoted to the one-dimensional case which is relatively easy since a rather weak (energy bounded) solution is also bounded in L^∞. This is no longer true in dimensions two or three and, while the existence of an energy bounded solution can be still proven rather easily (with some restrictions on f), its uniqueness or the existence of smoother solutions appear nontrivial.

Equation (1.2) behaves in a very different way with respect to (1.1) since there is no regularization of the solutions in finite time. This smoothing property can be restored if one adds a viscosity term of the form $-\alpha\Delta u_t$, with $\alpha > 0$ (see [38] for a physical justification). The viscous variant of (1.2) in dimension three was firstly analyzed in detail in [22]. In particular, the authors constructed a family of exponential attractors which is robust as (α, ϵ) goes to $(0, 0)$, provided that ϵ is dominated by α. For other recent contributions related to the viscous version of (1.2), the reader is referred to [5, 6, 25, 28]. Going back to (1.2), in the three-dimensional case, an analysis of the longtime behavior of the solutions based only on the existence result was carried out in [49]. Then, a nonisothermal phase-separation system with memory was considered in [50] (see also [31] for existence of weak solutions). This system can be easily reduced to (1.2) by neglecting the temperature effects and taking the memory kernel equal to a decreasing exponential in the remaining equation. However, the assumptions made on the memory kernel exclude this possibility since the decreasing exponential is not θ-sectorial with $\theta \in \left(0, \frac{\pi}{2}\right)$ (see [50, Def. 1 and assumption (P0)]). We recall that this hypothesis is crucial to ensure the parabolic nature of the integrodifferential equation.

Hence, many questions about (1.2) are still unanswered so far, namely, the uniqueness of weak solutions, the well-posedness in stronger settings, the construction of bounded absorbing sets, and the existence of global attractors as well as exponential attractors. The present contribution gives several answers in the two-dimensional case. It is worth observing that (1.2) presents some similarities with the Kirchhoff–Boussinesq equation recently analyzed in [10]. However, this analogy cannot be exploited due to the different nature of the nonlinear terms.

Here we are not concerned with the dependence on ϵ, thus we take $\epsilon = 1$. This dependence will be possibly studied in a future paper. Regarding the three-dimensional case we observe that the smallness of ϵ seems to play a crucial role if we want to extend the mentioned results (see [26]) as in the case of damped wave equation with supercritical nonlinearities treated in [56]. In two dimensions we can avoid the restriction on ϵ since we can take advantage of the Brézis–Gallouet inequality (cf. [7]). It is worth noting that this restricts our analysis to functions f with (at most) a cubic growth at infinity. In this sense, the situation we meet for the 2D fourth order Equation (1.1) is surprisingly similar to what happens for the 3D (second order) damped wave equation studied, e.g., in [1, 3, 24, 42], where an analogous growth restriction on f is assumed. Therefore, in this paper we will study the initial and boundary value problem

$$u_{tt}(t) + u_t(t) - \Delta(-\Delta u(t) + f(u(t))) = g, \quad \text{in } \Omega, t > 0, \tag{1.3}$$

$$u(t) = \Delta u(t) = 0, \quad \text{on } \partial\Omega, t > 0, \tag{1.4}$$

$$u(0) = u_0, \quad u_t(0) = u_1, \quad \text{in } \Omega, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^2$ is a given smooth and bounded domain. Here f is the derivative of a nonconvex smooth potential, while g is a known time-independent source term. The choice of the boundary conditions is somewhat artificial but it allows us to simplify the presentation (see also [21, 57, 58]). However, our arguments could be recasted when we deal with usual no-flux boundary conditions (like, e.g., in [11, 25]).

The paper is organized as follows. Section 2 is devoted to the statement of our hypotheses and the proof of well-posedness with initial data in $H^3 \times H^1$ (quasi-strong solutions). In Section 3, dissipativity and existence of the global attractor are demonstrated for the semiflow generated by the solutions we previously found. Then, a regularity property of the attractor is established in Section 4 while the existence of an exponential attractor is proven in Section 5. The class of energy bounded solutions is carefully analyzed in Section 6, showing, in particular, a well-posedness result and the existence of the global attractor.

2. Existence and Uniqueness of Quasi-Strong Solutions

Let us set $H := L^2(\Omega)$ and denote by $(\cdot, \cdot)$ the scalar product both in H and in $H \times H$, and by $\|\cdot\|$ the induced norm. The symbol $\|\cdot\|_X$ will indicate the norm in the generic real Banach space X. Next, we set $V := H_0^1(\Omega)$, so that $V' = H^{-1}(\Omega)$ is the topological dual of V. The space V is endowed with the scalar product

$$((v, z)) := \int_\Omega \nabla v \cdot \nabla z, \tag{2.1}$$

and the related norm. We also denote by $A : D(A) \to H$ the Laplace operator with homogeneous Dirichlet boundary condition. It is well known that A is a strictly positive operator with $D(A) = H^2(\Omega) \cap V$ (note that we shall always suppose

Ω smooth enough), so that we can define, for $s \in \mathbb{R}$, its powers $A^s : D(A^s) \to H$. Moreover, we introduce the scale of Hilbert spaces

$$\mathcal{V}_s := D(A^{\frac{s+1}{2}}) \times D(A^{\frac{s-1}{2}}), \tag{2.2}$$

so that we have, in particular, $\mathcal{V}_0 = V \times V'$ and $\mathcal{V}_1 = (H^2(\Omega) \cap V) \times H$. The spaces $\mathcal{V}_s$ are naturally endowed with the graph norm

$$\|(u, v)\|_s^2 := \|A^{\frac{s+1}{2}} u\|^2 + \|A^{\frac{s-1}{2}} v\|^2. \tag{2.3}$$

Our hypotheses on the nonlinear function f are the following:

$$f \in C^{2,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}), \quad f(0) = 0, \quad \exists r_0 \geq 0 : f(r)r \geq 0 \quad \forall |r| \geq r_0, \tag{2.4}$$

$$\exists \lambda \geq 0 : f'(r) \geq -\lambda \quad \forall r \in \mathbb{R}, \tag{2.5}$$

$$\exists M \geq 0 : |f''(r)| \leq M(1 + |r|) \quad \forall r \in \mathbb{R}. \tag{2.6}$$

We note by F the primitive of f such that $F(0) = 0$. Then, for any fixed final time $T > 0$, problem (1.3)–(1.5), noted in the sequel as Problem (P), can be written in the more abstract form

$$u_{tt} + u_t + A(Au + f(u)) = g, \quad \text{in } V', \text{ a.e. in } (0, T), \tag{2.7}$$

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \quad \text{a.e. in } \Omega, \tag{2.8}$$

where u_0, u_1 are given initial data. Formally testing (2.7) by $A^{-1}u_t$, one readily sees that the *energy* functional

$$\mathcal{E} : \mathcal{V}_0 \to \mathbb{R}, \quad \mathcal{E}(u, v) := \frac{1}{2}\|(u, v)\|_0^2 + \int_\Omega F(u) - \langle g, A^{-1}u \rangle, \tag{2.9}$$

can be associated with (2.7) and, due to (2.6), $\mathcal{E}$ is finite for all $(u, v) \in \mathcal{V}_0$, provided that $g \in V'$ (cf. (2.10) below; however, at this stage $g \in D(A^{-3/2})$ would suffice). Moreover, by (2.4), F is bounded from below (and $\mathcal{E}$ as well).

Remark 2.1. The last condition in (2.4) is assumed just to avoid further technicalities. Indeed, it can be relaxed by taking

$$\liminf_{|r| \to \infty} \frac{f(r)}{r} > -\lambda_1,$$

where λ_1 is the first eigenvalue of A. Note that, in this case, we can choose F such that

$$F(r) \geq -\frac{\kappa}{2} r^2,$$

for some $\kappa < \lambda_1$. Thus $\mathcal{E}$ is still bounded from below.

In order to distinguish the solutions according to their smoothness, we introduce the following terminology. Given some $T > 0$, a solution $(u, u_t) \in L^\infty(0, T; \mathcal{V}_0)$ to (2.7)–(2.8) will be named *energy solution*. Instead, $(u, u_t) \in L^\infty(0, T; \mathcal{V}_1)$ will be called *weak solution*. Note that energy solutions are weaker than weak solutions. Speaking of smoother solutions, $(u, u_t) \in L^\infty(0, T; \mathcal{V}_2)$ will be called *quasi-strong solution* and $(u, u_t) \in L^\infty(0, T; \mathcal{V}_3)$ will be a *strong solution*. In the latter case, u satisfies Equation (2.7) almost everywhere in $\Omega \times (0, T)$.

It seems natural to look first for energy solutions. Nonetheless, we prefer to investigate the class of quasi-strong solutions and then construct the energy solutions by an approximation-limit argument. Actually, we will see in Section 6 that some properties of energy solutions (like, e.g., uniqueness and asymptotic behavior) are rather delicate to handle. Regarding weak solutions, there are still some open questions (see Remark 6.8 below).

Our first result states then the well-posedness of Problem (P) in the class of quasi-strong solutions.

Theorem 2.2. *Let us assume* (2.4)–(2.6) *and*

$$g \in V', \tag{2.10}$$

$$(u_0, u_1) \in \mathcal{V}_2. \tag{2.11}$$

Then, there exists one and only one function

$$u \in W^{2,\infty}(0, T; V') \cap W^{1,\infty}(0, T; V) \cap L^\infty(0, T; D(A^{3/2})) \tag{2.12}$$

solving Problem (P).

Before proving the theorem, let us observe that, being $T > 0$ arbitrary, u can be thought to be defined for all times $t \in (0, \infty)$. Moreover, we remark that a solution to (P) will be indifferently noted in the sequel either as u or as a couple (u, u_t), the latter notation being preferred when we want to emphasize the role of some $\mathcal{V}_s$ as a phase space. We will also frequently write U for (u, u_t). Moreover, throughout the remainder of the paper the symbols c, κ, and c_i, $i \in \mathbb{N}$, will denote positive constants depending on the data f, g of the problem, but independent of the initial datum and of time. The value of c and κ is allowed to vary even within the same line. Analogously, $Q : \mathbb{R}^+ \to \mathbb{R}^+$ denotes a generic monotone function. Capital letters like C or C_i will be used to indicate constant which have other dependencies (in most cases, on the initial datum). Finally, the symbol c_Ω will denote some embedding constants depending only on the set Ω.

Proof of Theorem 2.2. To prove the regularity (2.12), we perform a number of *a priori* estimates. These may have just a formal character in this setting, but could be justified by working, e.g., in a Faedo–Galerkin approximation scheme and then taking the limit. The details of this standard procedure are omitted; however, a sketch will be given in the proof of Theorem 6.1 below.

Thus, let us start with the energy estimate. Let us set $U := (u, u_t)$ and $U_0 := (u_0, u_1)$, for brevity. Testing (2.7) by $A^{-1}u_t$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(U) + \|u_t\|_{V'}^2 \leq 0. \tag{2.13}$$

Then, let us take a (small) constant $\beta > 0$, test (2.7) by $\beta A^{-1}u$, and add the result to (2.13). Noting that, by (2.5), $f(r)r \geq F(r) - \lambda r^2/2$, it is then not difficult to infer the dissipativity of the energy, i.e.,

$$\mathscr{E}(U(t)) \leq \mathscr{E}(U_0)e^{-\kappa t} + Q(\|g\|_{V'}). \tag{2.14}$$

Hence, being by (2.6) $\int_\Omega F(u_0) \leq Q(\|u_0\|_V)$ and $\mathscr{E}(u, v) \leq Q(\|(u, v)\|_0) + Q(\|g\|_{V'})$, and recalling (2.9), we also derive

$$\|U(t)\|_0 \leq Q(\|U_0\|_0)e^{-\kappa t} + Q(\|g\|_{V'}). \tag{2.15}$$

Next, by (2.13) we obtain that $\|u_t\|_{V'}$ is summable over $(0, \infty)$. More precisely, integrating that relation from an arbitrary $t \geq 0$ to ∞, and using (2.14)–(2.15), we infer

$$\int_t^\infty \|u_t(s)\|_{V'}^2 \mathrm{d}s \leq \mathscr{E}(U(t)) - \mathscr{E}_\infty \leq Q(\|U_0\|_0)e^{-\kappa t} + Q(\|g\|_{V'}), \tag{2.16}$$

where $\mathscr{E}_\infty$ is the limit for $t \nearrow \infty$ of the (nonincreasing) function $t \mapsto \mathscr{E}(U(t))$. From this moment on, let us denote, for brevity, by c the right-hand side of (2.15). Let us then differentiate Problem (P) with respect to time. Again, this formal procedure can be justified in the Faedo–Galerkin approximation. Setting $v := u_t$, we obtain

$$v_{tt} + v_t + A(Av + f'(u)v) = 0, \tag{2.17}$$

$$v|_{t=0} = v_0 := u_1, \tag{2.18}$$

$$v_t|_{t=0} = v_1 := -u_1 - A^2u_0 - Af(u_0) + g. \tag{2.19}$$

Being $u_0 \in D(A^{3/2})$ by (2.11) and owing to (2.4), it is not difficult to check that $v_1 \in V' = D(A^{-1/2})$. More precisely, we have that

$$\|V_0\|_0 \leq Q(\|U_0\|_2), \tag{2.20}$$

(we have set here $V_0 := (v_0, v_1)$ and $V := (v, v_t)$). Next, let us test (2.17) by $A^{-1}(v_t + \beta v)$, with small $\beta > 0$ as before. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|V\|_0^2 + \beta\langle v_t, A^{-1}v\rangle + \frac{\beta}{2}\|v\|_{V'}^2\right) + (1-\beta)\|v_t\|_{V'}^2 + \beta\|v\|_V^2 + (f'(u)v, v_t) + \beta(f'(u)v, v) \leq 0. \tag{2.21}$$

A straightforward computation then shows that

$$(f'(u)v, v_t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(f'(u)v, v) - \frac{1}{2}(f''(u)v^2, v). \tag{2.22}$$

Let us then now pick β in (2.21) so small that

$$\frac{1}{2}\|V\|_0^2 + \beta\langle v_t, A^{-1}v\rangle + \frac{\beta}{2}\|v\|_{V'}^2 \geq \frac{1}{4}\|V\|_0^2. \tag{2.23}$$

Next, for $L > 0$ whose value is to be chosen later, we have by interpolation

$$L\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{V'}^2 + 2\beta L\|v\|_{V'}^2 \le \frac{\beta}{8}\|V\|_0^2 + c(L,\beta)\|v\|_{V'}^2 \le \frac{\beta}{8}\|V\|_0^2 + \mathrm{c}. \tag{2.24}$$

Subsequently, let us add the above inequality to (2.21) and then set

$$\mathcal{F} := \frac{1}{2}\|V\|_0^2 + \beta\langle v_t, A^{-1}v\rangle + \frac{\beta}{2}\|v\|_{V'}^2 + \frac{1}{2}(f'(u)v, v) + L\|v\|_{V'}^2, \tag{2.25}$$

and observe that for a suitable choice of L (depending on λ and on β taken before), by interpolation there holds

$$\frac{1}{2}(f'(u)v, v) + L\|v\|_{V'}^2 \ge -\frac{\lambda}{2}\|v\|^2 + L\|v\|_{V'}^2 \ge -\frac{1}{8}\|v\|_V^2. \tag{2.26}$$

Hence, it is easy to see that $\mathcal{F}$ satisfies, for some $\sigma > 0$ independent of the initial data,

$$\mathcal{F} \ge \sigma\|V\|_0^2. \tag{2.27}$$

Moreover, recalling (2.22) and possibly taking a smaller β, for some $\kappa > 0$ independent of the initial data we can rewrite (2.21) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + 2\kappa\mathcal{F} \le \mathrm{c} + \frac{1}{2}(f''(u)v^2, v). \tag{2.28}$$

At this point, let us test (2.7) by Au. Using (2.4)–(2.6), Sobolev embeddings and interpolation, we can estimate the nonlinear term this way:

$$\begin{aligned}\langle Af(u), Au\rangle &= (f'(u)\nabla u, \nabla\Delta u) \le \|f'(u)\nabla u\|\,\|u\|_{D(A^{3/2})}\\ &\le c(1+\|u\|_{L^4(\Omega)}^2)\|\nabla u\|_{L^\infty(\Omega)}\|u\|_{D(A^{3/2})}\\ &\le c(1+\|u\|_V^2)\|u\|_V^{1/2}\|u\|_{D(A^{3/2})}^{3/2} \le \delta\|u\|_{D(A^{3/2})}^2 + c_\delta\|u\|_V^2 + c_\delta\|u\|_V^{10}, \end{aligned}\tag{2.29}$$

whence, choosing δ small enough, we deduce that

$$\|U\|_2 \le c(1+\|V\|_0^5), \quad \text{a.e. in } (0,T). \tag{2.30}$$

Let us now recall the Brézis–Gallouet interpolation inequality [7, Lemma 2], holding for all $R > 0$ and $z \in D(A)$:

$$\|z\|_{L^\infty(\Omega)} \le c_\Omega\|z\|_V \log^{1/2}(1+R) + c_\Omega\|z\|_{D(A)}(1+R)^{-1}. \tag{2.31}$$

To apply the inequality, let then $\lambda_0 > 0$ depending on Ω be such that

$$\|z\|_{D(A)} \ge 2\lambda_0\|z\|_V \quad \forall z \in D(A). \tag{2.32}$$

Then, in case $\lambda_0\|z\|_V \ge 1$, take $1 + R = \|z\|_{D(A)}/\lambda_0\|z\|_V$ in (2.31), getting

$$\begin{aligned}\|z\|_{L^\infty(\Omega)} &\le c_\Omega\|z\|_V \log^{1/2}\left(\frac{\|z\|_{D(A)}}{\lambda_0\|z\|_V}\right) + c_\Omega\|z\|_V \le c_\Omega\|z\|_V\log^{1/2}\|z\|_{D(A)} + c_\Omega\|z\|_V\\ &\le c_\Omega\|z\|_V\log^{1/2}(1+\|z\|_{D(A)}) + c_\Omega\|z\|_V. \end{aligned}\tag{2.33}$$

Otherwise, simply choose $R = \|z\|_{D(A)}$, so that

$$\begin{aligned}\|z\|_{L^\infty(\Omega)} &\le c_\Omega \|z\|_V \log^{1/2}(1+\|z\|_{D(A)}) + c_\Omega \frac{\|z\|_{D(A)}}{1+\|z\|_{D(A)}} \\ &\le c_\Omega \|z\|_V \log^{1/2}(1+\|z\|_{D(A)}) + c_\Omega. \end{aligned} \tag{2.34}$$

Then, using (2.6), interpolation, and either (2.33) or (2.34), the right-hand side of (2.28) can be controlled as follows

$$\begin{aligned}\frac{1}{2}(f''(u)v^2, v) &\le c(1+\|u\|_{L^\infty(\Omega)})\|v\|^3_{L^3(\Omega)} \\ &\le c\Big(1+\|u\|_V + \|u\|_V \log^{1/2}(1+\|u\|_{D(A)})\Big)\|v\|_{V'}\|v\|_V^2 \\ &\le c\Big(1+\mathfrak{c}+\mathfrak{c}\log^{1/2}(1+\|u\|_{D(A)})\Big)\|v\|_{V'}\|v\|_V^2. \end{aligned} \tag{2.35}$$

Thus, estimating the norm of u in $D(A)$ with the help of (2.30) and recalling (2.27), inequality (2.28) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + 2\kappa\mathcal{F} &\le \mathfrak{c} + c\Big(1+\mathfrak{c}+\mathfrak{c}\log^{1/2}(1+\mathcal{F})\Big)\|v\|_{V'}\mathcal{F} \\ &\le \mathfrak{c} + \kappa\mathcal{F} + c\|v\|^2_{V'}\mathcal{F}\big(1+\mathfrak{c}+\mathfrak{c}\log(1+\mathcal{F})\big). \end{aligned} \tag{2.36}$$

Therefore, possibly replacing $\mathcal{F}$ with $\mathcal{F}+c$ for a suitable c and substituting the expression for $\mathfrak{c}$ from (2.15) (recall that $\mathfrak{c}$ is the quantity on the right-hand side and note that if the κ's in (2.15) and (2.36) do not coincide we can take the smaller), relation above (for the new $\mathcal{F}$) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + \kappa\mathcal{F} \le (1+\mathcal{F}\log\mathcal{F})(Q(\|U_0\|_0)e^{-\kappa t} + Q(\|g\|_{V'})), \tag{2.37}$$

whence the standard theory of ODEs, together with (2.30), implies that there exists a computable function $\mathcal{Q} : (\mathbb{R}^+)^3 \to \mathbb{R}^+$, monotone increasing in each of its arguments, such that

$$\|u\|_{L^\infty(0,t;D(A^{3/2}))} + \|u_t\|_{L^\infty(0,t;V)} + \|u_{tt}\|_{L^\infty(0,t;V')} \le \mathcal{Q}(\|U_0\|_2, \|g\|_{V'}, t). \tag{2.38}$$

By standard tools, the above estimate permits to remove the Faedo–Galerkin approximation and to pass to the weak* limit. In particular, the regularities (2.12) are obtained. This proves the existence part of Theorem 2.2.

To prove uniqueness, let us consider a couple of solutions u_1, u_2 to (P) in the above regularity setting and for the same initial data, write (2.7) for u_1 and u_2, take the difference, and test it by $A^{-1}u_t$, where $u := u_1 - u_2$. Using (2.6) and the regularity (2.12), a straightforward computation gives

$$\begin{aligned}\int_\Omega (f(u_1)-f(u_2))u_t &\le \frac{1}{2}\|u_t\|^2_{V'} + \frac{1}{2}\int_\Omega |f'(u_1)\nabla u_1 - f'(u_2)\nabla u_2|^2 \\ &\le \frac{1}{2}\|u_t\|^2_{V'} + C\|u\|^2_V, \end{aligned} \tag{2.39}$$

where C depends, of course, on the norms of u_1, u_2 specified in (2.12). Thus, Gronwall's lemma permits to conclude that $u_1 \equiv u_2$ as desired, which completes the proof of Theorem 2.2. □

3. Asymptotic Behavior of Quasi-Strong Solutions

We associate with Problem (P) the semiflow $\mathcal{S}$ acting on $\mathcal{V}_2$ and generated by the quasi-strong solutions provided by Theorem 2.2. We will also indicate by $S(t)$, $t \geq 0$, the semigroup operator defined by $\mathcal{S}$. Let us now prove some important properties of $\mathcal{S}$ and $S(t)$.

Theorem 3.1. *Let the assumptions of Theorem* 2.2 *hold. Then, the semiflow* $\mathcal{S}$ *is* uniformly dissipative. *Namely, there exists a constant* R_0 *independent of the initial data such that, for all bounded* $B \subset \mathcal{V}_2$, *there exists* $T_B \geq 0$ *such that* $\|S(t)b\|_2 \leq R_0$, *for all* $b \in B$ *and* $t \geq T_B$. *Moreover, any* $u \in \mathcal{S}$ *satisfies the additional time continuity property*

$$u \in C^2([0, T]; V') \cap C^1([0, T]; V) \cap C^0([0, T]; D(A^{3/2})). \tag{3.1}$$

Finally, given a sequence of initial data $\{(u_{0,n}, u_{1,n})\} \subset \mathcal{V}_2$ *tending to some* $(u_0, u_1) \in \mathcal{V}_2$ *in the sense specified below, and denoting by* u_n, u *the solutions emanating from* $(u_{0,n}, u_{1,n})$, (u_0, u_1), *respectively, we have that*

$$\begin{aligned}(u_{0,n}, u_{1,n}) &\to (u_0, u_1) \quad \text{weakly in } \mathcal{V}_2 \\ &\Rightarrow (u_n, u_{n,t}) \to (u, u_t) \quad \text{weakly star in } L^\infty(0, T; \mathcal{V}_2),\end{aligned} \tag{3.2}$$

$$\begin{aligned}(u_{0,n}, u_{1,n}) &\to (u_0, u_1) \quad \text{strongly in } \mathcal{V}_2 \\ &\Rightarrow (u_n, u_{n,t}) \to (u, u_t) \quad \text{strongly in } C^0([0, T]; \mathcal{V}_2),\end{aligned} \tag{3.3}$$

for any fixed $T \geq 0$.

Proof. Let us prove the existence of a bounded absorbing set first. Coming back to (2.36), we can now rewrite it in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + \kappa\mathcal{F} \leq (1 + \|v\|_{V'}^2 \mathcal{F} \log \mathcal{F})(Q(\|U_0\|_0)e^{-\kappa t} + Q(\|g\|_{V'})). \tag{3.4}$$

Next, recalling (2.15) and (2.16), we can take T_1 so large, only depending on $\|U_0\|_0$, that

$$Q(\|U_0\|_0)e^{-\kappa t} + Q(\|g\|_{V'}) \leq c_1, \quad \forall t \geq T_1 \tag{3.5}$$

$$\int_t^\infty \|v(s)\|_{V'}^2 \mathrm{d}s \leq c_2, \quad \forall t \geq T_1, \tag{3.6}$$

where c_1, c_2 depend on g but do not depend on U_0. Thus, setting $y := \log \mathcal{F} \geq 0$, (3.4) can be rewritten, for $t \geq T_1$, as

$$y' + \kappa \leq c_1 e^{-y} + c_1 \|v\|_{V'}^2 y. \tag{3.7}$$

Moreover, by (2.38), and possibly modifying the expression of $\mathcal{Q}$, we have

$$y(T_1) \leq \log(\mathcal{Q}(\|U_0\|_2, \|g\|_{V'}, T_1)) =: \eta, \tag{3.8}$$

where the value of η depends only on $\|U_0\|_2$ since so does T_1. Assume now that $c_1 > \kappa$ (if not, we can suitably modify its value). Setting $\zeta := \log(2c_1/\kappa) > 0$, we distinguish

$$\text{whether } y(T_1) > \zeta \quad \text{or} \quad y(T_1) \le \zeta. \tag{3.9}$$

If the first condition holds, in a right neighborhood of T_1 it is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(y + \frac{\kappa}{2}t\right) = y' + \frac{\kappa}{2} \le c_1\|v\|_{V'}^2 y \le c_1\|v\|_{V'}^2\left(y + \frac{\kappa}{2}t\right), \tag{3.10}$$

so that, recalling (3.6) and (3.8), we obtain

$$y(t) \le \left(\eta + \frac{\kappa}{2}T_1\right)e^{c_1c_2} - \frac{\kappa}{2}t, \tag{3.11}$$

which implies that for some time $\tau_1 > T_1$, still depending only on $\|U_0\|_2$, $y(\tau_1) = \zeta$.

Thus, we have essentially reduced us to the case when $y(T_1) \le \zeta$, which we now treat. Let us then set $\zeta_* := \log(c_1/\kappa) > 0$ and define $y_* := y \vee \zeta_*$. It is clear that y_* satisfies $y_*(T_1) \le \zeta$ and, for almost all $t \ge T_1$,

$$y_*' \le c_1 y_* \|v\|_{V'}^2, \tag{3.12}$$

so that, solving (3.12), noting that $y \le y_*$, and using (3.6), we have

$$y(t) \le y_*(t) \le \zeta \exp(c_1c_2), \quad \text{respectively } \forall t \ge T_1 \text{ or } \forall t \ge \tau_1. \tag{3.13}$$

Thus, we can conclude the proof of the "dissipative" part of Theorem 3.1 by taking $T_B := \max\{T_1, \tau_1\}$ (where the choice of T_1 or τ_1 corresponds now to the radius of the chosen bounded set $B \subset \mathcal{V}_2$). Note that the explicit value of the radius R_0 of the absorbing ball, i.e., the right-hand side of (3.13), could be explicitly computed by referring to (2.27) and (2.30).

Next, let us show (3.1). With this aim, let us first rewrite (2.7) in the form

$$u_{tt} + u_t + A^2u = f'(u)\Delta u + f''(u)|\nabla u|^2 + g =: G + g, \tag{3.14}$$

which makes sense thanks to (2.12) and (2.4)–(2.6). Actually, one can also easily prove that

$$G \in L^2(0, T; V) \tag{3.15}$$

(much more is true, in fact, but the above is sufficient for what follows). Then, given some $T > 0$, we take sequences such that

$$\{G_n\} \subset C^0([0, T]; D(A)), \quad G_n \to G \quad \text{strongly in } L^2(0, T; V), \tag{3.16}$$

$$\{(u_{0,n}, u_{1,n})\} \subset \mathcal{V}_3, \quad (u_{0,n}, u_{1,n}) \to (u_0, u_1) \quad \text{strongly in } \mathcal{V}_2, \tag{3.17}$$

$$\{g_n\} \subset V, \quad g_n \to g \quad \text{strongly in } V', \tag{3.18}$$

and, for all $n \in \mathbb{N}$, we consider the solution $U_n = (u_n, u_{n,t})$ to

$$u_{n,tt} + u_{n,t} + A^2 u_n = G_n + g_n, \tag{3.19}$$

coupled with the new initial datum $U_{0,n} = (u_{0,n}, u_{1,n})$. By the linear theory, this satisfies

$$u_n \in C^2([0, T]; H) \cap C^1([0, T]; D(A)) \cap C^0([0, T]; D(A^2)), \tag{3.20}$$

so it is suitable for the a priori estimates we need. Writing (3.19) for the couple of indexes n, m, taking the difference, temporarily setting $u := u_n - u_m$, testing by Au_t, and integrating over $(0, t)$ for $t \leq T$, we readily get

$$\begin{aligned} &\|U(t)\|_2^2 - 2\langle g_n - g_m, Au(t)\rangle + \int_0^t \|\nabla u_t\|^2 \\ &\quad \leq \|U_{0,n} - U_{0,m}\|_2^2 - 2\langle g_n - g_m, A(u_{0,n} - u_{0,m})\rangle + \int_0^t \|\nabla(G_n - G_m)\|^2. \end{aligned} \tag{3.21}$$

Thus, taking the supremum with respect to $t \in [0, T]$ we deduce that $\{u_n\}$ is a Cauchy sequence with respect to the latter two norms in (3.1). Since the convergence of u_{tt} can be proved by a comparison of terms in (3.19), this entails that u fulfills (3.1).

To conclude, let us examine the continuity properties of $S(t)$. First of all, (3.2) can be shown by using uniform boundedness in the norms indicated there, weak compactness, and lower semicontinuity of norms with respect to weak star convergence. Note that we do not need to extract subsequences since we have uniqueness of the limit.

The proof of (3.3) is equally simple but more technical. For this reason, we proceed by deriving formal estimates and just give the highlights of the procedure which could be used to make them rigorous. Thus, let us write (2.7) for u_n, u_m (u_n, u_m being now as in the statement), take the difference, and differentiate the resulting equation with respect to time. Testing by $A^{-1}u_{tt}$, where $u := u_n - u_m$, one then formally infers

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|U_t\|_0^2 + \|u_{tt}\|_{V'}^2 \leq \|f'(u_n)u_{n,t} - f'(u_m)u_{m,t}\|_V^2 \\ &\quad \leq 2\|f''(u_n)\nabla u_n u_{n,t} - f''(u_m)\nabla u_m u_{m,t}\|^2 + 2\|f'(u_n)\nabla u_{n,t} - f'(u_m)\nabla u_{m,t}\|^2, \end{aligned} \tag{3.22}$$

whence performing standard calculations (i.e., adding and subtracting some terms, exploiting uniform boundedness in the norms specified in (2.12), and using the properties of f as well as suitable Sobolev's embeddings), one can transform (3.22) into

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U_t\|_0^2 + \|u_{tt}\|_{V'}^2 \leq C_0(\|u_t\|_V^2 + \|u\|_{D(A)}^2), \tag{3.23}$$

where C_0 depends on the norms in (2.12) of u_n and u_m.

Next, let us write again (2.7) for u_n, u_m, take the difference, and test it by u. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \|u\|_{D(A)}^2 \leq c(\|u_{tt}\|_{V'}^2 + \|f(u_n) - f(u_m)\|^2) \leq C(\|u_{tt}\|_{V'}^2 + \|u\|^2), \tag{3.24}$$

whence, multiplying (3.24) by $2C_0$, and adding the result to (3.23), an application of Gronwall's lemma leads to the strong convergences of $u_{n,tt}$ to u_{tt} in $C^0([0,T];V')$ and of $u_{n,t}$ to u_t in $C^0([0,T];V)$. Note, indeed, that (3.3) and a comparison in (2.7) guarantee that $u_{n,tt}(0) \to u_{tt}(0)$ strongly in V'. Finally, the strong convergence of u_n to u in $C^0([0,T];D(A^{3/2}))$ can be proved by a further comparison of terms in the (difference) of (2.7).

Of course, the above procedure is not fully rigorous since the test function $A^{-1}u_{tt}$ is not admissible for the time derivative of (2.7). To overcome this problem, one could argue as in the proof of (3.1). Namely, setting

$$G_{n,m} := f(u_n) - f(u_m), \tag{3.25}$$

for fixed m and n, one notes that

$$G_{n,m} \in C^1([0,T];V) \cap C^0([0,T];D(A)) \tag{3.26}$$

and can approximate $G_{n,m}$ by a sequence $\{G^k_{n,m}\} \subset C^1([0,T];D(A^{3/2}))$ such that

$$G^k_{n,m} \xrightarrow{k} G_{n,m} \quad \text{strongly in } C^1([0,T];V) \cap C^0([0,T];D(A)). \tag{3.27}$$

Then, suitably approximating also the initial and source data (cf. (3.17)–(3.18)), and noting that the k-solution u_k is sufficiently regular, one can perform the estimates described above working on u_k and then take the limit with respect to k. The details are left to the reader. The proof is complete. □

The next theorem, whose proof relies on the so-called "energy method" (cf. [3, Sec. 2] for the theoretical background and a comparison with the second order case, see also [37]), states the asymptotic compactness in $\mathscr{V}_2$ of $\mathscr{S}$.

Theorem 3.2. *Let the assumptions of Theorem* 2.2 *hold. Then, the semiflow* $\mathscr{S}$ *associated to* (P) *is* asymptotically compact. *Namely, for any* $\mathscr{V}_2$*-bounded sequence* $\{(u_{0,n}, u_{1,n})\}$ *of initial data and any positively diverging sequence* $\{t_n\}$ *of times, there exists* $(\chi, \chi_1) \in \mathscr{V}_2$ *such that a subsequence of* $\{(u_n(t_n), u_{n,t}(t_n))\}$ *tends to* (χ, χ_1) strongly *in* $\mathscr{V}_2$ *(*u_n *is here the solution having* $(u_{0,n}, u_{1,n})$ *as initial datum).*

Proof. Let us first notice that, as a consequence of the first part of the previous proof (cf., in particular, (3.21)), the solution u to (P) satisfies for all s, t the equality

$$\frac{1}{2}\|U(t)\|_2^2 - \langle g, Au(t)\rangle + \int_s^t \|\nabla u_t\|^2 = \frac{1}{2}\|U(s)\|_2^2 - \langle g, Au(s)\rangle + \int_s^t \langle Au_t, G\rangle, \tag{3.28}$$

where G has been defined in (3.14) and $U = (u, u_t)$. Next, as we substitute G with its expression, we claim that, for all $s, t \in [0,T]$,

$$\begin{aligned}\int_s^t \langle Au_t, -f'(u)\Delta u\rangle = \frac{1}{2}\int_\Omega f'(u(t))|\Delta u(t)|^2 - \frac{1}{2}\int_\Omega f'(u(s))|\Delta u(s)|^2 \\ - \frac{1}{2}\int_s^t\int_\Omega f''(u)u_t|\Delta u|^2.\end{aligned} \tag{3.29}$$

To prove this, let us proceed once more by regularization. Actually, (3.29) surely holds for a more regular u_n. Assuming that

$$u_n \to u \quad \text{strongly in } C^1([0,T];V) \cap C^0([0,T];D(A^{3/2})), \tag{3.30}$$

we can write (3.29) for u_n and take the limit. This is straightforward as far as the right-hand side is concerned. On the other hand, the integrand on the left-hand side can be rewritten as

$$-(\nabla u_{n,t}, \nabla(f'(u_n)\Delta u_n)), \tag{3.31}$$

and it is easy to prove, using (3.30), that both terms in the scalar product converge to the expected limits in the strong topology of $C^0([0,T];H)$. Since

$$-(\nabla u_t, \nabla(f'(u)\Delta u)) = \langle Au_t, -f'(u)\Delta u\rangle \tag{3.32}$$

by definition of A, this concludes the proof of (3.29).

Consequently, u turns out to satisfy the following higher order energy equality

$$\begin{aligned}
&\frac{1}{2}\|U(t)\|_2^2 - \langle g, Au(t)\rangle + \int_s^t \|\nabla u_t\|^2 + \frac{1}{2}\int_\Omega f'(u(t))|\Delta u(t)|^2 \\
&\quad = \frac{1}{2}\|U(s)\|_2^2 - \langle g, Au(s)\rangle + \frac{1}{2}\int_\Omega f'(u(s))|\Delta u(s)|^2 \\
&\qquad + \frac{1}{2}\int_s^t\int_\Omega f''(u)u_t|\Delta u|^2 + \int_s^t \langle Au_t, f''(u)|\nabla u|^2\rangle.
\end{aligned} \tag{3.33}$$

This is the starting point to implement the so-called "energy method" introduced by Ball (cf. [3, Sec. 4], see also [37]) to prove asymptotic compactness, which is our next task.

To start with, let us define the functional

$$\mathcal{G}_0(t) := \frac{1}{2}\|U(t)\|_2^2 - \langle g, Au(t)\rangle + \frac{1}{2}\int_\Omega f'(u(t))|\Delta u(t)|^2. \tag{3.34}$$

Actually, at least if no danger of confusion occurs, we shall write indifferently $\mathcal{G}_0$, $\mathcal{G}_0(t)$ or $\mathcal{G}_0(u(t))$ in the sequel, with some abuse of language since in fact $\mathcal{G}_0$ depends both on u and on u_t. We shall use the same convention also for the other functionals defined below.

Then, writing (3.33) for $t = s + h$, dividing by h, and letting $h \to 0$, it is immediate to deduce that $\mathcal{G}_0$ is absolutely continuous on $[0,T]$ and there holds

$$\frac{d}{dt}\mathcal{G}_0 + \|\nabla u_t\|^2 = \frac{1}{2}\int_\Omega f''(u)u_t|\Delta u|^2 + \langle Au_t, f''(u)|\nabla u|^2\rangle, \quad \text{a.e. in } (0,T). \tag{3.35}$$

Next, let us test (2.7) by Au. The same procedure used before permits to justify the validity, a.e. in $(0,T)$, of the equality

$$\begin{aligned}
&\frac{d}{dt}\left((u_t, Au) + \frac{1}{2}\|\nabla u\|^2\right) - \|\nabla u_t\|^2 + \|u\|_{D(A^{3/2})}^2 \\
&\quad + \int_\Omega f'(u)|\Delta u|^2 - \langle g, Au\rangle = -\int_\Omega f''(u)|\nabla u|^2\Delta u.
\end{aligned} \tag{3.36}$$

Then, let us multiply (3.36) by 1/2 and sum the result to (3.35). We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{G}_0 + \frac{1}{2}(u_t, Au) + \frac{1}{4}\|\nabla u\|^2\right) + \frac{1}{2}\|U\|_2^2 + \frac{1}{2}\int_\Omega f'(u)|\Delta u|^2 - \frac{1}{2}\langle g, Au\rangle = \mathcal{H}_0, \tag{3.37}$$

where we have set

$$\mathcal{H}_0 := \frac{1}{2}\int_\Omega f''(u)u_t|\Delta u|^2 + \langle Au_t, f''(u)|\nabla u|^2\rangle - \frac{1}{2}\int_\Omega f''(u)|\nabla u|^2\Delta u. \tag{3.38}$$

Consequently, adding some terms to both hands sides of (3.37) we obtain the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G} + \mathcal{G} = \mathcal{H}, \quad \text{a.e. in } (0, T), \tag{3.39}$$

where we have set

$$\mathcal{G} := \mathcal{G}_0 + \tfrac{1}{2}(u_t, Au) + \tfrac{1}{4}\|\nabla u\|^2, \tag{3.40}$$

$$\mathcal{H} := \mathcal{H}_0 - \tfrac{1}{2}\langle g, Au\rangle + \tfrac{1}{2}(u_t, Au) + \tfrac{1}{4}\|\nabla u\|^2. \tag{3.41}$$

Thus, from (3.39) and for any $\tau, M \geq 0$, we obtain

$$\mathcal{G}(\tau + M) = \mathcal{G}(\tau)e^{-M} + \int_\tau^{\tau+M} e^{s-\tau-M}\mathcal{H}(s)\mathrm{d}s. \tag{3.42}$$

At this point, recalling the notation in the statement, let us set $v_n(t) := u_n(t_n + t - M - \tau)$ (so that, in particular, $v_n(\tau) = u_n(t_n - M)$ and $v_n(\tau + M) = u_n(t_n)$). Since $(u_{0,n}, u_{1,n})$ is bounded in $\mathcal{V}_2$, by uniform dissipativity it follows that u_n is uniformly bounded in the norms (2.12) (in a way which does not depend on T) by some constant C. The same, of course, holds also for v_n and for the values of the functional $\mathcal{G}$. Thus, at least a (nonrelabelled) subsequence of v_n tends to a solution v weakly star in the norm specified in (2.12) and for all $T > 0$. More precisely, we have

$$v_n \to v \quad \text{in } C_w^1([0, T]; V) \cap C_w^0([0, T]; D(A^{3/2})), \tag{3.43}$$

so that, in particular, there exist the limits

$$(\chi, \chi_1) := \lim_{n\to\infty}(v_n(\tau + M), v_{n,t}(\tau + M)) = \lim_{n\to\infty}(u_n(t_n), u_{n,t}(t_n)), \tag{3.44}$$

$$(\chi_{-M}, \chi_{1,-M}) := \lim_{n\to\infty}(v_n(\tau), v_{n,t}(\tau)) = \lim_{n\to\infty}(u_n(t_n - M), u_{n,t}(t_n - M)), \tag{3.45}$$

which have to be intended, at least in the meanwhile, in the weak topology of $\mathcal{V}_2$. Moreover, it is $(\chi, \chi_1) = (v(\tau + M), v_t(\tau + M))$ and $(\chi_{-M}, \chi_{1,-M}) = (v(\tau), v_t(\tau))$. At this point, writing (3.42) for v_n gives

$$\begin{aligned}\mathcal{G}(v_n(\tau + M)) - \mathcal{G}(v_n(\tau))e^{-M} &= \mathcal{G}(u_n(t_n)) - \mathcal{G}(u_n(t_n - M))e^{-M} \\ &= \int_\tau^{\tau+M} e^{s-\tau-M}\mathcal{H}(v_n(s))\mathrm{d}s.\end{aligned} \tag{3.46}$$

It is now a standard procedure to check that, as far as τ, M are fixed, (3.43) and suitable Sobolev's embeddings give

$$\int_\tau^{\tau+M} e^{s-\tau-M}\mathscr{H}(v_n(s))\mathrm{d}s \to \int_\tau^{\tau+M} e^{s-\tau-M}\mathscr{H}(v(s))\mathrm{d}s. \tag{3.47}$$

Thus, taking the supremum limit of (3.46) as (a proper subsequence of) n tends to ∞, we get

$$\begin{aligned}\limsup_{n\nearrow\infty}\mathscr{G}(u_n(t_n)) &\le Ce^{-M} + \limsup_{n\nearrow\infty}\int_\tau^{\tau+M} e^{s-\tau-M}\mathscr{H}(v_n(s))\mathrm{d}s\\ &= Ce^{-M} + \int_\tau^{\tau+M} e^{s-\tau-M}\mathscr{H}(v(s))\mathrm{d}s\\ &= Ce^{-M} + \mathscr{G}(v(\tau+M)) - \mathscr{G}(v(\tau))e^{-M}\\ &\le Ce^{-M} + \mathscr{G}(\chi),\end{aligned} \tag{3.48}$$

where in deducing the third equality we used that (3.42) is satisfied also by the limit solution v. Since the above holds for all $M > 0$ and with C independent of M, letting $M\nearrow\infty$ and using the immediate fact that $\mathscr{G}$ is sequentially weakly lower semicontinuous in $\mathscr{V}_2$, we eventually obtain that

$$\mathscr{G}(u_n(t_n)) \to \mathscr{G}(\chi). \tag{3.49}$$

Thus, since it is clear that just the weak convergence (3.44) entails

$$\begin{aligned}&-\langle g, Au_n(t_n)\rangle + \frac{1}{2}\int_\Omega f'(u_n(t_n))|\Delta u_n(t_n)|^2 + \frac{1}{2}(u_{n,t}(t_n), Au_n(t_n)) + \frac{1}{4}\|\nabla u_n(t_n)\|^2\\ &\quad\to -\langle g, A\chi\rangle + \frac{1}{2}\int_\Omega f'(\chi)|\Delta\chi|^2 + \frac{1}{2}(\chi_1, A\chi) + \frac{1}{4}\|\nabla\chi\|^2,\end{aligned} \tag{3.50}$$

comparing (3.49) with (3.50) we obtain

$$\|(u_n(t_n), u_{n,t}(t_n))\|_2 \to \|(\chi,\chi_1)\|_2. \tag{3.51}$$

This relation, together with the weak convergence (3.44), gives the desired strong convergence and concludes the proof of asymptotic compactness and of Theorem 3.2 as well. □

On account of Theorems 3.1 and 3.2, we thus deduce (see, e.g., [2, Thm. 3.3]).

Theorem 3.3. *Let the assumptions of Theorem* 2.2 *hold. Then, the semiflow* $\mathscr{S}$ *possesses the global attractor* $\mathscr{A}_2$.

4. Smoothness of the Global Attractor $\mathscr{A}_2$

We prove here a regularity property of the attractor constructed in the previous section. This fact will be used for constructing an exponential attractor in Section 5. A straightforward consequence of the results of this section is the existence of the global attractor which was already obtained in Theorem 3.2 by a different technique. We decided to keep the latter proof because it is simple and (hopefully) interesting in itself. On the contrary, the proof of Theorem 4.1, which relies on a new decomposition method partly related to that in [42], involves a number of technical complications. We also point out that, in Section 6, we will appeal to the same technique used for proving Theorem 3.2 to establish the existence of the global attractor for weak solution (see Theorem 6.4).

Theorem 4.1. *Let the assumptions of Theorem* 2.2 *hold. Additionally, let*

$$g \in H. \tag{4.1}$$

Then, the global attractor $\mathcal{A}_2$ for the semiflow $\mathcal{S}$ is bounded in $\mathcal{V}_3$.

Remark 4.2. On account of well-known results, we can infer that $\mathcal{A}_2$ consists of those points of $\mathcal{V}_3$ from which bounded complete trajectories originate. These are strong solutions to Problem (P).

Proof of Theorem 4.1. The proof is divided into several steps which are presented as separate lemmas. We start with a simple property whose proof is more or less straightforward.

Lemma 4.3. *Let $S : [0, +\infty)^2 \to [0, \infty)$, $S = S(t, R)$, be a continuous function such that*

(i) *There exists $R_0 \in [0, \infty)$ such that for all $R \in [0, \infty)$ there exists $T_R \in [0, \infty)$ such that $S(t, R) \leq R_0$ for all $t \geq T_R$;*
(ii) *$S(0, R) = R$ for all $R \in [0, \infty)$;*
(iii) *$R \mapsto S(t, R)$ is increasingly monotone for all $t \in [0, \infty)$.*

Then, there exists an increasingly monotone function $Q : [0, \infty) \to [0, \infty)$, $Q = Q(R)$, such that $S(t, R) \leq Q(R)e^{-t} + R_0$ for all $(t, R) \in [0, \infty)^2$. □

Actually, noting as $S(t, R)$ the $\mathcal{V}_2$-radius of $S(t)B(0, R)$ ($B(0, R)$ being the R-ball in $\mathcal{V}_2$), it is clear that S verifies the properties (i)–(iii). Thus, using Lemma 4.3, the dissipativity property of Theorem 3.1 is rewritten as

$$\|S(t)U_0\|_2 \leq Q(\|U_0\|_2)e^{-t} + R_0, \quad \text{where } R_0 = Q(\|g\|_{V'}). \tag{4.2}$$

Next lemma states that $\mathcal{V}_2$-solutions to (P) satisfy a dissipation property similar to (2.16), but with respect to a stronger norm.

Lemma 4.4. *Let the assumptions of* Theorem 4.1 *hold. Then, for any $\mathcal{V}_2$-solution $U = (u, u_t)$ to* (P) *there holds*

$$\int_0^\infty \|u_t(s)\|_V^2 \mathrm{d}s + \sup_{s\in[0,\infty)} \|u_{tt}(s)\|_{V'}^2 \leq Q(\|U_0\|_2) < \infty. \tag{4.3}$$

Proof. Let us differentiate (2.7) with respect to time and set $\theta := u_t$ and $\Theta := (\theta, \theta_t)$. We get, for $L > 0$ to be chosen later,

$$A^{-1}(\theta_{tt} + \theta_t) + A\theta + LA^{-1}\theta + f'(u)\theta = h := LA^{-1}u_t. \tag{4.4}$$

We now test (4.4) by $2\theta_t$. Simple computations lead to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[\|\Theta\|_0^2 + L\|\theta\|_{V'}^2 + (f'(u)\theta, \theta)] + \|\theta_t\|_{V'}^2 \\
&\quad \le \|h\|_V^2 + (f''(u)\theta^2, \theta) \le \|h\|_V^2 + C\|\theta\|_{L^4(\Omega)}^2\|\theta\| \\
&\quad \le \|h\|_V^2 + C\|\theta\|^2\|\theta\|_V \le \|h\|_V^2 + C\|\theta\|^2 \\
&\quad \le \|h\|_V^2 + C\|\theta\|_V\|\theta\|_{V'} \le \|h\|_V^2 + \frac{1}{4}\|\theta\|_V^2 + C\|\theta\|_{V'}^2, \qquad (4.5)
\end{aligned}$$

where we have used the duality pairing V'-V, Young's inequality, and, in the last passage, the uniform $\mathcal{V}_2$-boundedness of U and interpolation. Note that, here and in the rest of this Section, the constants C are allowed to depend (actually, at most in a polynomial way since f grows polynomially) on the $\mathcal{V}_2$-norm of the solution, which is uniformly bounded in time by (4.2). In fact, C is a quantity having the same expression as the right-hand side of (4.2), but we do *not* allow C to depend on L.

Next, we test (4.4) by $\theta/2$, inferring

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}(\theta_t, A^{-1}\theta) + \frac{1}{4}\|\theta\|_{V'}^2\right] - \frac{1}{2}\|\theta_t\|_{V'}^2 + \frac{1}{2}\|\theta\|_V^2 + \frac{L}{2}\|\theta\|_{V'}^2 + \frac{1}{2}(f'(u)\theta, \theta) \\
&\quad = \frac{1}{2}(h, \theta) \le \frac{1}{8}\|\theta\|_V^2 + c\|h\|_V^2. \qquad (4.6)
\end{aligned}$$

Summing (4.5) and (4.6), we then get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\Theta\|_0^2 + \left(L + \frac{1}{4}\right)\|\theta\|_{V'}^2 + (f'(u)\theta, \theta) + \frac{1}{2}(\theta_t, A^{-1}\theta)\right] \\
&\quad + \frac{1}{2}\|\theta_t\|_{V'}^2 + \frac{1}{8}\|\theta\|_V^2 + \frac{L - 2C}{2}\|\theta\|_{V'}^2 + \frac{1}{2}(f'(u)\theta, \theta) \le c\|h\|_V^2, \qquad (4.7)
\end{aligned}$$

still for C independent of L. Thus, noting by $\mathcal{Y}$ the quantity in square brackets, we notice that we can choose L so large (depending on C and λ in (2.6)) to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Y} + \kappa\mathcal{Y} \le c\|h\|_V^2, \qquad (4.8)$$

for some $\kappa > 0$. Hence, recalling that $h = LA^{-1}u_t$ and using (2.16) and the fact that $\mathcal{Y}(0) = Q(\|U_0\|_2)$, (4.3) follows immediately. □

We are now ready to decompose the solution u to (P) as the sum of a "compact" part

$$A^{-1}(v_{tt} + v_t) + Av + LA^{-1}v + f(v) = LA^{-1}u + g, \quad V|_{t=0} = 0, \qquad (4.9)$$

where $V := (v, v_t)$, and a "decaying" part

$$A^{-1}(w_{tt} + w_t) + Aw + LA^{-1}w + f(u) - f(v) = 0, \quad W|_{t=0} = U_0 := (u_0, u_1), \qquad (4.10)$$

where $W := (w, w_t)$ and U_0 belongs to a bounded absorbing set in $\mathcal{V}_2$ (cf. (4.2)). Note that the value of L in (4.9)–(4.10) will possibly differ from that in (4.4).

Lemma 4.5. *Let the assumptions of Theorem* 4.1 *hold. Then, L can be chosen so large that*

$$\|W(t)\|_0 \le Q(\|U_0\|_2)e^{-\kappa t}. \tag{4.11}$$

Proof. We proceed along the lines of the preceding proof. First, we test (4.10) by $2w_t$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\|W\|_0^2 + L\|w\|_{V'}^2 + 2\mathcal{I}_1\big] + \|w_t\|_{V'}^2 \le 2\mathcal{I}_2, \tag{4.12}$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ collect the terms coming from f. Namely, we have (recall that $U = V + W$)

$$\mathcal{I}_1 = (F(u-w) - F(u) + f(u)w, 1) \ge -\frac{\lambda}{2}\|w\|^2 \tag{4.13}$$

thanks to (2.5). To estimate $\mathcal{I}_2$, let us first notice that, performing the standard energy estimate (cf. (2.13)–(2.15)) on (4.9) (i.e., testing it by $v_t + \delta v$ for small $\delta > 0$) and using the energy estimate (2.15) for u to control the term on the right-hand side, we derive

$$\|V(t)\|_0^2 \le Q_L(\|U_0\|_0)e^{-\kappa t} + C_L, \tag{4.14}$$

where both C_L and Q_L depend on L since so does the right-hand side of (4.9). Comparing (2.15) and (4.14), we also get

$$\|W(t)\|_0^2 \le Q_L(\|U_0\|_0)e^{-\kappa t} + C_L. \tag{4.15}$$

Using now the uniform $\mathcal{V}_2$-bound on U, (2.6), (4.15), and standard interpolation and embeddings, we can estimate

$$\begin{aligned}\mathcal{I}_2 &= \big(f(u-w) - f(u) + f'(u)w, u_t\big) \le c\int_\Omega \big(1 + |u| + |w|\big)|w|^2|u_t| \\ &\le C_L\|u_t\|_V\|w\|_V^2 \le \frac{1}{16}\|w\|_V^2 + C_L\|u_t\|_V^2\|w\|_V^2,\end{aligned} \tag{4.16}$$

where the dependence on L of the constant C_L comes from (4.15).

Next, we test (4.10) by $w/2$, inferring

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\big(w_t, A^{-1}w\big) + \frac{1}{4}\|w\|_{V'}^2\right] - \frac{1}{2}\|w_t\|_{V'}^2 + \frac{1}{2}\|w\|_V^2 + \frac{L}{2}\|w\|_{V'}^2 \\ &\quad + \frac{1}{2}\big(f(u) - f(u-w), w\big) = 0.\end{aligned} \tag{4.17}$$

Now, using (2.5) and interpolation, it is not difficult to compute

$$\begin{aligned}\big(f(u) - f(u-w), w\big) = \big(f(u) - f(u-w), w\big) + \big(F(u-w) - F(u), 1\big) \\ - \big(F(u-w) - F(u), 1\big)\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{I}_1 + \big(F(u) - F(u-w) - f(u-w)w, 1\big) \\
&\geq \mathcal{I}_1 - \frac{\lambda}{2}\|w\|^2 \geq \mathcal{I}_1 - \frac{1}{4}\|w\|_V^2 - C\|w\|_{V'}^2,
\end{aligned} \tag{4.18}
$$

for some (new) $C > 0$. Thus, summing (4.12) and (4.17) we arrive at

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bigg[\|W\|_0^2 + \Big(L + \frac{1}{4}\Big)\|w\|_{V'}^2 + 2\mathcal{I}_1 + \frac{1}{2}\big(w_t, A^{-1}w\big)\bigg] \\
&\quad + \frac{1}{2}\|w_t\|_{V'}^2 + \frac{1}{2}\|w\|_V^2 + \frac{L-C}{2}\|w\|_{V'}^2 + \frac{1}{2}\mathcal{I}_1 \\
&\leq \frac{1}{4}\|w\|_V^2 + C_L\|u_t\|_V^2\|w\|_V^2.
\end{aligned} \tag{4.19}
$$

Finally, choosing L so large that

$$
L \geq 2C \quad \text{and} \quad \frac{L}{2}\|w\|_{V'}^2 + \mathcal{I}_1 + \frac{1}{4}\|w\|_V^2 \geq 0, \tag{4.20}
$$

rewriting (4.19) (with obvious notation) as

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Y} + \kappa\mathcal{Y} \leq m\mathcal{Y}, \tag{4.21}
$$

where

$$
m := C_L\|u_t\|_V^2 \in L^1(0,\infty) \tag{4.22}
$$

thanks to (4.3), the comparison principle for ODEs readily gives (4.11). □

Note now that, comparing (2.16) and (4.11), there follows in particular

$$
\int_t^\infty \|v_t(s)\|_{V'}^2 \mathrm{d}s \leq Q(\|U_0\|_2)e^{-\kappa t} + Q(\|g\|_{V'}). \tag{4.23}
$$

Thus, we can apply to (4.9) the procedure used in Theorem 3.1 to prove $\mathcal{V}_2$-dissipativity. Of course, the "source" term $LA^{-1}u_t$ in the right-hand side of the differentiated equation is easily controlled thanks to (2.16). Using also Lemma 4.3 we then get the estimate

$$
\|V(t)\|_2^2 + \|v_{tt}(t)\|_{V'}^2 \leq Q(\|U_0\|_2)e^{-t} + Q(\|g\|_{V'}). \tag{4.24}
$$

As a next step, we prove that the component V of the solution is compact in $\mathcal{V}_2$ and, more precisely, bounded in $\mathcal{V}_3$. From this point on, the further regularity (4.1) is needed.

Lemma 4.6. *Let the assumptions of Theorem* 4.1 *hold. Then we have*

$$
\|V(t)\|_3 \leq Q(\|U_0\|_2)e^{-\kappa t} + Q(\|g\|). \tag{4.25}
$$

Proof. We differentiate (4.9) in time and test the result by $A(v_{tt} + \delta v_t)$ for small $\delta > 0$. We do not give all the details, but just see how the nonlinear terms are controlled. Actually, performing some calculation and using (4.24) and interpolation, we get

$$\begin{aligned}\big(f'(v)v_t, Av_{tt}\big) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f'(v)|\nabla v_t|^2 - \frac{1}{2}\int_\Omega f''(v)v_t|\nabla v_t|^2 - \big(\mathrm{div}(f''(v)v_t\nabla v), v_{tt}\big)\\ &\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f'(v)|\nabla v_t|^2 - \frac{\delta}{4}\|v_t\|^2_{D(A)} - \frac{1}{4}\|v_{tt}\|^2 - C_\delta\|v_t\|^2_V, \end{aligned}\tag{4.26}$$

and, analogously,

$$\delta\big(f'(v)v_t, Av_t\big) \geq \delta\int_\Omega f'(v)|\nabla v_t|^2 - C\|v_t\|^2_V, \tag{4.27}$$

where in both formulas C (or C_δ) is a monotone function of $\|V(t)\|_2$ (and, more precisely, it depends at most polynomially on it). Thus, noting that the right-hand side term $L(u_t, v_{tt} + \delta v_t)$ can be estimated in a standard way, one arrives at an expression of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Y}_3 + \kappa\mathcal{Y}_3 \leq Q(\|V(t)\|_2) + Q(\|U(t)\|_2) \leq Q(\|U_0\|_2)e^{-\kappa t} + Q(\|g\|), \tag{4.28}$$

where (4.2) and (4.24) have been used in deducing the latter inequality, and the functional $\mathcal{Y}_3$ (upon possibly taking a larger L) satisfies

$$c_L\|V_t\|_1^2 \leq \mathcal{Y}_3 \leq C_L\|V_t\|_1^2, \tag{4.29}$$

where only C_L depends on the radius of the absorbing set. Noting now that, by standard elliptic regularity results applied to (4.9), we have

$$\|V(t)\|_3^2 \leq C\big(\|V_t(t)\|_1^2 + \|V(t)\|_2^2\big), \tag{4.30}$$

relation (4.25) comes then as an easy consequence of (4.28). The lemma is proved. □

Finally, we show that W is exponentially decaying in $\mathcal{V}_2$. Of course, this fact, together with (4.25), will give the desired property of the decomposition (4.9)–(4.10) and conclude the proof of Theorem 4.1.

Lemma 4.7. *Let the assumptions of Theorem* 4.1 *hold. Then we have*

$$\|W(t)\|_2 \leq Q(\|U_0\|_2)e^{-\kappa t}. \tag{4.31}$$

Proof. We differentiate (4.10) in time and test the result by $w_{tt} + \delta w_t$ for small $\delta > 0$. Still, the procedure is standard, but for the estimation of the nonlinear terms depending on f. Namely, we obtain on the left-hand side

$$\big((f(u) - f(u-w))_t, w_{tt} + \delta w_t\big). \tag{4.32}$$

Thus, defining

$$l = l(u, w) := \int_0^1 f'(su + (1-s)(u-w))\mathrm{d}s \geq -\lambda \tag{4.33}$$

so that $f(u) - f(u-w) = lw$, we clearly have

$$\big((f(u) - f(u-w))_t, w_{tt} + \delta w_t\big) = \big(l_t w + l w_t, w_{tt} + \delta w_t\big) \tag{4.34}$$

and

$$\big|\big(l_t w, w_{tt} + \delta w_t\big)\big| \le \|l_t w\|_V \|w_{tt} + \delta w_t\|_{V'} \le c\big(\|l_t\|_V \|w\|_{D(A)}\big)\|w_{tt} + \delta w_t\|_{V'}. \tag{4.35}$$

Now, let us notice that, by (4.24) and the analogue for U coming from Theorem 3.1 and Lemma 4.3,

$$\|W(t)\|_2^2 \le Q(\|U_0\|_2)e^{-t} + Q(\|g\|_{V'}). \tag{4.36}$$

In particular, $\|l_t\|_V \le C$, with C possibly depending on U_0, but independent of time. More precisely, using (4.11) and interpolation, we get, for all $\nu > 0$,

$$\|W(t)\|_{2-\nu}^2 \le Q(\|U_0\|_2)e^{-\kappa t}, \tag{4.37}$$

κ depending here on ν. Consequently (take $\nu = 1$), we can control the right-hand side of (4.35) so that

$$\big|\big(l_t w, w_{tt} + \delta w_t\big)\big| \le Q(\|U_0\|_2)e^{-\kappa t} + \frac{1}{4}\|w_{tt} + \delta w_t\|_{V'}^2 \tag{4.38}$$

and the latter term can be moved to the left-hand side and estimated directly. Finally, coming back to the remaining term in (4.34), we get

$$\big(l w_t, w_{tt} + \delta w_t\big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(l, w_t^2\big) + \delta\big(l, w_t^2\big) - \frac{1}{2}\big(l_t, w_t^2\big), \tag{4.39}$$

and the first two summands on the right-hand side are controlled once more thanks to (2.5), while the third is estimated for small $\nu > 0$ by

$$-\frac{1}{2}\big(l_t, w_t^2\big) \le c\|l_t\|_V \|w_t\|^2_{D\left(A^{\frac{1-\nu}{2}}\right)} \le C\|w_t\|^2_{D\left(A^{\frac{1-\nu}{2}}\right)} \le Q(\|U_0\|_2)e^{-\kappa t}, \tag{4.40}$$

thanks to (4.37). Thus, all the nonlinear terms are either (essentially) positive, or exponentially decaying. Then, (4.31) is proved, which concludes the proof of Lemma 4.7 and of Theorem 4.1. □

5. Exponential Attractors

This section is devoted to the proof of existence of an exponential attractor for the semiflow $\mathcal{S}$ consisting of the $\mathcal{V}_2$-solutions to Problem (P). More precisely, we will prove

Theorem 5.1. *Assume* (2.4)–(2.6) *and* (4.1). *Then, the semiflow $\mathcal{S}$ admits an exponential attractor $\mathcal{M}_2$. Namely, $\mathcal{M}_2$ is a positively invariant, compact subset of $\mathcal{V}_2$*

with finite fractal dimension with respect to the $\mathcal{V}_2$-metric and bounded in $\mathcal{V}_3$, such that, for any bounded $B \subset \mathcal{V}_2$, there exist $C_B > 0$ and $\kappa_B > 0$ such that

$$\operatorname{dist}_2(S(t)B, \mathcal{M}_2) \leq C_B e^{-\kappa_B t}, \tag{5.1}$$

where dist_2 *denotes the* Hausdorff semidistance *of sets with respect to the $\mathcal{V}_2$-metric.*

Before proving the theorem, we need a couple of preparatory lemmas.

Lemma 5.2. *Under the assumptions of Theorem* 5.1, *there exists a set $\mathcal{C}_3$ bounded in $\mathcal{V}_3$ which exponentially attracts any bounded set of $\mathcal{V}_2$ with respect to the $\mathcal{V}_2$-metric.*

Proof. It is a simple consequence of the decomposition made in Section 4. More in detail, it follows from relations (4.25) and (4.31). □

Lemma 5.3. *Under the assumptions of Theorem* 5.1, *there exists a set $\mathcal{B}_3$, bounded in $\mathcal{V}_3$ and positively invariant, which absorbs $\mathcal{C}_3$ and, consequently, exponentially attracts any bounded set of $\mathcal{V}_2$ with respect to the distance of $\mathcal{V}_2$.*

Proof. To prove the lemma, we basically need a dissipative estimate in $\mathcal{V}_3$. This can be obtained just by mimicking the proof of Lemma 4.6. Namely, one has to differentiate (2.7) in time and test the result by $u_{tt} + \delta u_t$ for small $\delta > 0$. This leads to an expression perfectly analogous to (4.28), with $\mathcal{Y}_3$ still satisfying (4.29), but with V everywhere replaced by U. This entails existence of a positively invariant and $\mathcal{V}_3$-bounded set $\mathcal{B}_3$, which eventually absorbs any $\mathcal{V}_3$-bounded set of data. Since this in particular happens for $\mathcal{C}_3$, the lemma is proved. □

Proof of Theorem 5.1. Let us start by considering initial data lying in the set $\mathcal{B}_3$ constructed above. Notice also that it is not restrictive to assume $\mathcal{B}_3$ to be weakly closed in $\mathcal{V}_3$. Let us then take a couple of solutions u_1, u_2 to Problem (P) whose initial data $(u_{0,1}, u_{1,1})$, $(u_{0,2}, u_{1,2})$ lie in $\mathcal{B}_3$. Since $\mathcal{B}_3$ is positively invariant, it is then clear that the functions $t \mapsto (u_i(t), u_{i,t}(t))$, for $i = 1, 2$, take values in $\mathcal{B}_3$. By the $\mathcal{V}_3$-analogue of (3.1), which can be proved in a standard way, we have that

$$u_i \in C^2([0,\infty); H) \cap C^1\big([0,\infty); D(A)\big) \cap C^0\big([0,\infty); D(A^2)\big), \tag{5.2}$$

still for $i = 1, 2$. Later on, the constants c will be allowed to depend on the choice of the initial datum in $\mathcal{B}_3$.

Let us now write Equation (2.7) for u_1 and u_2, and then take the difference. This gives

$$u_{tt} + u_t + A(Au + f(u_1) - f(u_2)) = 0, \tag{5.3}$$

where we have set $u := u_1 - u_2$. Let us then test (5.3) by $A^{-1}(u_t + \delta u)$. Setting

$$l = l(u_1, u_2) := \int_0^1 f'(\tau u_1 + (1-\tau)u_2)\mathrm{d}\tau \geq -\lambda \tag{5.4}$$

(cf. (2.5)) and writing $U := (u, u_t)$, standard manipulations lead us to the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|U\|_0^2 + \frac{\delta}{2}\|u\|_{V'}^2 + \frac{1}{2}\int_\Omega l(u_1, u_2)u^2 + \delta\langle u_t, A^{-1}u\rangle\right]$$
$$+ \delta\|u\|_V^2 + (1-\delta)\|u_t\|_{V'}^2 = -\delta\int_\Omega l(u_1, u_2)u^2 + \frac{1}{2}\int_\Omega l_t(u_1, u_2)u^2. \tag{5.5}$$

Then, adding the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L\|u\|_{V'}^2\right] \le L\big(\|u\|^2 + \|u_t\|_{D(A^{-1})}^2\big), \tag{5.6}$$

for L large enough, using (2.5) (this also permits to control the first term on the right-hand side of (5.5)), and noting that, by standard use of embeddings and interpolation,

$$\frac{1}{2}\int_\Omega l_t(u_1, u_2)u^2 \le c\int_\Omega \big(1 + |u_{1,t}| + |u_{2,t}|\big)u^2 \le \frac{\delta}{2}\|u\|_V^2 + c_\delta\|u\|^2, \tag{5.7}$$

relation (5.5) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{Y} + \mathcal{Z} \le c_{\delta,L}\big(\|u\|^2 + \|u_t\|_{D(A^{-1})}^2\big) = c_{\delta,L}\|U\|_{-1}^2, \tag{5.8}$$

with obvious meaning of $\mathcal{Y}$ and $\mathcal{Z}$.

Moreover, taking δ small enough and L large enough (the latter depending in particular on λ in (2.5)), it is clear that, for some c_3, c_4, κ also depending on the $\mathcal{V}_3$-radius of $\mathcal{B}_3$,

$$c_3\|U\|_0^2 \le \mathcal{Y} \le c_4\|U\|_0^2, \quad \mathcal{Z} \ge \kappa\mathcal{Y}, \tag{5.9}$$

whence, taking $\ell > 0$ and integrating (5.8) from $\tau \in [0, \ell]$ to 2ℓ, we get

$$\mathcal{Y}(2\ell) + \kappa\int_\tau^{2\ell} \mathcal{Y}(s)\mathrm{d}s \le \mathcal{Y}(\tau) + c_5\int_\tau^{2\ell} \|U(s)\|_{-1}^2\mathrm{d}s. \tag{5.10}$$

(the $c_{\delta,L}$ in (5.8) has been noted as c_5 for later convenience). Now, let us apply the following straightforward fact (see, e.g., [43, Lemma 3.2]):

Lemma 5.4. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{W}$ a Banach space such that $\mathcal{H}$ is compactly embedded into $\mathcal{W}$. Then, for any $\gamma > 0$, there exist a finite-dimensional orthonormal projector $P : \mathcal{H} \to \mathcal{H}$ and a positive constant K, both depending on γ and such that, for all $w \in \mathcal{H}$,*

$$\|w\|_{\mathcal{W}}^2 \le \gamma\|w\|_{\mathcal{H}}^2 + K\|Pw\|_{\mathcal{H}}^2. \tag{5.11}$$

□

We apply Lemma 5.4 with $\mathcal{H} = \mathcal{V}_0$ and $\mathcal{W} = \mathcal{V}_{-1}$. Thus, with the notation above, we have in particular

$$\|U\|_{-1}^2 \le \gamma\|U\|_0^2 + K\|PU\|_0^2. \tag{5.12}$$

Let us now introduce the set of ℓ-trajectories associated with $\mathcal{V}_0$-solutions of Problem (P) as

$$\mathcal{U}_\ell := \{U = (u, u_t) \in C^0([0, \ell], \mathcal{V}_0) : U \text{ solves (P) on } [0, \ell]\}. \tag{5.13}$$

The set $\mathcal{U}_\ell$ is endowed with the metric of $L^2(0, \ell; \mathcal{V}_0)$. For brevity, we will write

$$\|U\|_\ell \quad \text{in place of } \|U\|_{L^2(0,\ell;\mathcal{V}_0)}. \tag{5.14}$$

Note that, in general, $\mathcal{U}_\ell$ is not complete with respect to the chosen metric. However, thanks to (6.16) of Theorem 6.3, it is not difficult to see that, if $\{U_n\} \subset \mathcal{U}_\ell$ is a sequence such that $\{U_n(0)\}$ is *bounded* in $\mathcal{V}_0$, and U_n tends in $L^2(0, \ell; \mathcal{V}_0)$ to some function U, then still it is $U \in \mathcal{U}_\ell$. This is in fact our case since we can restrict ourselves to the subset $\mathcal{X}_\ell$ of the elements of $\mathcal{U}_\ell$ whose initial values lie in $\mathcal{B}_3$. Actually, being $\mathcal{B}_3$ *weakly* closed in $\mathcal{V}_3$, it is easy to prove that $\mathcal{X}_\ell$ is a complete metric space with respect to the $L^2(0, \ell; \mathcal{V}_0)$-metric. We can then define the shift operator

$$\mathcal{L} = \mathcal{L}_\ell : \mathcal{X}_\ell \to \mathcal{X}_\ell, \quad \mathcal{L}(U)(\cdot) := U(\ell + \cdot). \tag{5.15}$$

Integrating now (5.10) with respect to τ from 0 to ℓ and using (5.9), (5.12), we infer

$$\begin{aligned} &c_3\ell\|U(2\ell)\|_0^2 + c_3\kappa\ell\int_\ell^{2\ell}\|U(s)\|_0^2\mathrm{d}s \\ &\quad \le c_4\int_0^\ell\|U(\tau)\|_0^2\mathrm{d}\tau + c_5\gamma\ell\int_0^{2\ell}\|U(s)\|_0^2\mathrm{d}s + c_5K\ell\int_0^{2\ell}\|PU(s)\|_0^2\mathrm{d}s. \end{aligned} \tag{5.16}$$

Then, dividing (5.16) by ℓ and using the notation (5.15) we obtain

$$(c_3\kappa - c_5\gamma)\|\mathcal{L}U\|_\ell^2 \le \left(\frac{c_4}{\ell} + c_5\gamma\right)\|U\|_\ell^2 + c_5K\big(\|PU\|_\ell^2 + \|P\mathcal{L}U\|_\ell^2\big). \tag{5.17}$$

Now, let us choose in turn γ and ℓ such that

$$c_5\gamma \le \frac{c_3\kappa}{17}, \quad \frac{c_4}{\ell} \le \frac{c_3\kappa}{17}. \tag{5.18}$$

Thus, for some K' depending on all other constants, (5.17) gives, for all $u \in \mathcal{X}_\ell$,

$$\|\mathcal{L}U\|_\ell^2 \le \frac{1}{8}\|U\|_\ell^2 + K'\big(\|PU\|_\ell^2 + \|P\mathcal{L}U\|_\ell^2\big). \tag{5.19}$$

Consequently, the semiflow $\mathcal{S}$ associated with Problem (P) enjoys the *generalized squeezing property* introduced in [43, Def. 3.1] on the set $\mathcal{B}_3$. Recalling [44, Lemma 2.2], we infer that the discrete dynamical system on $\mathcal{X}_\ell$ generated by $\mathcal{L}$ admits an exponential attractor $\mathcal{M}_{\mathrm{discr}}$, which is compact and has finite fractal dimension in $\mathcal{V}_0$ and exponentially attracts $\mathcal{B}_3$ in the $\mathcal{V}_0$-metric.

To pass from $\mathcal{M}_{\mathrm{discr}}$ to a *regular* exponential attractor for the original semiflow, we proceed by noting a number of facts:

(a) The evaluation map $e : \mathcal{Z}_\ell \to \mathcal{V}_0$ given by $e : U \mapsto U(\ell)$ is Lipschitz continuous. To prove this, one can, e.g., multiply (5.8) by t and integrate in time between 0 and ℓ.

(b) The semigroup operator $S(t)$ is uniformly Lipschitz continuous on $[0, \ell]$ with respect to the metric of $\mathcal{V}_0$. This is easily shown by integrating once more (5.8) between 0 and an arbitrary $t \leq \ell$ and using Gronwall's lemma.

(c) For each solution $U \in \mathcal{Z}_\ell$ and all $0 \leq s \leq t \leq \ell$, by (2.12) there holds

$$\|U(t) - U(s)\|_0^2 \leq \left| \int_s^t \|U_t(\tau)\|_0 \mathrm{d}\tau \right|^2 \leq c|t - s|^2. \tag{5.20}$$

In other words, the ℓ-trajectories lying in $\mathcal{Z}_\ell$ are uniformly Lipschitz continuous in time.

(d) Properties (a)–(c) allow us to apply, e.g., [34, Thm. 2.6] to deduce that there exists an invariant compact subset $\mathcal{M}_0 \subset \mathcal{V}_0$, of finite fractal dimension with respect to the $\mathcal{V}_0$-topology, which exponentially attracts $\mathcal{B}_3$ still with respect to the $\mathcal{V}_0$-topology. More precisely, since the elements of $\mathcal{Z}_\ell$ take values in the $\mathcal{V}_3$-bounded and positively invariant set $\mathcal{B}_3$, setting $\mathcal{M}_2 := \mathcal{M}_0 \cap \mathcal{B}_3$, we have that $\mathcal{M}_2$ is bounded in $\mathcal{V}_3$ and, by interpolation, it is compact and has finite fractal dimension in $\mathcal{V}_2$ (in fact, in $\mathcal{V}_s$ for any $s < 3$). Moreover, it exponentially attracts $\mathcal{B}_3$ and, by interpolation, this happens even with respect to the $\mathcal{V}_2$-metric;

(e) Finally, we see that $\mathcal{M}_2$ exponentially attracts any set B bounded in $\mathcal{V}_2$. Actually, we know from point (d) that $\mathcal{M}_2$ exponentially attracts $\mathcal{B}_3$ and from Lemma 5.3 that $\mathcal{B}_3$ exponentially attracts any such B. Note that the exponential attraction holds in both cases with respect to the $\mathcal{V}_2$-metric. To conclude, we can thus apply the transitivity property of exponential attraction introduced in [15, Thm. 5.1]. To do this, we have to check (cf. [15, (5.1)]) that the semigroup operators $S(t)$ are uniformly Lipschitz continuous on bounded balls B of $\mathcal{V}_2$, with the Lipschitz constant having the form $c_6 e^{c_7 t}$, where c_6 and c_7 depend only on B. To prove this fact, we can rewrite (3.21) for two solutions u^1 and u^2 originating from initial data $U_0^1 = (u_0^1, u_1^1)$ and $U_0^2 = (u_0^2, u_1^2)$, respectively. We obtain (recall also (3.14))

$$\|U(t)\|_2^2 + \int_0^t \|\nabla U_t\|^2 \leq \left\|U_0^1 - U_0^2\right\|_2^2 + \int_0^t \|\nabla(G^1 - G^2)\|^2.$$

Then, it is not difficult to recover the wanted estimate. The proof is completed. □

6. Energy Solutions

We finally consider the class of energy solutions. As we shall see, in this case the dissipation integral (2.16) will not be used.

We start by establishing existence and uniqueness of solutions.

Theorem 6.1. *Let us assume* (2.4)–(2.6) *and* (2.10), *together with*

$$(u_0, u_1) \in \mathcal{V}_0. \tag{6.1}$$

Then, there exists one and only one *function*

$$u \in W^{1,\infty}(0, T; V') \cap L^\infty(0, T; V) \tag{6.2}$$

which solves Problem (P).

Remark 6.2. It will be clear from the proof that the growth restriction (2.6) is required only for uniqueness. Actually, existence in the class $\mathscr{V}_0$ holds for any polynomial growth (cf., e.g., [3, Thm. 1.1]).

Proof of Theorem 6.1. Let us start with the proof of existence, which follows closely [49, Sec. 4] and is reported just for later convenience. We let $\{\lambda_j\} \subset (0, \infty)$, $j \in \mathbb{N}$, be the sequence of eigenvalues of A, increasingly ordered and with possible repetitions according to the multiplicities. Correspondingly, we let $\{z_j\}$ be a (complete) systems of eigenvectors, which is chosen to be orthonormal in H and orthogonal in V. We set $Z_N := \operatorname{span}\{z_1, \dots, z_N\}$ and denote by P_N the orthogonal projector onto Z_N. Of course, P_N can be thought to act on any of the spaces $D(A^s)$, $s \in \mathbb{R}$. We then consider the Faedo–Galerkin approximation of Problem (P), i.e.,

$$A^{-1}\big(v_{N,tt} + v_{N,t}\big) + Av_N + P_N f(v_N) = P_N A^{-1} g, \tag{6.3}$$

$$V_N|_{t=0} = V_{0,N} := P_N(u_0, u_1), \tag{6.4}$$

where both relations are intended as equalities in Z_N and we have set, for brevity, $V_N := (v_N, v_{N,t})$. It is easy to show that Problem (6.3)–(6.4) admits one and only one solution, which satisfies the energy estimate (cf. (2.13)) uniformly with respect to N. Then, standard compactness tools and the growth restriction (2.6) (note that at this level any polynomial growth of f would be admissible, cf. Remark 6.2) permit to take the limit of (6.3)–(6.4) as at least a subsequence of N goes to ∞ and, as a consequence, get existence of one $\mathscr{V}_0$-solution $U = (u, u_t)$ to Problem (P) satisfying in particular (6.2).

To get uniqueness, following the method developed in [48] (see also [10] and references therein), we will prove that, as $U = (u, u_t)$ is *any* solution to (P) in the regularity class (6.2), the whole sequence V_N converges to U. Of course, this entails uniqueness of U. With this aim, we let $U_N := P_N U$ (i.e., $u_N := P_N u$ and $u_{N,t} := P_N u_t$) and consider the projection of Equation (2.7). Then, it is clear that the difference $W_N = (w_N, w_{N,t}) := U_N - V_N$ satisfies

$$A^{-1}\big(w_{N,tt} + w_{N,t}\big) + Aw_N + P_N\big(f(u_N) - f(v_N)\big) = P_N\big(f(u_N) - f(u)\big), \tag{6.5}$$

$$W_N|_{t=0} = 0. \tag{6.6}$$

Testing (6.5) by $A^{-1}w_{N,t}$, we readily get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W_N\|_{-1}^2 + \|w_{N,t}\|_{D(A^{-1})}^2 \\
&\quad = \big\langle P_N\big(f(v_N) - f(u_N)\big), A^{-1}w_{N,t}\big\rangle + \big\langle P_N\big(f(u_N) - f(u)\big), A^{-1}w_{N,t}\big\rangle \\
&\quad \le \|f(u_N) - f(v_N)\| \|w_{N,t}\|_{D(A^{-1})} + \|f(u_N) - f(u)\|_{V'} \|w_{N,t}\|_{V'}.
\end{aligned} \tag{6.7}$$

Let us then notice that, by (2.6) and for fixed but arbitrary $\epsilon > 0$, one has

$$\begin{aligned}\|f(u_N) - f(u)\|_{V'} &\leq c\left\| \int_0^1 f'(\tau u_N + (1-\tau)u)\mathrm{d}\tau(u_N - u)\right\|_{L^{1+\epsilon}(\Omega)} \\ &\leq C\|u_N - u\| \leq C\lambda_N^{-1/2}\|u_N - u\|_V \leq C\lambda_N^{-1/2}. \qquad (6.8)\end{aligned}$$

Here and below, the constant C is allowed to depend on the $L^\infty(0, T; \mathcal{V}_0)$-norms of U and V_N (of course, they are bounded independently of N). Thus, using once more (2.6) and the Brézis–Gallouet inequality (cf. (2.33) or (2.34)), the remaining term in (6.7) can be controlled as

$$\begin{aligned}\|f(u_N) - f(v_N)\| &\leq C\big(1 + \|u_N\|^2_{L^\infty(\Omega)} + \|v_N\|^2_{L^\infty(\Omega)}\big)\|w_N\| \\ &\leq C\big(1 + \log(1 + \|u_N\|_{D(A)}) + \log(1 + \|v_N\|_{D(A)})\big)\|w_N\| \\ &\leq C\log\lambda_N\|w_N\|. \qquad (6.9)\end{aligned}$$

Thus, using (6.8) and (6.9), (6.7) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W_N\|^2_{-1} + \|w_{N,t}\|^2_{D(A^{-1})} \leq C\log\lambda_N\|w_N\|\|w_{N,t}\|_{D(A^{-1})} + C\lambda_N^{-1/2}, \qquad (6.10)$$

whence in particular

$$\frac{\mathrm{d}}{\mathrm{d}t}\|W_N\|^2_{-1} \leq C_1\log\lambda_N\|W_N\|^2_{-1} + C_2\lambda_N^{-1/2}, \qquad (6.11)$$

and, by Gronwall's lemma,

$$\|W_N(t)\|^2_{-1} \leq C_2 t\lambda_N^{C_1 t - \frac{1}{2}}, \qquad (6.12)$$

so that, taking, e.g., $t_* := 1/4C_1$, we readily obtain that, as (the whole sequence) $N \nearrow \infty$,

$$W_N \to 0 \quad \text{strongly in } L^\infty(0, t_*; \mathcal{V}_{-1}), \qquad (6.13)$$

whence, since we already know that $U_N \to U$, we obtain $V_N \to U$ by comparison. Finally, restarting the procedure from the time t_* (and noting that the value of the "new" t_* does not change since the functions U_N, V_N stay bounded in $\mathcal{V}_0$ uniformly in time), we deduce uniqueness on the whole of $(0, T)$, which concludes the proof. □

Therefore, the energy solutions constitute a new semiflow $\mathcal{S}_0$. The following analogue of Theorem 3.1 establishes some properties of $\mathcal{S}_0$ and of the associated semigroup operator S_0.

Theorem 6.3. *Let the assumptions of Theorem* 6.1 *hold. Then, the semiflow $\mathcal{S}_0$ is* uniformly dissipative. *Namely, there exists a constant R_0 independent of the initial data such that, for all bounded $B \subset \mathcal{V}_0$, there exists $T_B \geq 0$ such that $\|S_0(t)b\|_0 \leq R_0$, for all $b \in B$ and $t \geq T_B$. Moreover, any $u \in \mathcal{S}_0$ satisfies the additional time continuity property*

$$u \in C^2([0, T]; D(A^{-3/2})) \cap C^1([0, T]; V') \cap C^0([0, T]; V) \qquad (6.14)$$

as well as the energy equality

$$\mathscr{E}(u, u_t)(t) - \mathscr{E}(u, u_t)(s) = -\int_s^t \|u_t(r)\|_{V'}^2 \mathrm{d}r \quad \forall s, t \in [0, T]. \tag{6.15}$$

Finally, given a sequence of initial data $\{(u_{0,n}, u_{1,n})\} \subset \mathscr{V}_0$ *tending to some* $(u_0, u_1) \in \mathscr{V}_0$ *in the sense specified below, and denoting by* u_n, u *the solutions emanating from* $(u_{0,n}, u_{1,n}), (u_0, u_1)$, *respectively, we have that*

$$(u_{0,n}, u_{1,n}) \to (u_0, u_1) \text{ weakly in } \mathscr{V}_0$$
$$\Rightarrow (u_n, u_{n,t}) \to (u, u_t) \text{ weakly star in } L^\infty(0, T; \mathscr{V}_0), \tag{6.16}$$
$$(u_{0,n}, u_{1,n}) \to (u_0, u_1) \text{ strongly in } \mathscr{V}_0$$
$$\Rightarrow (u_n, u_{n,t}) \to (u, u_t) \text{ strongly in } C^0([0, T]; \mathscr{V}_0), \tag{6.17}$$

for any fixed $T \geq 0$.

Proof. We start by showing (6.14). Let u be the $\mathscr{V}_0$-solution to (P) and set $v := e^{t/2}u$ so that

$$v_t = e^{t/2}u_t + \frac{1}{2}e^{t/2}u = e^{t/2}u_t + \frac{v}{2}, \quad v_{tt} = e^{t/2}u_{tt} + \frac{1}{2}e^{t/2}u_t + \frac{v_t}{2} = e^{t/2}u_{tt} + v_t - \frac{v}{4}. \tag{6.18}$$

Let us now (formally) multiply (2.7) by $e^{t/2}A^{-1}v_t$. After some calculations we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{Y}(t) = \Phi(t) := e^t \int_\Omega \big(2F(u) - f(u)u\big) - e^t\langle g, A^{-1}u\rangle, \tag{6.19}$$

where we have set

$$\mathscr{Y}(t) := \|v_t\|_{V'}^2 + \|v\|_V^2 - \frac{1}{4}\|v\|_{V'}^2 + 2e^t \int_\Omega F(u) - 2e^t\langle g, A^{-1}u\rangle. \tag{6.20}$$

Of course, (6.19) could make no sense, because $e^{t/2}A^{-1}v_t$ is not smooth enough to be used as a test function. However, if we let u_n be a class of $\mathscr{V}_2$-solutions suitably approximating u and define v_n accordingly, it is clear that then (6.19) holds at least for v_n.

More precisely, we can suppose that, given $s, t \in [0, T]$, with $s < t$, there holds at least

$$u_n \to u \quad \text{weakly star in } W^{1,\infty}(s, t; V') \cap L^\infty(s, t; V). \tag{6.21}$$

Thus, by (2.6) and Lebesgue's dominated convergence theorem, it readily follows that

$$\Phi_n \to \Phi \quad \text{strongly in } L^1(s, t) \tag{6.22}$$

(with obvious meaning of Φ_n, cf. (6.19)). To proceed, we additionally assume that

$$(u_n, u_{n,t})(s) \to (u, u_t)(s) \quad \text{strongly in } V \times V'. \tag{6.23}$$

Clearly, this can be done as we consider s as the initial time and choose the approximation u_n accordingly. Then, integrating (6.19) (written for u_n) over (s, t), taking the supremum limit as $n \nearrow \infty$, and using (6.21) and the trivial fact that $\mathcal{Y}$ (seen as a functional of the couple (v, v_t)) is sequentially weakly lower semicontinuous in $\mathcal{V}_0$, one gets that for the limit solution u there holds

$$\mathcal{Y}(t) \leq \mathcal{Y}(s) + \int_s^t \Phi(r)\mathrm{d}r. \tag{6.24}$$

To prove the converse inequality, one simply repeats the procedure by considering t as the initial time and noting that (P) is solvable backward in time (of course, this prevents dissipation, but still there is global boundedness in the energy norm). In particular, the approximation u_n can be still chosen to fulfill (6.21), while in place of (6.23) we can ask that

$$(u_n, u_{n,t})(t) \to (u, u_t)(t) \quad \text{strongly in } V \times V'. \tag{6.25}$$

Thus, we finally get the *equal sign* in (6.24), which, due to arbitrariness of s, t, readily implies that $\mathcal{Y}$ (written for the limit solution u and regarded as a function of time) is absolutely continuous over $[0, T]$. To conclude, we observe that from (6.2) we know that

$$v \in C^1_w([0, T]; V') \cap C^0_w([0, T]; V), \tag{6.26}$$

so that the latter three summands in $\mathcal{Y}$ (cf. (6.20)) are strongly continuous in time. Then, by comparison, the function $t \mapsto \|v_t(t)\|^2_{V'} + \|v(t)\|^2_V$ is also continuous in time. This fact, joint with (6.26), immediately gives (6.14) (as before, the continuity of u_{tt} can be shown by a further comparison of terms).

We now show that u satisfies (6.15), i.e., the energy equality for the original energy $\mathcal{E}$. We give just the highlights of the argument and leave the details to the reader. Setting $\mathcal{Y}_0 := e^{-t}\mathcal{Y}$ and $\Phi_0 := e^{-t}\Phi$, and noting that $\mathcal{Y}$, and hence $\mathcal{Y}_0$, is absolutely continuous in time, we infer from (6.19)

$$\mathcal{Y}_0' + \mathcal{Y}_0 = \Phi_0 \quad \text{a.e. in } (0, T). \tag{6.27}$$

Next, let us integrate (6.27) between s and t, and use (6.18)–(6.20) (in particular, $\mathcal{Y}_0$ and Φ_0 have to be rewritten in terms of u, u_t rather than v, v_t). Performing standard manipulations and subtracting from the resulting formula the outcome of (2.7) tested by $A^{-1}u$, one then gets (6.15) by simple computations. It is maybe worth pointing out that (6.14), or even (6.2), is sufficient to test (2.7) by $A^{-1}u$ (while it does *not* permit to test it by $A^{-1}u_t$), so the latter argument is rigorous.

As we know the energy equality to hold, the dissipativity of S_0 can be standardly obtained by writing (6.15) in the differential form (which is possible almost everywhere by absolute continuity of $\mathcal{E}$) and adding the result of (2.7) tested by $\delta A^{-1}u$ for small $\delta > 0$. We leave once more the details to the reader.

Finally, we have to prove (6.16) and (6.17). The first is standard. To show the latter, we proceed along the lines of [3, Proof of Thm. 3.6]. Namely, setting as usual $U_n := (u_n, u_{n,t})$, we let by contradiction $\epsilon > 0$ and $\{t_n\} \subset [0, T]$ such that $\|U_n(t_n) - U(t_n)\|_0 \geq \epsilon$ at least for a subsequence (here we shall not relabel subsequences). We can also assume $t_n \to t$ for some $t \in [0, T]$. Note that then, by weak convergence, $U_n(t_n) \to U(t)$ weakly in $\mathcal{V}_0$.

Now, we claim it suffices to prove that $|\mathcal{E}(U_n(t_n)) - \mathcal{E}(U(t))| \to 0$. Indeed, since the other terms in $\mathcal{E}$ have a lower order, this would entail

$$\big|\|U_n(t_n)\|_0^2 - \|U(t)\|_0^2\big| \to 0 \tag{6.28}$$

and consequently that $U_n(t_n)$ goes to $U(t)$ *strongly* in $\mathcal{V}_0$. Now, by the energy equality it is

$$\big|\mathcal{E}(U(t_n)) - \mathcal{E}(U(t))\big| \le \left| \int_{t_n}^{t} \|u_t\|_{V'}^2 \right| \to 0, \tag{6.29}$$

so that U is strongly continuous in time and in particular $U(t_n) \to U(t)$ strongly in $\mathcal{V}_0$. Thus, we would end up with

$$\big\|U_n(t_n) - U(t_n)\big\|_0 \to 0, \tag{6.30}$$

a contradiction.

Let us now prove the claim. First, by sequential weak lower semicontinuity of $\mathcal{E}$, we have

$$\mathcal{E}(U(t)) \le \liminf_{n\nearrow\infty} \mathcal{E}(U_n(t_n)). \tag{6.31}$$

Conversely, by the energy equality, we obtain

$$\begin{aligned}
\limsup_{n\nearrow\infty} \mathcal{E}(U_n(t_n)) &= \lim_{n\nearrow\infty} \mathcal{E}(U_{0,n}) - \liminf_{n\nearrow\infty} \int_0^{t_n} \|u_{n,t}\|_{V'}^2 \\
&= \mathcal{E}(U_0) - \liminf_{n\nearrow\infty} \int_0^{t} \|u_{n,t}\|_{V'}^2 - \lim_{n\nearrow\infty} \int_t^{t_n} \|u_{n,t}\|_{V'}^2 \\
&\le \mathcal{E}(U_0) - \int_0^{t} \|u_t\|_{V'}^2 - \lim_{n\nearrow\infty} \int_t^{t_n} \|u_{n,t}\|_{V'}^2 = \mathcal{E}(U(t)),
\end{aligned} \tag{6.32}$$

provided that the last integral on the second and third row does go to 0, which is true for $t_n \to t$ and

$$\limsup_{n\nearrow\infty} \|u_{n,t}\|_{L^2(0,T;V')}^2 \le \mathcal{E}(U_0) - \liminf_{n\nearrow\infty} \mathcal{E}(U_n(T)) \le \mathcal{E}(U_0) - \mathcal{E}(U(T)) = \|u_t\|_{L^2(0,T;V')}^2, \tag{6.33}$$

i.e., $u_{n,t}$ goes to u_t *strongly* in $L^2(0, T; V')$. The proof is thus complete. □

Still paralleling the $\mathcal{V}_2$-case, we finally have the

Theorem 6.4. *Let the assumptions of Theorem* 6.1 *hold. Then, the semiflow $\mathcal{S}_0$ associated with* (P) *is asymptotically compact. Thus $\mathcal{S}_0$ possesses the global attractor $\mathcal{A}_0$.*

Proof. The proof is analogous to that of Theorem 3.2 and, in fact, even technically simpler. Indeed, having the $\mathcal{V}_0$-energy equality (6.15) at our disposal, we can still implement Ball's energy method [3, Sec. 4], which leads us to show the asymptotic compactness of S_0 in the phase space $\mathscr{V}_0$. Then we can conclude as in Section 3. □

Remark 6.5. It is not difficult to check that the results of this section still hold when $f \in C^1(\mathbb{R}; \mathbb{R})$ only (compare with (2.4)).

Remark 6.6. The semiflow $\mathscr{S}_0$ is generated by a gradient system (see (2.9)). Hence $\mathscr{A}_0$ coincides with the unstable manifold of the set of equilibria. We recall that this set can have a very complicated structure (see, e.g., [4, 33, 52–54]). Moreover, it is clear that $\mathscr{A}_2 \subset \mathscr{A}_0$. However, the converse is far less trivial (see [27]).

Remark 6.7. Any energy solution given by Theorem 6.1 converges to a unique equilibrium, provided that f is real analytic. This fact can be proven, using Theorem 6.4 and the Łojasiewicz–Simon inequality, arguing as in [25]. A convergence rate estimate can also be obtained.

Remark 6.8. Well-posedness of (2.7) can be also shown in the space $\mathscr{V}_1$, i.e., for weak solutions. Actually, the uniqueness part of Theorem 2.2 still holds with no change in the proof. Concerning existence, the main bound should be obtained by testing (2.7) by $u_t + \beta u$, for small $\beta > 0$ (compare this with (2.21) below). However, this estimate does not seem to have a dissipative character since one apparently cannot take advantage of the dissipation integral (2.16). In this sense, the $\mathscr{V}_1$-theory for Problem (P) seems less complete than the $\mathscr{V}_2$ and $\mathscr{V}_0$ theories discussed above. We also notice that, both in the $\mathscr{V}_2$ and in the $\mathscr{V}_1$ setting, it seems possible to obtain global well posedness for nonlinearities f having up to a fourth order growth (rather than a cubic growth as stated in (2.6)). However, in this case we can no longer prove dissipativity, even for quasi-strong solutions.

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