ON A CLASS OF DOUBLY NONLINEAR EVOLUTION EQUATIONS*

PIERLUIGI COLLI† and AUGUSTO VISINTIN‡

Abstract. The abstract equation $A\dfrac{du}{dt} + Bu \ni f$ is considered for A and B nonlinear maximal monotone operators in a Hilbert space H, with A bounded, B unbounded and such that its domain $D(B)$ is contained in a Banach space V compactly embedded in H.

Existence results are proved for the related initial-value problem, requiring that either A or B be the subdifferential of a convex and lower semicontinuous function, and assuming suitable coerciveness conditions. Arguments are based on monotonicity and compactness techniques.

Key words : nonlinear evolution equations, maximal monotone operators, initial-value problems.

AMS (MOS) subject classifications : 35G25, 35K22, 39B70.

Abbreviated title : DOUBLY NONLINEAR EVOLUTION EQUATIONS

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(see the address below)

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† Dipartimento di Matematica, Università di Pavia, Strada Nuova, 65, 27100 PAVIA, Italy.

‡ Dipartimento di Matematica dell' Università, 38050 POVO (Trento), Italy and Istituto di Analisi Numerica del C.N.R., PAVIA, Italy.

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P. Colli A. Visintin

Dipartimento di Matematica
Università di Pavia
Strada Nuova, 65
27100 Pavia, Italy

Dip. di Matematica dell'Università
38050 Povo (TN), Italy
and Istituto di Analisi Numerica
del C.N.R., Pavia, Italy

1. Introduction.

In this paper we study the equation

$$A\frac{du}{dt} + Bu \ni f \qquad \text{in } H, \tag{1.1}$$

where H is a Hilbert space, A and B are maximal monotone (multivalued) operators in H, with A bounded and B unbounded, and f is a datum. Obviously equations of this type are meant for applications to nonlinear partial differential equations, more precisely *possibly degenerate parabolic equations*. So, although in the next sections we shall deal with equation (1.1) in abstract form, here we will illustrate our results on an example in Sobolev spaces.

Let Ω be a bounded domain of $\mathbf{R}^M$ $(M \geq 1)$ and let α, β be two maximal monotone graphs of $\mathbf{R}^N \times \mathbf{R}^N$ $(N \geq 1)$ and of $\mathbf{R}^{M\times N} \times \mathbf{R}^{M\times N}$, respectively. Given a function $f : Q := \Omega\times]0,T[\to \mathbf{R}^N$, we look for a function $u : Q \to \mathbf{R}^N$ such that

$$\alpha\left(\frac{\partial u}{\partial t}\right) - \operatorname{div}\,(\beta(\nabla u)) \ni f \tag{1.2}$$

in the sense of distributions, and satisfying suitable initial and boundary conditions. More precisely, (1.2) must be understood as equivalent to the system

$$w + v = f, \quad w \in \alpha\left(\frac{\partial u}{\partial t}\right), \quad v \in -\operatorname{div}\,(\beta(\nabla u)) \qquad \text{in } \mathcal{D}'(Q). \tag{1.3}$$

We can prove existence of a solution of the corresponding initial-boundary value problem, uniqueness being an open question if both α and β are nonlinear.

First we require that α be coercive and with linear growth at infinity, and β the subdifferential of a proper, convex lower semicontinuous function $\psi : \mathbf{R}^{M\times N} \to \mathbf{R}$ such that for some $a > 0,\ b \in \mathbf{R}$,

$$\psi(v) \geq a|v|^2 - b \qquad \forall\, v \in \mathbf{R}^{M\times N}.$$

This setting corresponds to a *nonlinear relaxation dynamics* for a potential system. Here one can consider an approximate equation (depending on a parameter $\varepsilon > 0$) and multiply it by $\dfrac{\partial u_\varepsilon}{\partial t}$; this yields the estimate

$$u_\varepsilon \text{ is uniformly bounded in } H^1(0,T;L^2(\Omega)^N) \cap L^\infty(0,T;H^1(\Omega)^N). \tag{1.4}$$

Hence, possibly extracting a subsequence, u_ε weakly star converges to some u in that space. By means of a standard monotonicity and compactness procedure, one can then show that

$$v := -\lim_{\varepsilon\searrow 0} \operatorname{div}\,(\beta_\varepsilon(\nabla u_\varepsilon)) \in -\operatorname{div}\,(\beta(\nabla u)) \qquad \text{in } \mathcal{D}'(Q). \tag{1.5}$$

Finally, as β is *cyclically monotone* (being a subdifferential [**9**]), by using (1.2) and (1.4), one can prove that

$$w := \lim_{\varepsilon\searrow 0} \alpha_\varepsilon\left(\frac{\partial u_\varepsilon}{\partial t}\right) \in \alpha\left(\frac{\partial u}{\partial t}\right) \qquad \text{in } \mathcal{D}'(Q). \tag{1.6}$$

Our other existence results require α to be linearly bounded and equal to the subdifferential of a proper, convex lower semicontinuous function $\varphi : \mathbf{R}^N \to \mathbf{R}$ (without any coerciveness assumption), and β to be strongly monotone and either Lipschitz continuous or equal to the subdifferential of a proper, convex lower semicontinuous function $\psi : \mathbf{R}^{M\times N} \to \mathbf{R}$. In this setting the function φ can correspond to a *dissipation potential.* Also here a convenient approximation is introduced, then the equation is differentiated in time and multiplied by $\dfrac{\partial u_\varepsilon}{\partial t}$; this yields the estimate

$$(1.7) \qquad u_\varepsilon \text{ is uniformly bounded in } H^1(0,T;H^1(\Omega)^N),$$

which allows us to take the limit in the approximate equation without much difficulty.

Equations of the form (1.1) occur in several physical models. For instance in thermodynamics, denoting by u the vector of *generalized displacements* and by F that of *generalized forces*, from the *second principle of thermodynamics* it follows that the so-called *phenomenological laws* are of the form

$$(1.8) \qquad F = Bu,$$

with B monotone and $B0 = 0$ [**15**]. In a neighbourhood of $u = 0$ one can assume that B is linear; moreover, by *Onsager relations*, B is also self-adjoint, hence *cyclically monotone*, that is $B = \partial\psi$, with ψ convex potential. We allow ψ to be nonquadratic. Now one can introduce the assumption of *normal dissipativity* [**17**], requiring the existence of another convex function φ, named *dissipation potential*, such that

$$(1.9) \qquad \partial\varphi\left(\frac{du}{dt}\right) = -F.$$

Thus by (1.8) and (1.9) we have

$$(1.10) \qquad \partial\varphi\left(\frac{du}{dt}\right) + \partial\psi(u) \ni 0.$$

More generally the presence of an exterior thermodynamic force $-f$ would yield a right hand side f.

It does not seem that equation (1.1) has yet been studied, unless one of the operators is linear. In fact, the case with A linear (and self-adjoint) is well known and has been first detailed by Brezis [**9**] (see also [**4**], [**20**], and Chapter 3 of [**10**] for a review of the related literature), while equations with B linear arise from heat control problems, e.g., and have been studied by Duvaut and Lions [**12**,**13**]. Otherwise, the authors just know of an equation of the form $\alpha\left(\dfrac{\partial u}{\partial t}\right) + \beta(u) \ni f$, coupled with a degenerate diffusion equation, studied by Blanchard, Damlamian and Guidouche [**7**]. More concern has been devoted to equations of the form

$$\frac{d}{dt}(Au) + Bu = f \qquad \text{in } H, \tag{1.11}$$

still with A and B nonlinear and B unbounded in H. These have been examined by Raviart [**19**], Grange and Mignot [**16**], Bamberger [**2**], Benilan [**5**], Barbu [**3**], Di Benedetto and Showalter [**11**], and by Bernis [**6**], as well as other authors. General partial differential equations of this form have been also studied by Alt and Luckhaus [**1**], who actually considered a much more general setting for systems, and by Blanchard and Francfort [**8**].

The plan of this paper is as follows. In section 2 we give a precise formulation of the abstract initial value problem for equation (1.1), state the existence theorems and recall some preliminary results. Proofs are then given in sections 3 and 4.

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2. Statement of results and preliminaries.

Let H be a real Hilbert space which we identify with its dual, and V a reflexive Banach space dense and compactly embedded in H. Thus $V \subset H \subset V'$, where V' is the dual space of V. We denote by $(\cdot\,,\cdot)$ either the scalar product in H or the duality pairing between V' and V, and by $|\cdot|$ the norm in H.

Let A and B be maximal monotone operators in H with domains

$D(A)$ and $D(B)$. Our existence results require at least one of the operators B or A to be *cyclically monotone*, hence equal to the subdifferential of a proper, convex and lower semicontinuous function (see [**9**, Thm. 2.5]). The operator A is always supposed to be bounded in H, so that A^{-1} is surjective and $D(A) \equiv H$ (see [**9**, Thm. 2.3]), while B will be unbounded and such that $D(B) \subset V$. Besides we assume some coerciveness for one of the two operators. Here below we state the precise results.

First we suppose B to be cyclically monotone and A coercive and with linear growth in H. Namely we assume that

$$(i) \qquad \exists\, C_1 > 0 \,:\, \forall\, u \in H,\ \forall\, \xi \in Au \qquad (\xi, u) \geq C_1 \left(|u|^2 - 1\right),$$

$$(ii) \qquad \exists\, C_2 > 0 \,:\, \forall\, u \in H,\ \forall\, \xi \in Au \qquad |\xi| \leq C_2 \left(|u| + 1\right),$$

(*iii*) B *is the subdifferential of a proper, convex and lower semicontinuous function* $\psi \,:\, H \to]-\infty, +\infty]$,

(*iv*) $D(\psi) \subset V$ *and there exist* $C_3, C_4 > 0$, $p_1, p_2 > 0$ *such that*

$$\psi(u) \,\geq\, C_3 \, \|u\|_V^{p_1} - C_4 \left(|u| + 1\right)^{p_2} \qquad \forall\, u \in V,$$

$$(v) \qquad f \,\in\, L^2(0, T; H),$$

$$(vi) \qquad u_0 \,\in\, D(\psi).$$

Theorem 2.1. *Under the assumptions (i)-(vi), there exists a triple* $u \in H^1(0, T; H) \cap L^\infty(0, T; V)$, $w, v \in L^2(0, T; H)$, *such that for a.e.* $t \in]0, T[$

$$w(t) + v(t) \,=\, f(t), \tag{2.1}$$

$$w(t) \,\in\, Au'(t), \tag{2.2}$$

$$v(t) \,\in\, Bu(t), \tag{2.3}$$

where $u' = \dfrac{du}{dt}$, *and*

$$u(0) = u_0. \tag{2.4}$$

Remark 2.1. Assumptions (i)-(ii) restrict the behaviour of A at infinity, but allow for the presence of *horizontal segments* in this graph.

Our second existence theorem deals with the case where A is cyclically monotone. Now we do not require any coerciveness on A, but suppose B to be strongly monotone and Lipschitz continuous in V. Namely we assume that

(*vii*) *A is the subdifferential of a proper, convex and lower semicontinuous function* $\varphi : H \to]-\infty, +\infty]$, *and* *$A$ is bounded in H (i.e. maps bounded sets into bounded sets),*

(*viii*) $B : V \to V'$ *is Lipschitz continuous, i.e.*

$$\exists\, C_5 > 0 \,:\, \forall\, u_1, u_2 \in V \qquad \|Bu_1 - Bu_2\|_{V'} \le C_5 \|u_1 - u_2\|_V \,,$$

(*ix*) B *is strongly monotone, i.e.*

$$\exists\, C_6 > 0 \,:\, \forall\, u_1, u_2 \in V \qquad (Bu_1 - Bu_2, u_1 - u_2) \ge C_6 \|u_1 - u_2\|_V^2 \,,$$

(*x*) $$f \in H^1(0,T;V'),$$

(*xi*) $$u_0 \in V,$$

(*xii*) $$f(0) - Bu_0 \in D(\varphi^*),$$

where φ^* is the *convex conjugate* of φ (defined in the next proposition 2.4).

Remark 2.2. It is easy to see (cf., e.g., [**18**, pp. 171-173]) that by (viii) and (ix) the restriction of the operator B to $D(B) \subset V$ taking values in H

is maximal monotone in H and surjective: for any $g \in H$ the equation $Bu = g$ has one and only one solution $u \in V$.

Theorem 2.2. *Assume (vii)-(xii) hold. Then there exist* $u \in H^1(0,T;V)$ *and* $w \in L^\infty(0,T;H) \cap H^1(0,T;V')$ *satisfying (2.4), (2.2) and*

$$w(t) + Bu(t) \;=\; f(t) \qquad \text{in } V' \tag{2.5}$$

for a.e. $t \in]0,T[$.

Our third existence result is a variation of theorem 2.2, in which also B is required to be cyclically monotone, but not necessarily Lipschitz continuous.

Remark 2.3. Let B be as in assumption (iii) and let $D(\psi) \subset V$, so that ψ is proper, convex and lower semicontinuous also in V. Denote by $\widetilde{B}$ the subdifferential of ψ restricted to V and by $\widetilde{D} \subset V$ its domain. Obviously $\widetilde{B} : \widetilde{D} \to V'$ is an extension of B and $D(B) \subset \widetilde{D} \subset D(\psi)$.

Theorem 2.3. *Assume that (vii), (iii) and (x) hold. Let* $D(\psi) \subset V$ *and let the operator* $\widetilde{B}$ *of remark 2.3 be strongly monotone in* $\widetilde{D}$ *in the sense of assumption (ix). Besides let* $u_0 \in D(B)$ *and* $f(0) - B^0 u_0 \in D(\varphi^*)$, *where* $B^0 u_0$ *is the element of* Bu_0 *with minimal norm (cf. the next proposition 2.2). Under these assumptions, there exists a triple* $u \in H^1(0,T;V)$, $w \in L^\infty(0,T;H)$, $v \in L^\infty(0,T;V')$, *satisfying (2.2) in* H, *both (2.1) and (2.3) (with* B *replaced by* $\widetilde{B}$*) in* V' *for a.e.* $t \in]0,T[$, *and (2.4).*

Remark 2.4 (cf. also remarks 2.2 and 2.3). By comparison in the corresponding equations, it is straightforward to see that in theorem 2.2 (resp. 2.3) if $f \in L^2(0,T;H)$, then $Bu \in L^2(0,T;H)$ (resp. $v \in L^2(0,T;H)$), and (2.5) (resp. (2.2) and (2.3)) holds in H for a.e. $t \in]0,T[$.

Remark 2.5 (concerning uniqueness). It is not difficult to show that there is at most one solution of problem (2.1)-(2.4), if at least one of A or B is linear and self-adjoint in H, and moreover at least one of these operators is *strictly monotone* (that is, for instance for A, $(\xi_1 - \xi_2, u_1 - u_2) > 0$ for any $u_1, u_2 \in D(A)$, $u_1 \neq u_2$, and for any $\xi_1 \in Au_1$, $\xi_2 \in Au_2$). Indeed it suffices to multiply the difference of the equations (2.1) corresponding to u_1 and u_2 by $u_1 - u_2$ if A is linear and self-adjoint, by $u_1' - u_2'$ in the other case, and then to integrate in time. Since one of the operators is strictly monotone, the conclusion easily follows.

Theorems 2.1-2.3 will be proved in the next sections. Here we list some remarks and results which will be utilized later on. We refer to [**9**] and [**14**] for their proofs and for additional related material.

Remark 2.6. Let A, B be maximal monotone operators in H : then A and B induce on $L^2(0,T;H)$ two maximal monotone operators (still denoted by A and B) defined by $w \in Au'$, $v \in Bu$ if and only if (2.2) and (2.3) hold for a.e. $t \in]0,T[$.

Proposition 2.1. *Let* $\mathcal{A}$ *be a maximal monotone operator in some Hilbert space* $\mathcal{X}$ *and let* $\{y_n\,,\, z_n\}$ *be such that* $z_n \in \mathcal{A}y_n$ *for any* $n \in \mathbf{N}$, $y_n \to y$, $z_n \to z$ *weakly in* $\mathcal{X}$, *and* $\limsup\,(z_n, y_n)_{\mathcal{X}} \leq (z,y)_{\mathcal{X}}$ *as* $n \to \infty$. *Then* $z \in \mathcal{A}y$ *and* $(z_n, y_n)_{\mathcal{X}} \to (z,y)_{\mathcal{X}}$ *as* $n \to \infty$.

Proposition 2.2. *Let* B *be a maximal monotone operator in* H, *and denote by* I *the identity in* H. *For any* $\varepsilon > 0$ *the resolvent of* B

$$J_\varepsilon := (I + \varepsilon B)^{-1}$$

is a contraction defined on all H. *Moreover the Yoshida approximation*

$$B_\varepsilon := \frac{1}{\varepsilon}(I - J_\varepsilon)$$

is a monotone and Lipschitz continuous mapping defined on all H *and has the following properties:*

$$B_\varepsilon u \in BJ_\varepsilon u \qquad \forall\, u \in H,\ \forall\, \varepsilon > 0, \tag{2.6}$$

$$\forall\, u \in D(B) \qquad B_\varepsilon u \to B^0 u \quad as \quad \varepsilon \searrow 0, \tag{2.7}$$

where $B^0 u \in Bu$ *is such that* $|B^0 u| = \min\{|\xi| \,:\, \xi \in Bu\}$.

Proposition 2.3. *Let* ψ *and* B *satisfy (iii). For any* $\varepsilon > 0$ *and any* $u \in H$, *define*

$$\psi_\varepsilon(u) := \min_{z\in H}\left\{\frac{1}{2\varepsilon}|u-z|^2 + \psi(z)\right\}.$$

Then ψ_ε *is convex, Fréchet-differentiable in* H *and its subdifferential* $\partial\psi_\varepsilon$ *coincides with* B_ε. *Moreover*

$$\psi_\varepsilon(u) = \frac{\varepsilon}{2}|B_\varepsilon u|^2 + \psi(J_\varepsilon u) \qquad \forall\, u \in H,\ \forall\, \varepsilon > 0, \tag{2.8}$$

$$\forall\, u \in H \qquad \psi_\varepsilon(u) \nearrow \psi(u) \quad \text{as} \quad \varepsilon \searrow 0. \tag{2.9}$$

Proposition 2.4. *Let* $\phi : H \to]-\infty,+\infty]$ *be a proper, convex and lower semicontinuous function. The convex conjugate of* ϕ, *defined for any* $z \in H$ *by*

$$\phi^*(z) := \sup_{u\in H}\{(z,u) - \phi(u)\},$$

is such that $\partial\phi^* = (\partial\phi)^{-1}$. *Moreover the following three conditions are equivalent:*

$$u \in \partial\phi^*(z), \qquad z \in \partial\phi(u), \qquad \phi(u) + \phi^*(z) = (z,u).$$

Proposition 2.5. *Let* ϕ *be as in proposition 2.4. If* $u \in H^1(0,T;H)$, $v \in L^2(0,T;H)$ *and* $v(t) \in \partial\phi(u(t))$ *for a.e.* $t \in]0,T[$, *then the function* $t \mapsto \phi(u(t))$ *is absolutely continuous on* $[0,T]$, *and for a.e.* $t \in]0,T[$

$$\frac{d}{dt}\phi(u(t)) = (w, u'(t)) \qquad \forall\, w \in \partial\phi(u(t)). \tag{2.10}$$

Proposition 2.6. *Let* $\phi : V \to]-\infty,+\infty]$ *be proper, convex and lower semicontinuous . Then the function* $\Phi : L^2(0,T;V) \to]-\infty,+\infty]$, *defined by*

$$\Phi(u) = \begin{cases} \displaystyle\int_0^T \phi(u(t))dt & \textit{if} \ \ \phi(u) \in L^1(0,T), \\ +\infty & \textit{elsewhere}, \end{cases}$$

is proper, convex lower semicontinuous and

$$\partial\Phi(u) = \left\{v \in L^2(0,T;V') : v(t) \in \partial\phi(u(t)) \ \textit{for a.e.} \ t \in]0,T[\right\}.$$

Moreover let $\{u_n\,,\,v_n\}$ *be such that* $v_n \in \partial\phi(u_n)$ *for* $n \in \mathbf{N}$, $u_n \to u$ *weakly in* $L^2(0,T;V)$, $v_n \to v$ *weakly in* $L^2(0,T;V')$, *and*

$$\limsup_{n\to\infty} \int_0^T (v_n(t), u_n(t))dt \le \int_0^T (v(t), u(t))dt.$$

Then $v \in \partial\Phi(u)$.

Remark 2.7 ([**9**, Prop. 2.14]). Let A and φ satisfy (vii). Then A is bounded if and only if

$$\frac{\varphi^*(u)}{|u|} \to +\infty \qquad \text{as} \quad u \in D(\varphi^*), \quad |u| \to +\infty. \tag{2.11}$$

3. Proof of Theorem 2.1.

First we regularize problem (2.1)-(2.4). To this aim we introduce a sequence $\{f_\varepsilon\}$ such that

$$f_\varepsilon \in C^0([0,T];H) \qquad \forall\, \varepsilon > 0, \tag{3.1}$$

$$f_\varepsilon \to f \qquad \text{strongly in } L^2(0,T;H) \text{ as } \varepsilon \searrow 0, \tag{3.2}$$

and consider the following approximate problem

$$\varepsilon u_\varepsilon'(t) + w_\varepsilon(t) + B_\varepsilon u_\varepsilon(t) = f_\varepsilon(t) \qquad \forall\, t \in [0,T], \tag{3.3}$$

$$w_\varepsilon(t) \in A u_\varepsilon'(t) \qquad \forall\, t \in [0,T], \tag{3.4}$$

$$u_\varepsilon(0) = u_0, \tag{3.5}$$

where B_ε is defined as in proposition 2.2. Note that the operator $\varepsilon I + A$ is strongly monotone in H. Since $(\varepsilon I + A)^{-1}$ and B_ε are Lipschitz continuous in H , it is not difficult to see (using, for instance, the Contraction Mapping Principle step by step in time) that, by (3.1), problem (3.3)-(3.5) has a unique solution $u_\varepsilon \in C^1([0,T];H)$ satisfying (3.5) and

$$u_\varepsilon'(t) - (\varepsilon I + A)^{-1}(f_\varepsilon(t) - B_\varepsilon u_\varepsilon(t)) = 0 \qquad \forall\, t \in [0,T], \tag{3.6}$$

while $w_\varepsilon \in C^0([0,T];H)$ is given by

$$(3.7) \qquad w_\varepsilon(t) \,=\, f_\varepsilon(t) - \varepsilon u_\varepsilon'(t) - B_\varepsilon u_\varepsilon(t) \qquad \forall\, t \in [0,T].$$

To derive a priori estimates, we take the scalar product of (3.3) with $u_\varepsilon'(t)$ and integrate in time from 0 to any $s \in]0,T]$. Using (iii) and proposition 2.3, we obtain

$$(3.8) \quad \varepsilon \int_0^s |u_\varepsilon'(t)|^2 dt \,+\, \int_0^s (w_\varepsilon(t), u_\varepsilon'(t)) dt \,+\, \psi_\varepsilon(u_\varepsilon(s)) \,= \\ = \psi_\varepsilon(u_0) \,+\, \int_0^s (f_\varepsilon(t), u_\varepsilon'(t)) dt \qquad \forall\, s \in [0,T].$$

Using (3.4), (i) and (2.9), standard calculations lead to

$$(3.9) \quad \frac{C_1}{2} \int_0^T |u_\varepsilon'(t)|^2 dt \,+\, \psi_\varepsilon(u_\varepsilon(s)) \,\le\, \psi(u_0) \,+\, C_1 T \,+\, \frac{1}{2C_1} \, \|f_\varepsilon\|^2_{L^2(0,T;H)} \,.$$

In the sequel we shall denote by $C_7, C_8, \ldots$ positive constants independent of ε. Note that, thanks to the Hahn-Banach Theorem,

$$\exists\, z_\psi \in H, \;\exists\, c_\psi \in \mathbf{R} \;:\; \forall\, u \in H \qquad \psi(u) \,\ge\, (z_\psi, u) + c_\psi\,,$$

and that

$$|u_\varepsilon(T)|^2 \,\le\, 2 \left\{ |u_0|^2 \,+\, T \int_0^T |u_\varepsilon'(t)|^2 dt \right\}.$$

Then, from (vi), (3.2), (3.9), (2.8) and by the contraction property of J_ε (cf. proposition 2.2), it follows that

$$(3.10) \quad |u_\varepsilon(T)|^2 \,+\, \|u_\varepsilon'\|^2_{L^2(0,T;H)} \,\le\, C_7 \,-\, \frac{2(T+1)}{C_1}\, \psi_\varepsilon(u_\varepsilon(T)) \,\le \\ \le\, C_7 \,+\, \frac{2(T+1)}{C_1} |z_\psi|\, |u_\varepsilon(T)| \,+\, C_8\,.$$

Hence, by standard arguments and using again the fact that J_ε is a contraction on H, we infer that (see, e.g., [**9**, Appendice])

$$(3.11) \qquad \|u_\varepsilon\|_{H^1(0,T;H)} \,+\, \|J_\varepsilon u_\varepsilon\|_{H^1(0,T;H)} \,\le\, C_9\,.$$

Owing to (iv), (2.8), (3.8) and (3.11), we deduce that

$$(3.12) \qquad \|J_\varepsilon u_\varepsilon\|^2_{L^\infty(0,T;V)} \,\le\, C_{10}\,,$$

where the constant C_{10} depends on C_3, C_4, p_1, p_2 and C_9. Finally by (ii), (3.4) and (3.11), we have

$$\|w_\varepsilon\|_{L^2(0,T;H)} \leq \sqrt{2}\, C_2 \left(C_9 + \sqrt{T}\right) \tag{3.13}$$

and , using (3.11), (3.13) and (3.2), by comparison in (3.3) we obtain

$$\|B_\varepsilon u_\varepsilon\|_{L^2(0,T;H)} \leq C_{11}\,. \tag{3.14}$$

By the a priori estimates (3.11)-(3.14), there exist $u \in H^1(0,T;H) \cap L^\infty(0,T;V)$, $w, v \in L^2(0,T;H)$ and subsequences, still denoted by u_ε, w_ε, such that

$$J_\varepsilon u_\varepsilon \to u, \qquad w_\varepsilon \to w, \qquad B_\varepsilon u_\varepsilon \to v \qquad \text{weakly (star)} \tag{3.15}$$

in the corresponding spaces, as ε goes to 0. Since (cf. proposition 2.2)

$$u_\varepsilon - J_\varepsilon u_\varepsilon = \varepsilon B_\varepsilon u_\varepsilon \to 0 \qquad \text{strongly} \tag{3.16}$$

in $L^2(0,T;H)$, by a standard compactness result due to Aubin (see, e.g., [**18**, p. 58]) from (3.11), (3.12) and (3.16) it follows that

$$J_\varepsilon u_\varepsilon \to u \qquad \text{strongly in } L^2(0,T;H), \tag{3.17}$$

$$u_\varepsilon \to u \qquad \text{strongly in } L^2(0,T;H) \text{ and weakly in } H^1(0,T;H) \tag{3.18}$$

as ε goes to 0. It remains to show that u, w, v satisfy (2.1)-(2.4). Taking the scalar product of (3.3) with any $z \in H$ and integrating in time, we obtain

$$\varepsilon(u_\varepsilon(t), z) + \int_0^t (w_\varepsilon(s) + B_\varepsilon u_\varepsilon(s) - f_\varepsilon(s), z)ds = \varepsilon(u_0, z) \qquad \forall\, z \in H,\ \forall\, t \in [0,T]. \tag{3.19}$$

Taking the limit as $\varepsilon \searrow 0$ and using (3.2), (3.15), (3.18), we have

$$\int_0^t (w(s) + v(s) - f(s), z)ds = 0 \qquad \forall\, z \in H,\ \forall\, t \in [0,T], \tag{3.20}$$

which is equivalent to (2.1). Thanks to remark 2.6, proposition 2.1 and (2.6), to get (2.2) and (2.3) it suffices to prove that

$$\limsup_{\varepsilon\searrow 0} \int_0^T (w_\varepsilon(t), u_\varepsilon'(t))dt \le \int_0^T (w(t), u'(t))dt, \tag{3.21}$$

$$\limsup_{\varepsilon\searrow 0} \int_0^T (B_\varepsilon u_\varepsilon(t), J_\varepsilon u_\varepsilon(t))dt \le \int_0^T (v(t), u(t))dt. \tag{3.22}$$

Now (3.22) follows directly from (3.15) and (3.17). To show (3.21), we utilize the equations (3.3) and (2.1) to infer that (see also proposition 2.5)

$$\begin{aligned}\int_0^T (w_\varepsilon(t), u_\varepsilon'(t))dt = & -\varepsilon \int_0^T |u_\varepsilon'(t)|^2 dt - \psi_\varepsilon(u_\varepsilon(T)) + \\ & + \psi_\varepsilon(u_0) + \int_0^T (f_\varepsilon(t), u_\varepsilon'(t))dt,\end{aligned} \tag{3.23}$$

$$\int_0^T (w(t), u'(t))dt = -\psi(u(T)) + \psi(u_0) + \int_0^T (f(t), u'(t))dt. \tag{3.24}$$

Hence, owing to (vi), (2.9), (3.2) and (3.18), (3.21) is equivalent to

$$\liminf_{\varepsilon\searrow 0} \psi_\varepsilon(u_\varepsilon(T)) \ge \psi(u(t)). \tag{3.25}$$

But this easily follows from the lower semicontinuity of ψ because of (2.8) and (3.15).

4. Proofs of Theorems 2.2 and 2.3.

First we prove theorem 2.2.

Proof of theorem 2.2. In order to approximate problem (2.5), (2.2), (2.4), we shall use the following simple result.

Remark 4.1. Let X and Y be two Banach spaces such that $X \subset Y$ with dense and continuous inclusion. Then for any $y \in Y$ there exists a sequence $\{y_\varepsilon \in X\}$ such that

$$y_\varepsilon \to y \qquad \text{strongly in } Y \text{ as } \varepsilon \searrow 0, \tag{4.1}$$

(4.2) $$\lim_{\varepsilon\searrow 0} \varepsilon^{1/2} \|y_\varepsilon\|_X = 0.$$

Indeed, given a sequence $\{y_\varepsilon\}$ such that $y_\varepsilon \in X$ for any $\varepsilon > 0$ and verifying (4.1), if $\{y_\varepsilon\}$ does not satisfy (4.2) it is sufficient to select a subsequence which has the desired property.

Lemma 4.1. *Let B, u_0 and f satisfy (viii)-(xii). Then there exist two approximating sequences $\{u_0^\varepsilon\}$ and $\{f_\varepsilon\}$ such that*

(4.3) $$Bu_0^\varepsilon \in H \quad \forall\, \varepsilon > 0 \quad (\text{i.e. } u_0^\varepsilon \in D(B) \text{ in } H),$$

(4.4) $$\begin{cases} u_0^\varepsilon, J_\varepsilon u_0^\varepsilon \to u_0 & \textit{strongly in } V \\ B_\varepsilon u_0^\varepsilon \to Bu_0 & \textit{strongly in } V' \end{cases} \qquad \textit{as } \varepsilon \searrow 0,$$

(4.5) $$f_\varepsilon \in C^1([0,T];H), \qquad f_\varepsilon(0) = f(0) - Bu_0 + B_\varepsilon u_0^\varepsilon,$$

(4.6) $$f_\varepsilon \to f \qquad \textit{strongly in } H^1(0,T;V') \quad \textit{as } \varepsilon \searrow 0,$$

(4.7) $$\lim_{\varepsilon\searrow 0} \varepsilon^{1/2} \left\{ |B_\varepsilon u_0^\varepsilon| + \|f_\varepsilon\|_{C^1([0,T];H)} \right\} = 0,$$

where J_ε, B_ε are defined as in proposition 2.2.

Proof. Set $b_0 = Bu_0 \in V'$ and let $\{b_0^\varepsilon\}$, with $b_0^\varepsilon \in H$ for any $\varepsilon > 0$, be a sequence as in remark 4.1 (it exists since H is dense in V'). Setting $u_0^\varepsilon = B^{-1} b_0^\varepsilon$ (cf. remark 2.2) and $z_\varepsilon = J_\varepsilon u_0^\varepsilon$, by proposition 2.2 we have

(4.8) $$Bz_\varepsilon = B_\varepsilon u_0^\varepsilon, \qquad \frac{1}{\varepsilon}(z_\varepsilon - u_0^\varepsilon) + Bz_\varepsilon = 0.$$

Hence, since B is strongly monotone (cf. (ix)), $u_0^\varepsilon \to u_0$ strongly in V and

(4.9) $$\frac{1}{\varepsilon}|z_\varepsilon - u_0^\varepsilon|^2 + C_6 \|z_\varepsilon - u_0^\varepsilon\|_V^2 \le -(b_0^\varepsilon, z_\varepsilon - u_0^\varepsilon) \le$$
$$\le \frac{1}{2\varepsilon}|z_\varepsilon - u_0^\varepsilon|^2 + \frac{\varepsilon}{2}|b_0^\varepsilon|^2 \qquad \forall\, \varepsilon > 0.$$

Therefore $z_\varepsilon \to u_0$ strongly in V. Using the Lipschitz continuity of B, $B_\varepsilon u_0^\varepsilon = Bz_\varepsilon \to Bu_0$ stongly in V', and so (4.4) is proved. By (4.8) and (4.9) we have

(4.10) $$\varepsilon\, |B_\varepsilon u_0^\varepsilon|^2 = \frac{1}{\varepsilon}\, |z_\varepsilon - u_0^\varepsilon|^2 \to 0 \qquad \text{as } \varepsilon \searrow 0.$$

Taking now a sequence $\{g_\varepsilon \in C^0(0,T;H)\}$ approximating $f' \in L^2(0,T;V')$ in the sense of remark 4.1, it is easy to see that the sequence $\{f_\varepsilon\}$ defined for any $\varepsilon > 0$ by

$$f_\varepsilon(t) := f(0) - Bu_0 + B_\varepsilon u_0^\varepsilon + \int_0^t g_\varepsilon(s)ds, \qquad t \in [0,T],$$

satisfies (4.5)-(4.7) (cf. (xii) and (4.10)).

Consider now the following approximation of problem (2.5), (2.2), (2.4): find u_ε verifying (3.3) and (3.4) for any $t \in [0,T]$, and such that

$$(4.11) \qquad u_\varepsilon(0) = u_0^\varepsilon,$$

where f_ε and u_0^ε are given by lemma 4.1. As we have seen in section 3, this problem has one and only one solution, with u_ε and w_ε given by (3.6) and (3.7). Since $(\varepsilon I + A)^{-1}$ and B_ε are Lipschitz continuous and $f_\varepsilon,\, u_\varepsilon \in C^1([0,T];H)$, here we have the additional regularity

$$(4.12) \qquad u_\varepsilon \in C^{1,1}([0,T];H), \qquad w_\varepsilon,\, B_\varepsilon u_\varepsilon \in C^{0,1}([0,T];H)$$

(cf., e.g., [**9**, Appendice]). Then, differentiating (3.3) with respect to t, we obtain

$$(4.13) \qquad \varepsilon u_\varepsilon''(t) + w_\varepsilon'(t) + (B_\varepsilon u_\varepsilon)'(t) = f_\varepsilon'(t) \qquad \text{for a.e. } t \in]0,T[,$$

and, writing (3.6) and (3.7) for $t = 0$, we have (see also (4.11) and (4.5))

$$(4.14) \qquad u_\varepsilon'(0) = (\varepsilon I + A)^{-1}(f(0) - Bu_0),$$

$$(4.15) \qquad w_\varepsilon(0) = f(0) - Bu_0 - \varepsilon u_\varepsilon'(0).$$

In order to deduce a priori estimates, we take the scalar product of (4.13) with $u_\varepsilon'(t)$ and integrate in time, getting

$$(4.16) \qquad \frac{\varepsilon}{2}|u_\varepsilon'(t)|^2 + (w_\varepsilon'(t), u_\varepsilon'(t)) + \int_0^t ((B_\varepsilon u_\varepsilon)'(s), u_\varepsilon'(s))\, ds \leq$$
$$\leq \frac{\varepsilon}{2}|u_\varepsilon'(0)|^2 + \int_0^t |f_\varepsilon'(s)|\, |u_\varepsilon'(s)|\, ds \qquad \forall\, t \in [0,T].$$

Since $u_\varepsilon - J_\varepsilon u_\varepsilon = \varepsilon B_\varepsilon u_\varepsilon$ and $B_\varepsilon u_\varepsilon = B J_\varepsilon u_\varepsilon$ (cf. proposition 2.2), note that (see, e.g., [**9**, Appendice]) by (ix) we have

$$\int_0^t ((B_\varepsilon u_\varepsilon)'(s), u_\varepsilon'(s))\, ds = \tag{4.17}$$

$$= \int_0^t \lim_{h\searrow 0} \frac{1}{h^2}\{(B_\varepsilon u_\varepsilon(s+h) - B_\varepsilon u_\varepsilon(s), J_\varepsilon u_\varepsilon(s+h) - J_\varepsilon u_\varepsilon(s)) +$$

$$+ \varepsilon |B_\varepsilon u_\varepsilon(s+h) - B_\varepsilon u_\varepsilon(s)|^2\} ds \geq$$

$$\geq \int_0^t \lim_{h\searrow 0} \left\{ C_6 \left\| \frac{J_\varepsilon u_\varepsilon(s+h) - J_\varepsilon u_\varepsilon(s)}{h} \right\|_V^2 + \right.$$

$$\left. + \varepsilon \left| \frac{B_\varepsilon u_\varepsilon(s+h) - B_\varepsilon u_\varepsilon(s)}{h} \right|^2 \right\} ds =$$

$$= \int_0^t \left\{ C_6 \left\|(J_\varepsilon u_\varepsilon)'(s)\right\|_V^2 + \varepsilon \left|(B_\varepsilon u_\varepsilon)'(s)\right|^2 \right\} ds \qquad \forall\, t \in [0, T[.$$

Besides, as $u_\varepsilon'(t) \in A^{-1}(w_\varepsilon(t)) \equiv \partial\varphi^*(w_\varepsilon(t))$ for a.e. $t \in]0, T[$ by (vii), (3.4) and proposition 2.4, with the help of proposition 2.5 we infer

$$(w_\varepsilon'(t), u_\varepsilon'(t)) = \frac{d}{dt}\varphi^*(w_\varepsilon(t)) \qquad \text{for a.e. } t \in]0, T[. \tag{4.18}$$

Then from (4.16)-(4.18) it follows that

$$\frac{\varepsilon}{2}\left|u_\varepsilon'(t)\right|^2 + \varphi^*(w_\varepsilon(t)) + \int_0^t \left\{ C_6 \left\|(J_\varepsilon u_\varepsilon)'(s)\right\|_V^2 + \varepsilon \left|(B_\varepsilon u_\varepsilon)'(s)\right|^2 \right\} ds \leq \tag{4.19}$$

$$\leq \frac{\varepsilon}{2}\left|u_\varepsilon'(0)\right|^2 + \varphi^*(w_\varepsilon(0)) + \int_0^t (f_\varepsilon'(s), u_\varepsilon'(s)) ds \qquad \forall\, t \in [0, T].$$

Since $u_\varepsilon = J_\varepsilon u_\varepsilon + \varepsilon B_\varepsilon u_\varepsilon$ (cf. proposition 2.2), by (4.6) and (4.7) we have

$$\int_0^t (f_\varepsilon'(s), u_\varepsilon'(s)) ds \leq \tag{4.20}$$

$$\leq C_{12} + \int_0^t \left\{ \frac{C_6}{2} \left\|(J_\varepsilon u_\varepsilon)'(s)\right\|_V^2 + \frac{\varepsilon}{2} \left|(B_\varepsilon u_\varepsilon)'(s)\right|^2 \right\} ds \qquad \forall\, t \in [0, T]$$

(recall that we denote by C_i' s constants independent of ε). As $u_\varepsilon'(0) \in \partial\varphi^*(w_\varepsilon(0))$ (cf. (vii), (3.4) and proposition 2.4), by (xii) and (4.15) we obtain

$$\frac{\varepsilon}{2}\left|u_\varepsilon'(0)\right|^2 + \varphi^*(w_\varepsilon(0)) \leq \tag{4.21}$$

$$\le \frac{\varepsilon}{2} |u_\varepsilon'(0)|^2 + \varphi^*(f(0) - Bu_0) + (u_\varepsilon'(0), w_\varepsilon(0) - f(0) + Bu_0) \le C_{13} .$$

From (4.19)-(4.21) and (2.11) it follows that

$$(4.22) \qquad \varepsilon^{1/2} \left\{ \|u_\varepsilon'\|_{C^0([0,T];H)} + \|(B_\varepsilon u_\varepsilon)'\|_{L^2(0,T;H)} \right\} + \|(J_\varepsilon u_\varepsilon)'\|_{L^2(0,T;V)} + \\ + \|w_\varepsilon\|_{C^0([0,T];H)} \le C_{14} ,$$

then by (4.22) and (4.4) we have

$$(4.23) \qquad \|J_\varepsilon u_\varepsilon\|_{H^1(0,T;V)} \le C_{15} ,$$

and by the Lipschitz continuity of B (cf. (viii))

$$(4.24) \quad \|(B_\varepsilon u_\varepsilon)'\|^2_{L^2(0,T;V')} \le \lim_{h \searrow 0} \int_0^{T-h} \frac{C_5^2}{h^2} \|J_\varepsilon u_\varepsilon(s+h) - J_\varepsilon u_\varepsilon(s)\|_V^2 \, dt \le \\ \le (C_5 C_{14})^2 .$$

Finally by comparison in (3.3) we get

$$(4.25) \qquad \|B_\varepsilon u_\varepsilon - f_\varepsilon\|_{C^0([0,T];H)} \le C_{16} .$$

Owing to (4.22)-(4.25), (4.6) we have that, possibly extracting subsequences,

$$(4.26) \qquad J_\varepsilon u_\varepsilon \to u \qquad \text{weakly in } H^1(0,T;V),$$

$$(4.27) \qquad w_\varepsilon \to w \qquad \text{weakly star in } L^\infty(0,T;H),$$

$$(4.28) \quad B_\varepsilon u_\varepsilon - f_\varepsilon \to v - f \qquad \text{weakly star in } L^\infty(0,T;H) \cap H^1(0,T;V').$$

By using (4.26)-(4.28) and standard compactness results including Aubin's Lemma (see, e.g., [**18**, p. 58]), we deduce that

$$(4.29) \qquad J_\varepsilon u_\varepsilon \to u \qquad \text{strongly in } C^0([0,T];H),$$

$$(4.30) \qquad B_\varepsilon u_\varepsilon - f_\varepsilon \to v - f \qquad \text{strongly in } L^2(0,T;V'),$$

and by (3.16), (4.6), (4.7) and (4.25),

$$(4.31) \qquad u_\varepsilon \to u \qquad \text{strongly in } C^0([0,T];H),$$

$$(4.32) \qquad B_\varepsilon u_\varepsilon \to v \qquad \text{strongly in } L^2(0,T;V').$$

Recalling now the proof of theorem 2.1, here the passage to the limit is straightforward, once we have shown that

$$(4.33) \qquad v(t) = Bu(t) \qquad \text{for a.e. } t \in]0,T[$$

and that (3.21) holds. But (4.33) follows from (3.22) (consequence of (4.26) and (4.32)) and from the Lipschitz continuity of B (see, e.g., [**18**, p. 161]). Besides, using (3.3) and (3.16), we have

$$(4.34) \qquad \limsup_{\varepsilon\searrow 0} \int_0^T (w_\varepsilon(t), u_\varepsilon'(t))dt =$$

$$= -\liminf_{\varepsilon\searrow 0} \int_0^T \{\varepsilon|u_\varepsilon'(t)|^2 + (B_\varepsilon u_\varepsilon(t) - f_\varepsilon(t), u_\varepsilon'(t))\}dt \le$$

$$\le -\liminf_{\varepsilon\searrow 0} \left\{ \int_0^T (B_\varepsilon u_\varepsilon(t) - f_\varepsilon(t), (J_\varepsilon u_\varepsilon)'(t))\, dt - \frac{\varepsilon}{2}|B_\varepsilon u_0^\varepsilon|^2 + \right.$$

$$\left. + \frac{\varepsilon}{2}|B_\varepsilon u_\varepsilon(T)|^2 - \int_0^T \varepsilon\,(f_\varepsilon(t), (B_\varepsilon u_\varepsilon)'(t))\, dt \right\}.$$

Then, by (4.6), (4.7), (4.22), (4.26) and (4.32), we deduce that

$$(4.35) \qquad \limsup_{\varepsilon\searrow 0} \int_0^T (w_\varepsilon(t), u_\varepsilon'(t))dt \le -\int_0^T (Bu(t) - f(t), u'(t))dt,$$

that is (3.21), since $w = f - Bu$ a.e. in $]0,T[$ by (2.5).

Proof of theorem 2.3. This follows in parts the previous one, hence we shall just point out the main differences.
As $u_0 \in D(B)$, in (4.11) we take $u_0^\varepsilon = u_0$ and

$$f_\varepsilon(t) := f(0) - B^0 u_0 + B_\varepsilon u_0 + \int_0^t g_\varepsilon(s)ds, \qquad t \in [0,T],$$

for any $\varepsilon > 0$, where $g_\varepsilon \in C^0([0,T];H)$ approximates $f' \in L^2(0,T;V')$ in the sense of remark 4.1. Since $\widetilde{B}$ is strongly monotone, by (2.7) we have that (4.3)-(4.7) still hold with Bu_0 replaced by $B^0 u_0$. The estimates (4.22), (4.23) and (4.25) (but not (4.24)) are deduced in the same way as before. Hence we still have (4.26), (4.27), (4.29), (4.31) and

$$(4.36) \qquad B_\varepsilon u_\varepsilon \to v \qquad \text{weakly star in } L^\infty(0,T;V'),$$

$$B_\varepsilon u_\varepsilon - f_\varepsilon \;\to\; v - f \qquad \text{weakly star in } L^\infty(0,T;H). \tag{4.37}$$

In the passage to the limit,

$$v(t) \in \widetilde{B}u(t) \qquad \text{for a.e. } t \in]0,T[\tag{4.38}$$

is a consequence of proposition 2.6, and of the fact that

$$\limsup_{\varepsilon \searrow 0} \int_0^T (B_\varepsilon u_\varepsilon(t), J_\varepsilon u_\varepsilon(t))dt = \tag{4.39}$$

$$= \limsup_{\varepsilon \searrow 0} \int_0^T (B_\varepsilon u_\varepsilon(t) - f_\varepsilon(t), J_\varepsilon u_\varepsilon(t))dt + \lim_{\varepsilon \searrow 0} \int_0^T (f_\varepsilon(t), J_\varepsilon u_\varepsilon(t))dt \le$$

$$\le \int_0^T (v(t), u(t))dt$$

by (4.37), (4.29), (4.6) and (4.26). In order to obtain (3.21), note that (iii), propositions 2.2, 2.3 and 2.5 yield

$$\liminf_{\varepsilon \searrow 0} \int_0^T (B_\varepsilon u_\varepsilon(t) - f_\varepsilon(t), u_\varepsilon'(t))dt =$$

$$= \liminf_{\varepsilon \searrow 0} \Big\{ \psi(J_\varepsilon u_\varepsilon(T)) - \psi(J_\varepsilon u_0) - \frac{\varepsilon}{2}|B_\varepsilon u_0|^2 + \frac{\varepsilon}{2}|B_\varepsilon u_\varepsilon(T)|^2 - \\ - \int_0^T (f_\varepsilon(t), (J_\varepsilon u_\varepsilon)'(t) + \varepsilon(B_\varepsilon u_\varepsilon)'(t))\, dt \Big\}.$$

Then, using the lower semicontinuity of ψ, (4.29), (4.6), (4.7) and (4.22), we have

(4.40)

$$\liminf_{\varepsilon \searrow 0} \int_0^T (B_\varepsilon u_\varepsilon(t) - f_\varepsilon(t), u_\varepsilon'(t))dt \ge \psi(u(T)) - \psi(u_0) - \int_0^T (f(t), u'(t))dt$$

and we can conclude as in the proof of theorem 3.1.

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