GLOBAL SOLUTIONS TO A GENERALIZED CAHN-HILLIARD EQUATION WITH VISCOSITY

ELENA BONETTI
Dipartimento di Matematica, Università di Pavia
Via Ferrata, 1, I-27100 Pavia, Italy

WOLFGANG DREYER
WIAS – Weiestrass Institut für Angewandte Analysis und Stochastik
Mohrenstrasse, 39, D-10117 Berlin, Germany

GIULIO SCHIMPERNA
Dipartimento di Matematica, Università di Pavia
Via Ferrata, 1, I-27100 Pavia, Italy

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Abstract. We address a viscous Cahn-Hilliard equation describing the phase separation process in a binary alloy. Such a material is subject to the influence of internal and external mechanical stresses, whose contribution is assumed to be known. The physical modelling refers to a recent work by Dreyer and Müller. The main features and difficulties of this model are given by a highly nonlinear fourth order elliptic term, a strong constraint imposed by the presence of a double obstacle energy potential, and the dependence of the mobility matrix on the concentration variable (i.e., the unknown of the problem). We are able to prove global existence of solutions of the corresponding initial boundary value problem by using approximation and compactness tools. Hence, we perform some asymptotic analysis of the problem and obtain, at the limit, two models of independent physical interest.

1. Introduction

In this paper we study a model describing the isothermal diffusive phase separation in a binary metallic alloy filling a smooth container \( \Omega \subset \mathbb{R}^d \), \( 1 \leq d \leq 3 \), during a finite time interval \([0, T]\). The physical modelling refers to a recent work by Dreyer and Müller [12] that analyzes the thermomechanical behavior of a metallic solder composed of tin and lead. Indeed,
practical experiments show that if such a mixture is cooled below a critical temperature $\theta_c$, then the alloy evolves from a stable and uniform mixed state to states with spatially separated phases. More precisely, a sequence of two phenomena occurs. At a first step, some regions of one pure or almost pure phase tend to nucleate inside the mixture (phase separation), while on a larger time scale those regions start growing and possibly assume a geometric structure characteristic of the solder. This second process is usually called \textit{coarsening}.

Hence, taking the temperature $\theta$ as a constant datum (assumed $< \theta_c$, of course), the evolution process can be described by just one state variable – name it $\chi$ – that is related to the volumetric fractions of the two “phases”. Indeed, as it is physically consistent to assume that no void nor overlapping can occur between the phases, we require that

$$-1 \leq \chi \leq 1,$$

where we have to notice that the pure configurations, given by $\chi = \pm 1$, do not really account for pure tin or pure lead, but just correspond to two thermomechanical states (given by the different crystalline structures of the components) proper of the solder, where, anyway, either of the metals is percentually prevailing [12, 14].

The natural mathematical setting for describing the kinetics of this decomposition in terms of the concentration variable $\chi$ refers to the so-called Cahn-Hilliard equation [9]. Nonetheless, the model we address has a far more detailed structure, mainly as it has to account also for the internal and external mechanical forces acting on $\Omega$, which are involved in the diffusion process of $\chi$. In particular, these forces collect the contribution of microscopical movements and external traction. Thus, in the original model of [12], a Cahn-Hilliard type equation for $\chi$ is coupled with a standard equilibrium equation for the strain fields. In addition, in the resulting system most of the physical coefficients are assumed to be strongly dependent on the concentration, as it will be detailed in the sequel.

As usual, a phenomenological theory describing the phase separation process is provided by considering a Ginzburg-Landau free energy functional $\mathcal{G}$, that is given by the sum of two contributions: a first term, say $\psi$, which represents the homogeneous free energy, and a gradient term accounting for the surface energy of the interfaces separating the phases. A typical expression for this kind of energy functional is the following one,

$$\mathcal{G}(\chi) = \frac{a}{2} \int_{\Omega} |\nabla \chi|^2 + \int_{\Omega} \psi(\chi),$$

(1.2)
under the constraint that the mean value of $\chi$ on $\Omega$ is fixed. It is well-known that the square of the coefficient $a$, which accounts for the surface tension, is proportional to the thickness of the phase transition layer.

In this setting, one can write the energy balance system – that has to account for the conservation of $\chi$ – by stating a gradient flow problem,

$$\partial_t \chi = \text{div}(M(\chi)\nabla w),$$  \hspace{1cm} (1.3)

where the generalized chemical potential $w$ is simply given by $w = \partial G/\partial \chi$. In the model by Dreyer and Müller, however, the coefficient $a$ is no longer constant since it is assumed to be a convex combination with respect to $\chi$ of the coefficients characterizing the surface tensions of the two pure phases. Hence, performing some simplifications as in [12, Appendix], we can deal with the following expression for $w$

$$w = -a(\chi)\Delta \chi + \psi'(\chi).$$  \hspace{1cm} (1.4)

Note indeed that a strong nonlinearity in the highest order diffusion term results from the coupling of (1.3) and (1.4).

By assuming the physically meaningful no-flux boundary conditions

$$M(\chi)\nabla w \cdot \mathbf{n} = \nabla \chi \cdot \mathbf{n} = 0,$$  \hspace{1cm} (1.5)

where $\mathbf{n}$ denotes the outward normal unit vector to the boundary $\Gamma$ of the domain $\Omega$, we prescribe that the total mass flux through $\Gamma$ is zero, provided that the local mass flux density $J$ is given by the natural expression

$$J = -M(\chi)\nabla w.$$  \hspace{1cm} (1.6)

Hence, by (1.3) and (1.5), $\chi$ turns out to be a conserved order parameter.

As we have already specified, the model by Dreyer and Müller associates to (1.3–1.4) the balance equation of movements which, under the quasi-static assumption and neglecting any volume forces, turns out to be

$$\text{div}\sigma = 0 \quad \text{in} \quad \Omega, \quad \text{coupled, e.g., with} \quad \sigma \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma.$$  \hspace{1cm} (1.7)

Here, the stress tensor $\sigma$ is related to the linearized strain tensor $\varepsilon(\mathbf{u})$ as usual by $\sigma = \partial \psi/\partial \varepsilon$ and now $\psi$ has to collect also the elastic contributions, of course. Thus, in order to detail the equations (1.3), (1.4), and (1.7) in terms of $\mathbf{u}$ and $\chi$, we should specify the form of the free energy $\psi$. Nonetheless, for the sake of simplicity, we prefer not to insist on this point and refer again to [12] for further aspects of the modelling. In addition, the equation (1.7), that turns out to assume the form (see also [7, Sec. 2])

$$\text{div}(K(\chi)\nabla \mathbf{u} - Y(\chi)) = 0,$$  \hspace{1cm} (1.8)
will not be analyzed in the present paper. Note anyway that in the expression above the matrices $K$ (positive definite) and $Y$ are assumed to be linear functions of the concentration.

Let us now state the mathematical formulation of the equation for the chemical potential $w$ (see (1.4)) in detail. Actually, in order to guarantee (1.1), some constraint has to be included in the energy $\psi$. Hence, as in [22] we prescribe a double obstacle energy potential in view of using the subdifferential approach of, e.g., [17] for the mathematical analysis of the resulting equations. On a practical viewpoint, the idea is to consider $\psi$ defined for any value of $\chi$, even for those that are physically unfeasible, but to assign the value $+\infty$ to the free energy when $|\chi| > 1$. Moreover, we also have to let the chemical potential $w$ account for the contribution of the mechanical forces. In particular, as in [7], we assume it contain two nonlinear forcing terms, coupling displacements and concentration, the latter of which is quadratic with respect to the strain. This ansatz is motivated by the dependence of the stiffness tensor on $\chi$. Accounting for these considerations, the precise form of $w$ turns out to be the following,

$$w \in -a(\chi)\Delta \chi + \partial I(\chi) + \tilde{g}(\chi) + z(\chi)\varepsilon(u) + \frac{1}{2}\varepsilon(u)C'(\chi)\varepsilon(u),$$

where $C'$ is the derivative of the stiffness tensor $C$ with respect to $\chi$, $\tilde{g}$ corresponds to the derivative of the polynomial part of the free energy, $z(\chi)$ is a linear tensor valued function, and $\partial I$ is the subdifferential of the indicator function $I$ of the interval $[-1, 1]$ and it accounts for the required constraint (1.1) for the phases. Namely, we have that $I(r) = 0$ if $r \in [-1, 1]$ and $I(r) = +\infty$ otherwise, so that $\partial I(r) = (-\infty, 0]$ if $r = -1$, $\partial I(r) = \{0\}$ if $-1 < r < 1$, $\partial I(r) = [0, +\infty)$ if $r = 1$, and $\partial I(r)$ is the empty set otherwise.

We note that the resulting system given by (1.3), (1.8), and (1.9) (coupled with suitable initial and boundary conditions) where the mobility and the surface tension do depend on $\chi$, is highly nonlinear and, as far as we know, no related existence result for it has been proved yet. Actually, the paper [7] deals with a slightly simplified model, that keeps the dependence of $a$ on $\chi$ in (1.4), but assumes the mobility matrix $M$ not depend on $\chi$ and also takes into account some viscosity effect in the dynamics of the diffusion process. Namely, a term $\partial_t \chi$, possibly multiplied by a small positive parameter, is added in the above expression of $w$ (1.9) (cf. [21]). Nonetheless, let us note that the resulting model retains its physical consistence and is in accordance
with experiments, at least on a small time scale. In particular, we recall a paper by Gurtin [16] where the author relies on a proposed micro-force balance to show that this smoothing term is thermodynamically consistent. Under the previous assumptions, the authors of [7] are able to prove existence of a unique solution in one space dimension and some partial results in dimension two, holding provided the quadratic term in $\varepsilon(u)$ is neglected in (1.9). In this concern, we also have to quote the results of Garcke et al. [14, 15], who deal with a generalization of the system given by (1.3), (1.8), and (1.9) to the case of a multicomponent alloy. They are able to get existence and uniqueness for the non-viscous problem, in the case when both $M$ and $a$ do not depend on $\chi$ and the quadratic term involving $\varepsilon(u)$ is present. Note that the paper [15] also describes some numerical simulations. Finally, we have to mention a recent work by Miranville [20], studying a coupling between the standard Cahn-Hilliard and elasticity systems and hence considering a much more regular case.

Due to the mathematical difficulties of our complete system, in this paper we restrict our analysis to equations (1.3) and (1.9), actually assuming the strain dependent forcing term in (1.9) as a given and sufficiently regular datum. In this setting, we are able to keep the dependence on $\chi$ of both $M$ and $a$, which is a physically significant characteristic of the model. Nonetheless, at a first study, we also retain the viscosity term in the chemical potential $w$. This yields the following equations,

$$\partial_t \chi - \text{div}(M(\chi) \nabla w) = 0, \quad (1.10)$$

$$w = \partial_t \chi - a(\chi) \Delta \chi + g(\chi) + \partial I(\chi) + f, \quad (1.11)$$

coupled with (1.5), where $\partial I$ has the same meaning as in (1.9), and the other mechanically induced contributions are collected by the sum $g + f$. As the reader can see, relations (1.10–1.11) can be included in the framework of the (viscous) Cahn-Hilliard system [9]. A comprehensive review of related mathematical and numerical results is reported in [5, 6]. We remark that in our setting a double obstacle potential is imposed to the variable $\chi$ as in [22, 17]. Moreover, we should point out that the dependence of the mobility matrix $M$ on concentration, prescribed by Cahn and Hilliard themselves in [9], has been already considered in this framework by [10, 13], the latter paper including the case when it can be degenerate.

Hence, in relation to [7], we stress once more that our system is rather different since $M$ in (1.10) may depend on concentration. Nonetheless, note that we require it cannot degenerate as it was assumed in [13]. On the other hand, we extend in some sense the previous approaches as we assume the
coefficient $a$ of the interfacial energy be strictly positive but depending on the concentration variable $\chi$. In this setting, we are able to prove an existence result for an abstract version of (1.10–1.11) in the three dimensional case by use of a Faedo-Galerkin approximation and an a priori estimates – passage to the limit procedure. However, we have to notice that no uniqueness result seems to hold for our system, mainly due to the dependence on $\chi$ of the mobility $M$ (cf. also [13]).

On a second step, we study the asymptotic behavior of the solutions as some physically significant parameters are let tend to zero. More precisely, for any $\varepsilon > 0$ we can replace (1.11) by the following equivalent relation

$$w = \varepsilon^{\lambda} \partial_t \chi - \varepsilon^{\nu} a_\varepsilon(\chi) \Delta \chi + g(\chi) + \partial I_{[-1,1]}(\chi) + f,$$  \hspace{1cm} (1.12)

where $\lambda$ and $\nu$ are given nonnegative parameters, and then study two singular limit problems of the system (1.10) and (1.12). Indeed, we let $\varepsilon \searrow 0$ and consider the two cases of ($\lambda = 0$, $\nu = 1$) and ($\lambda = 1$, $\nu = 0$). It is understood that $a_\varepsilon(\cdot)$ has the same properties of $a(\cdot)$ for any $\varepsilon > 0$, and in the second problem we are supposing it uniformly tend to a positive constant $a_0$. In both cases, we are able to prove the weak convergence of the solutions, holding up to the extraction of suitable subsequences, to some new functions solving the natural limit problems.

Let us note that these investigations turn out to be meaningful both from a mathematical and a physical point of view, due to the relevance of the limit problems. Indeed, in the first case ($\lambda = 0$, $\nu = 1$) we are able to get a thermodynamically consistent sharp interface limit still accounting for a dissipative effect thanks to the viscosity term in (1.12); in the case of ($\lambda = 1$, $\nu = 0$), instead, we derive a variant of the Cahn-Hilliard equation with nonconstant mobility studied in [13]. However, we have to note that both these asymptotic results are proved under stronger regularity assumptions on the data $f, g$ with respect to the hypotheses of the original problem.

Let us now give the outline of the paper. In the next section we provide some analytical preliminaries that are required for stating the precise mathematical abstract formulation of the problem. This is presented in Section 3 together with our main existence result. In Section 4, we standardly approximate the system by use of the Yosida approximation of $\partial I$ and then show the local solvability of the regularized problem through a Faedo-Galerkin scheme. In Section 5, we perform suitable a priori estimates not depending on the approximating parameters, in order to pass to the limit by compactness arguments. Finally, in Section 6, the above two singular limit problems
are considered and the related convergence results are established again by compactness methods.

We finally thank Professor Gianni Gilardi and Dr. Ulisse Stefanelli for the helpful discussions especially concerning the mathematical approximation of the problem.

2. Mathematical preliminaries

In order to deal with a rigorous variational formulation of the system (1.10–1.11), we first need to introduce some notations and mathematical tools. Let \( \Omega \subset \mathbb{R}^d \), \( 1 \leq d \leq 3 \), be a smooth, bounded, and connected domain and let \( T > 0 \) be an assigned final time. Set also \( \Gamma := \partial \Omega \), \( \Sigma := \Gamma \times (0, T) \), \( Q_t := \Omega \times (0, t) \) for \( t \in (0, T] \), and \( Q := Q_T \). Then, we can introduce the Hilbert spaces \( H := L^2(\Omega) \) and \( V := H^1(\Omega) \). The latter space is endowed with the usual scalar product

\[
\left( \langle v, w \rangle \right) := \int_{\Omega} \nabla v \cdot \nabla w \, dx.
\]

We identify \( H \) with its dual space, in order that the compact inclusion \( H \subset V' \) holds and \((V, H, V')\) form a Hilbert triplet [18, p. 202]. We also denote by \((\cdot, \cdot)\) the scalar product of both \( H \) and \( H^d \) and by \(|\cdot|\) the associated norm. Finally, we indicate by \( \langle \cdot, \cdot \rangle \) the duality pairing between \( V' \) and \( V \).

Due to the Neumann conditions (1.5), we will be concerned with functionals with zero mean value. Thus, for any \( \zeta \in V' \), let us set

\[
\zeta_\Omega := \frac{1}{|\Omega|} \langle \zeta, 1 \rangle; \quad V'_0 := \{ \zeta \in V' : \zeta_\Omega = 0 \}, \quad V_0 := V \cap V'_0. \tag{2.3}
\]

We remark that the above notation \( V'_0 \) is suggested just by the sake of convenience; \( V'_0 \) is merely seen as a (closed) subspace of \( V' \). Hence, \( V_0 \) and \( V'_0 \) in the sequel will be intended to inherit their norms from \( V \) and \( V' \), respectively.

**Definition 2.1.** For any \( 0 < \alpha \leq \mu \), we indicate by \( \mathcal{M}(\alpha, \mu) \) the class of the \( d \times d \) symmetric matrices specified by

\[
D \in \mathcal{M}(\alpha, \mu) \iff \alpha |\xi|^2 \leq D \xi \cdot \xi \leq \mu |\xi|^2 \quad \text{for all} \; \xi \in \mathbb{R}^d. \tag{2.4}
\]

Let now and in the sequel of this section

\[
M : \Omega \to \mathcal{M}(\alpha, \mu) \tag{2.5}
\]
be a measurable field of symmetric matrices. (2.6)

Then, we can associate to $M$ the elliptic operator

$$A_M : V \rightarrow V', \quad \langle A_M u, v \rangle := \int_{\Omega} M \nabla u \cdot \nabla v \, dx$$

for all $u, v \in V$; (2.7)

in particular, if $I$ is the identity matrix a.e. in $\Omega$, we put $A := A_I$.

Clearly, $A_M$ maps $V$ onto the space $V_0'$ of zero-mean valued functionals.

Since the kernel of $A_M$ consists of the a.e.-constant functions, it follows that the restriction of $A_M$ to $V_0$ is an isomorphism of $V_0$ onto $V_0'$. Let us denote by $N_M$ its inverse. By the symmetry of $A_M$, one can readily check that for any $v \in V$ and $\zeta \in V_0'$ there holds

$$\langle A_M v, N_M \zeta \rangle = \langle A_M N_M \zeta, v \rangle = \langle \zeta, v \rangle.$$ (2.8)

We also recall the Poincaré-Wirtinger inequality in the form

$$|v|^2 \leq C_\Omega |\nabla v|^2$$

for any $v \in V_0$, (2.9)

where the constant $C_\Omega > 0$ depends only on the domain $\Omega$. We can prove the following preliminary result:

**Lemma 2.2.** For all $\zeta \in V_0'$ and for all $M$ fulfilling (2.5) and (2.6), we have that

$$\|N_M \zeta\|_V \leq C_{\alpha, \Omega} \|\zeta\|_{V'},$$ (2.10)

where $C_{\alpha, \Omega} > 0$ only depends on $\alpha, \Omega$.

**Proof.** Set $v = N_M \zeta \in V_0$, so that $A_M v = \zeta$. Then, by virtue of (2.9) and (2.1) (cf. (2.4)), we have

$$\|\zeta\|_{V'} \|v\|_V \geq \langle \zeta, v \rangle = \langle A_M v, v \rangle = \int_{\Omega} M \nabla v \cdot \nabla v \, dx$$

$$\geq \alpha |\nabla v|^2 \geq \frac{\alpha}{1 + C_\Omega} \|v\|_V^2. \quad \Box$$ (2.11)

As one would expect, $N_M$ enjoys the following coercivity property,

**Lemma 2.3.** There exists $C_{\alpha, \Omega, \mu} > 0$ such that, for any $M$ fulfilling (2.5–2.6) and $\zeta \in V_0'$, there holds

$$\langle \zeta, N_M \zeta \rangle \geq C_{\alpha, \Omega, \mu} \|\zeta\|^2_{V'}.$$ (2.12)

**Proof.** Setting again $v := N_M \zeta$ and proceeding as in the previous proof, we obtain (cf. (2.11))

$$\langle \zeta, N_M \zeta \rangle \geq \frac{\alpha}{1 + C_\Omega} \|N_M \zeta\|^2_{V'}.$$ (2.13)
Since for any $z \in V$,
\[
    \langle \zeta, z \rangle = \langle A_M v, z \rangle \leq \mu |\nabla v| |\nabla z| \leq \mu \|v\|_V \|z\|_V,
\]
then
\[
    \|\zeta\|_{V'} \leq \mu \|v\|_V = \mu \|N_M \zeta\|_V,
\]
from which, owing to (2.13), we can get (2.12) taking, e.g., $C_{\alpha,\Omega,\mu} = \alpha/((1 + C_{\Omega})\mu^2)$.

For the reader’s convenience, let us now recall some properties of maximal monotone operators in Hilbert spaces. Referring to the classical textbooks by Barbu [2] and Brezis [8] for the general definitions, we just focus our attention on maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$. Hence, let $j : \mathbb{R} \to [0, +\infty]$ be a given convex and l.s.c. (lower semicontinuous) function. We define the domain of $j$ as the set $D(j) := \{r \in \mathbb{R} : j(r) < +\infty\}$ and suppose for simplicity that $j(0) = 0$, so that $j$ is proper ($D(j) \neq \emptyset$). We remind that the subdifferential $\partial j$ [8, Ex. 2.1.3, p. 21] of $j$ is a maximal monotone, possibly multivalued, operator on $\mathbb{R}$.

Given a bounded set $\Omega \subset \mathbb{R}^d$, $j$ induces in a natural way a convex function $j$ on the infinite-dimensional space $H = L^2(\Omega)$, i.e.,
\[
    J(u) := \begin{cases} 
        \int_\Omega j(u(x)) \, dx & \text{if } j(u) \in L^1(\Omega), \\
        +\infty & \text{otherwise},
    \end{cases}
\]
whose subdifferential can be characterized as follows [8, Prop. 2.16, p. 47 and Ex. 2.1.3, p. 21],
\[
    w \in \partial J(u) \iff w(x) \in \partial j(u(x)) \text{ for a.e. } x \in \Omega. \quad (2.15)
\]

We conclude by stating two lemmas that extend to any maximal monotone graph $\beta$ two properties that are obvious in the case when $\beta$ is a regular function. First, we have [8, Lemme 3.3, p. 73]:

**Lemma 2.4.** Let $T > 0$, $u \in H^1(0, T; H)$, $\xi \in L^2(0, T; H)$, $\xi(t) \in \partial J(u(t))$ a.e. in $(0, T)$. Then, $J \circ u \in W^{1,1}(0, T)$ and
\[
    (\xi(t), \partial_t u(t)) = \frac{d}{dt} J(u)(t) \quad \text{for a.e. } t \in (0, T). \quad (2.16)
\]

Finally, for a proof of the next very natural property see [19, Lemma 4.1].

**Lemma 2.5.** Let $u, \xi \in H$, $\xi \in \partial J(u)$. Let also $0 < \alpha \leq \mu$ and let $M$ be specified by (2.5) and (2.6) and $A_M$ by (2.7). Finally, let $A_M u \in H$. Then $(A_M u, \xi) \geq 0$. 
3. Existence result

Using the notations introduced in the previous section, we can state at once our basic hypotheses on the data. In particular, we assume

\begin{align}
& f \in L^2(0, T; H), \\
& \chi_0 \in V, \quad \text{with } \chi_0(x) \in [-1, 1] \text{ for a.e. } x \in \Omega, \\
& \chi_\Omega := (\chi_0)_\Omega \in (-1, 1), \\
& g \in C^1([-1, 1]; \mathbb{R}), \\
& B_0, B_1 \in \mathcal{M}(\alpha, \mu), \\
& a_0, a_1 \in \mathbb{R}, \quad \alpha \leq a_0 \leq \mu, \quad \alpha \leq a_1 \leq \mu,
\end{align}

where \(0 < \alpha \leq \mu\) are given constants. We denote by \(j = I_{[-1,1]}\) the indicator function of the closed interval \([-1,1]\) and set \(\beta := \partial j\) in \(\mathbb{R} \times \mathbb{R}\). Hence, defining \(J\) as in (2.14), we notice that assumption (3.2) entails \(J(\chi_0) = 0\) and, in particular, \(\chi_0 \in D(J)\). In addition, note that (3.3) means our analysis does not consider initial configurations characterized by \(\chi_0 \equiv 1\) or \(\chi_0 \equiv -1\) a.e. in \(\Omega\).

For any measurable \(v : Q \to \mathbb{R}\), we define the functions \(a = a(v) : Q \to \mathbb{R}\) and \(B = B(v) : Q \to \mathbb{R}^{d \times d}\) as follows (cf. (3.5–3.6)),

\begin{align}
& a(v(x,t)) := a_0 \quad \text{and} \quad B(v(x,t)) := B_0 \quad \text{if } v(x,t) < -1, \\
& a(v(x,t)) := a_1 \left(\frac{1 + v(x,t)}{2}\right) + a_0 \left(\frac{1 - v(x,t)}{2}\right) \quad \text{and} \quad
\end{align}

\begin{align}
& B(v(x,t)) := B_1 \left(\frac{1 + v(x,t)}{2}\right) + B_0 \left(\frac{1 - v(x,t)}{2}\right) \quad \text{if } -1 \leq v(x,t) \leq 1, \\
& a(v(x,t)) := a_1 \quad \text{and} \quad B(v(x,t)) := B_1 \quad \text{if } v(x,t) > 1.
\end{align}

Moreover, following the notation introduced in (2.7), for a.e. \(t \in (0, T)\) we associate to \(B(v)(\cdot, t)\) the symmetric elliptic operator \(B_v = B_v(t) : V \to V'\) specified by \(B_v := A_{B(v)}\). As we have already pointed out in the previous section, for a.e. \(t \in (0, T)\) its restriction to the space \(V_0\) turns out to be an isomorphism of \(V_0\) onto \(V_0'\); hence, we can denote its inverse map by \(\mathcal{N}_v = \mathcal{N}_{B(v)} : V_0' \to V_0\) (cf. (2.8)). The measurability of \(v\) guarantees that \(B(v) \in L^\infty(0, T; \mathbb{R}^{d \times d})\) for any \(v\) and it is also clear that \(B(v)(x, t) \in \mathcal{M}(\alpha, \mu)\) for a.e. \((x, t) \in Q\).

In the following lemma, \(\mathcal{L}(X, Y)\) denotes the space of the linear and continuous operators from the Banach spaces \(X\) to the Banach space \(Y\), endowed with the usual supremum norm.
Lemma 3.1. For any measurable function \( v : Q \rightarrow \mathbb{R} \) and any \( p \in [1, +\infty] \), we have that the operators
\[
B_v : L^p(0,T;V) \rightarrow L^p(0,T;V_0') \quad \text{and} \quad \mathcal{N}_v : L^p(0,T;V_0) \rightarrow L^p(0,T;V_0)
\]
defined by \((B_vz)(t) := B_v(t)z(t)\) and \((\mathcal{N}_v\zeta)(t) := \mathcal{N}_v(t)\zeta(t)\) for \( z \in V \), \( \zeta \in V_0' \), and a.e. \( t \in (0,T) \), are well defined. In addition \( B_v \) is a (surjective) isomorphism from \( L^p(0,T;V_0) \) onto \( L^p(0,T;V_0') \). Finally, we have that
\[
\|B_v\|_{L^p(0,T;V),L^p(0,T;V_0')} \leq \mu, \quad \|\mathcal{N}_v\|_{L^p(0,T;V_0'),L^p(0,T;V_0)} \leq C_{\alpha,\Omega}, \quad (3.8)
\]
where the constant \( C_{\alpha,\Omega} \) is the same as in (2.10) and, in particular, does not depend on the function \( v \).

Proof. First of all, we prove that \( B_vz \) is measurable. To this aim, we notice that, for any \( u \in V \), we can write
\[
\langle B_vz, u \rangle = \int_\Omega B(v) \nabla z \cdot \nabla u \, dx \quad \text{a.e. in} \ (0,T)
\]
and this function is measurable in time, of course. As a consequence, \( B_vz \) is weakly measurable. Since \( V_0' \) is a separable space, the strong measurability follows by Pettis’ theorem \([8, \text{p. 137}]\). Now, by a direct check, it is easy to see that \( B_v \) maps \( L^p(0,T;V) \) onto \( L^p(0,T;V_0') \) and that its kernel consists precisely of the functions \( v \in L^p(0,T;V) \) such that \( v \) is constant in space a.e. in time. A passage to the quotient readily yields that \( B_v \) is an isomorphism of \( L^p(0,T;V_0) \) onto \( L^p(0,T;V_0') \) and also the first of (3.8) is now immediate. Given \( \zeta \in L^p(0,T;V_0') \), of course \( \mathcal{N}_v\zeta \) is measurable, since \( \mathcal{N}_v \) turns out to be the inverse of \( B_v \) (viewed here as a map of \( L^p(0,T;V_0') \) onto \( L^p(0,T;V_0) \)). Finally, the second of (3.8) is a direct consequence of (2.10). \( \square \)

We are now able to state our main existence result:

Theorem 3.2. Assume the hypotheses (3.1–3.6) on data. Then, there exists a triplet of functions \((\chi, w, \xi)\), with
\[
\chi \in H^1(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \quad (3.9)
\]
\[
-1 \leq \chi \leq 1 \quad \text{a.e. in} \ Q, \quad (3.10)
\]
\[
w \in L^2(0,T;V), \quad (3.11)
\]
\[
\xi \in L^2(0,T;H), \quad (3.12)
\]

satisfying the system
\[
\partial_t \chi + B_\chi w = 0 \quad \text{in} \ L^2(0,T;V'), \quad (3.13)
\]
\[
w = \partial_t \chi + a(\chi)AX + \xi + g(\chi) + f \quad \text{a.e. in} \ Q, \quad (3.14)
\]
\[ \xi \in \beta(\chi) \text{ a.e. in } Q, \]  
\[ \chi(0) = \chi_0 \text{ a.e. in } \Omega. \]  

Moreover, \( \chi \) fulfills the relation

\[ (\chi(t))_{\Omega} = \chi_{\Omega} \text{ for a.e. } t \in (0, T). \]  

The proof of this theorem will be accomplished throughout the next two sections. For the present, we observe that the constraint (3.10) guarantees that \( \chi \) is physically meaningful as a concentration variable in the sense specified in the introduction.

**Remark 3.3.** Let us point out that the regularity of \( w \) (3.11) might be improved, at least in case \( B_0, B_1 \) are reduced to be scalar. Indeed, using (3.9) and recalling (3.7), it is not difficult to prove that \( B(\chi) \) is Hölder continuous in \( \Omega \) for a.e. \( t \in (0, T) \). Then, by the definition of \( B_\chi \) and taking advantage of the result [11, Theorem 1.2, p. 210], still holding in our homogeneous Neumann case, one can show that \( w \in L^2(0, T; H^{1+\delta}(\Omega)) \) for some \( \delta > 0 \).

### 4. Approximation

We first introduce a regularization of our system, by replacing \( \beta \) with its Yosida approximation \( \beta_\varepsilon \) [8, p. 28], i.e., \( \beta_\varepsilon(r) = \varepsilon^{-1}(r+1) \) if \( r \leq -1 \), \( \beta_\varepsilon(r) = 0 \) if \( -1 < r < 1 \), and \( \beta_\varepsilon(r) = \varepsilon^{-1}(r-1) \) if \( r \geq 1 \). We also denote by \( j_\varepsilon \) the primitive of \( \beta_\varepsilon \) such that \( j_\varepsilon(0) = 0 \). Clearly, the solutions to the regularized system will be allowed to take their values not only in \([-1, 1]\). Hence, we extend \( g \) to the whole real line, still denoting by \( g \) the extended function, in such a way that \( g \in W^{1,\infty}(\mathbb{R}) \) (cf. (3.4)). Proceeding similarly as in [13], we introduce a Faedo-Galerkin approximation scheme to prove an existence theorem for the regularized statement.

Thus, for \( i \in \mathbb{N} \), \( i \geq 1 \), we take a complete system \( \{v_i\} \) of eigenfunctions of the problem

\[ v_i \in V, \quad Av_i = \lambda_i v_i \text{ in } V'. \]  

According to the general spectral theory, the eigenvalues \( \lambda_i \) can be increasingly ordered and counted according to their multiplicities in order to form a real divergent sequence. Moreover, the respective eigenvectors turn out to form an orthogonal basis both in \( V \) and in \( H \) and may be assumed to be normalized in the norm of \( H \). At this point, we can set \( V_n := \text{span}\{v_1, \ldots, v_n\} \) and \( V_\infty := \bigcup_{n=1}^\infty V_n \). Clearly, \( V_\infty \) is a dense subspace of \( V \). We also remark that, as the first eigenvalue in our sequence is \( \lambda_1 = 0 \), \( V_1 \) consists precisely of the constant functions on \( \Omega \); as a consequence, \( 1 \in V_n \) for every \( n \).
For any $n \in \mathbb{N}$, we look for a couple of functions of the form
\[
\chi = \chi^n_\varepsilon = \sum_{i=1}^n c_i(t)v_i, \quad w = w^n_\varepsilon = \sum_{i=1}^n d_i(t)v_i, \tag{4.2}
\]
solving the approximate version of system (3.9–3.12) that we now introduce.
When no danger of confusion occurs, the simpler notation $(\chi, w)$ will be generally preferred for the above functions. Note that, in (4.2), $c_i$ and $d_i$ are thought to be suitably regular real valued functions of time, whose dependence on $\varepsilon, n$, is not stressed, for simplicity. We shall indicate by $c, d$ the vectors $\{c_i\}_{i=1}^n, \{d_i\}_{i=1}^n$.

Owing to the regularity of $\beta_\varepsilon$, it is convenient to collect some terms, by introducing the notation
\[
h^n_\varepsilon(\chi^n_\varepsilon, f) := \beta_\varepsilon(\chi^n_\varepsilon) + g(\chi^n_\varepsilon) + f. \tag{4.3}
\]
This allows us to state the following:

**Problem 4.1.** For any $\varepsilon > 0, n \geq 1$, we look for a pair of functions $(\chi^n_\varepsilon, w^n_\varepsilon)$ of the form (4.2) such that for any $t$ it is
\[
\begin{align*}
\langle \partial_t \chi^n_\varepsilon, v \rangle + \int_\Omega B(\chi^n_\varepsilon)\nabla w^n_\varepsilon \cdot \nabla v &= 0 \quad \text{for all } v \in V_n, \tag{4.4} \\
\langle \partial_t \chi^n_\varepsilon + a(\chi^n_\varepsilon)A \chi^n_\varepsilon + h^n_\varepsilon, v \rangle &= \langle w^n_\varepsilon, v \rangle \quad \text{for all } v \in V_n, \tag{4.5} \\
\langle \chi^n_\varepsilon(0), v \rangle &= \langle \chi_0, v \rangle \quad \text{for all } v \in V_n. \tag{4.6}
\end{align*}
\]

Note that, by (4.1), $A \chi^n_\varepsilon$ is smooth; thus, the duality $\langle a(\chi^n_\varepsilon)A \chi^n_\varepsilon, v \rangle$ in (4.5) makes sense. We are able to prove the following result,

**Theorem 4.2.** For any $n \in \mathbb{N}$, there exist $T_0 \in (0, T], c \in C^1([0, T_0]; \mathbb{R}^n)$, $d \in C^0([0, T_0]; \mathbb{R}^n)$, such that the functions $(\chi^n_\varepsilon, w^n_\varepsilon)$ defined by (4.2) solve Problem 4.1 in the time interval $[0, T_0]$. Moreover, this solution is unique.

**Proof.** We aim to apply the standard existence theorems for ODE’s. Thus, it is convenient to rewrite system (4.4–4.5) into a more compact form. With this purpose, we define the matrices
\[
M(c) = (m_{ij}(c)), \quad m_{ij}(c) := \int_\Omega B\left(\sum_{k=1}^n c_k v_k\right)\nabla v_i \cdot \nabla v_j \, dx, \tag{4.7}
\]
\[
N(c) = (n_{ij}(c)), \quad n_{ij}(c) := \int_\Omega a\left(\sum_{k=1}^n c_k v_k\right)v_i v_j \, dx, \tag{4.8}
\]
\[
\Lambda = (\lambda_{ij}), \quad \lambda_{ij} := \lambda_i \delta_{ij}, \tag{4.9}
\]

where $\delta_{ij}$ are the Kronecker symbols. Now, if $n$ is fixed, we choose $v = v_j$ for $j = 1, \ldots, n$ and substitute the expression (4.2) to the unknowns $(\chi^{n}_\varepsilon, w^{n}_\varepsilon)$ in (4.4–4.5). Performing direct calculations, we actually derive the system

\begin{align*}
c' + M(c)d &= 0, \quad (4.10) \\
d &= c' + N(c)\Lambda c + h(c), \quad (4.11)
\end{align*}

where we have set (cf. (4.3))

\begin{equation}
h(c) = \{h_j(c)\}_{j=1,\ldots,n}, \quad h_j(c) := \int_{\Omega} h^{n}_\varepsilon \left( \sum_{k=1}^{n} c_k v_k, f \right) v_j \, dx. \quad (4.12)
\end{equation}

Recalling (2.4), (3.5), and (3.7), we see that, whatever is the value assumed by $c$, the matrix $M(c)$ is strongly elliptic, so that the inverse $(I + M(c))^{-1}$ is a contraction. Hence, substituting (4.11) into (4.10), we finally derive the equation

\begin{equation}
c' = -(I + M(c))^{-1} M(c)(N(c)\Lambda c + h(c)). \quad (4.13)
\end{equation}

Notice that, by the Lipschitz continuity of the functions $B, g, a$, and $\beta_\varepsilon$, the right hand side of the above expression depends on $c$ in a locally Lipschitz continuous way. Indeed, this fact is clear as far as the factors $M(c)$ and $N(c)\Lambda c + h(c)$ are concerned. As for the inverse matrix $(I + M(c))^{-1}$, one might explicitly write its components in terms of the components of $M(c)$ and notice that the determinant of $M(c)$ is bounded from below independently of the value assumed by $c$. Thus, applying Cauchy’s Theorem for ODE’s, we find a small time $T_0$ such that (4.13) holds for $t \in [0, T_0]$ and the Cauchy condition (4.6) is fulfilled by the corresponding function $\chi^{n}_\varepsilon$ defined in (4.2). This gives the desired solution to Problem 4.1, since now $d$ can be constructed by using (4.11) and $w^{n}_\varepsilon$ is given by (4.2) again. \hfill \Box

5. Proof of Theorem 3.2

5.1. Basic a priori estimate. We first remove the Faedo-Galerkin approximation by using an a priori estimate-passage to the limit procedure with respect to the parameters $n$ and $\varepsilon$. Moreover, since we can prove a bound for the solution of the approximated system uniform with respect to time, we also infer that the limit solution is actually defined over the whole $(0, T)$. For this reason, the computations are performed by directly working in $(0, T)$. In the following, $c$ and $c_\varepsilon$ are positive constants that might vary from row to row. However, every $c$ has to depend only on the data $\Omega, T, \alpha, \mu, \chi_0, g, f$, while, in addition, $c_\varepsilon$ is allowed to depend also on $\varepsilon$. 
Hence, for \( n \geq 1 \), let us work on the solution to Problem 4.1 and, for simplicity, let us omit the indices \( n \) and \( \varepsilon \). Take \( t \in (0, T] \) and choose \( v = w \) in (4.4) and \( v = \partial_t \chi \) in (4.5). Indeed, note that \( \partial_t \chi(s), w(s) \in V_n \) for every \( s \in [0, t] \) and consequently they are admissible test functions. Then, integrate over \((0, t]\) and sum the obtained expressions so that two terms cancel. Using (3.5) (cf. also (2.4)), one readily infers

\[
\alpha \| \nabla w \|_{L^2(Q_t)}^2 + \| \partial_t \chi \|_{L^2(Q_t)}^2 + \int_0^t \int_\Omega h \partial_t \chi \, dx \, ds + I_1 = 0,
\]

where

\[
I_1 := \int_0^t \int_\Omega A \chi \partial_t \chi \, dx \, ds.
\]

Recalling (3.6) and by the elementary Young inequality

\[
rs \leq \sigma r^2 + s^2 / 4\sigma \quad \text{for any } r, s \in \mathbb{R}, \sigma > 0,
\]

we can estimate \( I_1 \) as follows,

\[
|I_1| \leq \mu^2 \| A \chi \|_{L^2(Q_t)}^2 + \frac{1}{4} \| \partial_t \chi \|_{L^2(Q_t)}^2.
\]

Hence, by (4.3) and the definition of \( j_\varepsilon \), we observe that

\[
\int_0^t \int_\Omega h \partial_t \chi \, dx \, ds = \int_\Omega j_\varepsilon(\chi(t)) \, dx - \int_\Omega j_\varepsilon(\chi_0) \, dx + \int_0^t \int_\Omega (g(\chi) + f) \partial_t \chi \, dx \, ds.
\]

Moreover, by the boundedness of \( g \) we see that

\[
\int_0^t \int_\Omega (g(\chi) + f) \partial_t \chi \, dx \, ds \leq c + \frac{1}{4} \| \partial_t \chi \|_{L^2(Q_t)}^2.
\]

The main problem is the control of \( I_1 \) (5.4). For this purpose, take again (4.5) and test it by choosing \( v = AX \) (cfr. (4.1)). Then, integrating as before over \((0, t]\), using (3.6), (3.7), and (5.3), and splitting the term with \( g + f \) as above, we can immediately obtain

\[
\alpha \| AX \|_{L^2(Q_t)}^2 + \int_0^t \int_\Omega \beta_\varepsilon(\chi) AX \, dx \, ds + \frac{1}{2} |\nabla \chi(t)|^2 \\
\leq \int_0^t \int_\Omega (g(\chi) + f) AX \, dx \, ds + \int_0^t \int_\Omega \nabla w \cdot \nabla \chi \, dx \, ds + \frac{1}{2} |\nabla \chi_0|^2 \\
\leq c + \frac{\alpha}{2} \| AX \|_{L^2(Q_t)}^2 + \frac{1}{4\sigma} \| \nabla \chi \|_{L^2(Q_t)}^2 + \sigma \| \nabla w \|_{L^2(Q_t)}^2 + \frac{1}{2} |\nabla \chi_0|^2.
\]
Hence, owing to the monotonicity of $\beta_\varepsilon$, we have that
\[
\int_0^t \int_\Omega \beta_\varepsilon(x) A\chi\, dx\, ds \geq 0. \tag{5.8}
\]
Then, we can multiply expression (5.7) by a constant $m > 0$ (to be chosen later) and sum the result to (5.1). Taking (5.5), (5.6), and (5.8) into account, we readily derive
\[
\alpha \|\nabla w\|^2_{L^2(Q_t)} + \frac{m}{2} |\nabla \chi(t)|^2 + J_\varepsilon(\chi(t)) + \frac{1}{2} \|\partial_t \chi\|^2_{L^2(Q_t)} + \frac{m\alpha}{2} \|A\chi\|^2_{L^2(Q_t)} \\
\leq c(1 + m) \mu^2 \|\nabla \chi\|^2_{L^2(Q_t)} + m\sigma \|\nabla w\|^2_{L^2(Q_t)} + J_\varepsilon(\chi_0), \tag{5.9}
\]
where $J_\varepsilon$ is defined as in (2.14) but replacing $\beta$ by $\beta_\varepsilon$. Then, choosing, e.g., $m = 4\mu^2/\alpha$ and $\sigma = \alpha/2m$, we can apply the Gronwall lemma in order to prove a uniform bound for the function $t \mapsto |\nabla \chi(t)|^2$. Indeed, observing that, by (3.2), $J_\varepsilon(\chi_0) = 0$ and using the standard elliptic regularity results, we eventually derive
\[
\|\chi\|_{H^1(0,T;H)} + \|\chi\|_{L^2(0,T;H^2(\Omega))} + \|\chi\|_{L^\infty(0,T;V)} \leq c, \tag{5.10}
\]
\[
\|\nabla w\|_{L^2(Q)} \leq c, \tag{5.11}
\]
\[
\|J_\varepsilon(\chi)\|_{L^1(0,T)} \leq c. \tag{5.12}
\]
Furthermore, the Lipschitz continuity of $\beta_\varepsilon$ readily yields
\[
\|\beta_\varepsilon(\chi)\|_{L^2(0,T;H)} \leq c_\varepsilon. \tag{5.13}
\]
At this point, taking $v = w$ in (4.5), performing standard computations, and referring to (5.10–5.11) and (5.13), we finally get
\[
\|w\|_{L^2(0,T;V)} \leq c_\varepsilon. \tag{5.14}
\]

5.2. Passage to the limit as $n \to \infty$. We come back to the full notation $(\chi^n_\varepsilon, w^n_\varepsilon)$ and note that, from this point on, all convergence relations will be intended to hold up to the extraction of suitable subsequences, generally not relabeled. Thus, we observe that weak and weak star compactness results applied to (5.10) and (5.14) entail that there exist $(\chi_\varepsilon, w_\varepsilon)$ such that, as $n \to \infty$, the following properties hold,
\[
\chi^n_\varepsilon \rightharpoonup \chi_\varepsilon \quad \text{weakly in } H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)), \tag{5.15}
\]
\[
\chi^n_\varepsilon \rightharpoonup^* \chi_\varepsilon \quad \text{weakly--}^* \text{ in } L^\infty(0,T;V), \tag{5.16}
\]
\[
w^n_\varepsilon \rightharpoonup w_\varepsilon \quad \text{weakly in } L^2(0,T;V). \tag{5.17}
\]
Then, standard interpolation and compact embedding results for vector-valued functions (see, e.g., [23, Sec. 8]) ensure that
\[ \chi_n^\varepsilon \to \chi_\varepsilon \quad \text{strongly in } C^0([0,T];H). \] (5.18)
Choose now \( v \in V_\infty \). Clearly, since \( v \) is fixed, it can be taken as a test function in (4.4–4.6) for sufficiently large \( n \). Let us see that we can actually take the limits as \( n \to \infty \). Indeed, by the Lipschitz continuity of \( \beta_\varepsilon, a, b, \) and \( g \), the convergence above allow us to infer that (actually, much more is true)
\[ h(\chi_n^\varepsilon, f) \to h(\chi_\varepsilon, f) \quad \text{strongly in } L^2(0,T;H), \] (5.19)
\[ a(\chi_n^\varepsilon) \to a(\chi_\varepsilon) \quad \text{strongly in } L^2(Q) \text{ and weakly–}^* \text{ in } L^\infty(Q), \] (5.20)
\[ B(\chi_n^\varepsilon) \to B(\chi_\varepsilon) \quad \text{strongly in } L^2(Q) \text{ and weakly–}^* \text{ in } L^\infty(Q). \] (5.21)
By (5.20–5.21), (5.15), and (5.17), we readily see that
\[ B(\chi_n^\varepsilon) \nabla w_n^\varepsilon \to B(\chi_\varepsilon) \nabla w_\varepsilon \quad \text{weakly in } L^2(Q), \] (5.22)
\[ a(\chi_n^\varepsilon)A\chi_n^\varepsilon \to a(\chi_\varepsilon)A\chi_\varepsilon \quad \text{weakly in } L^2(Q). \] (5.23)
Hence, on account of (5.15–5.23), both the relations (4.4) and (4.5) pass to the limit. Moreover, since \( V_\infty \) is dense in \( V \), the limit system turns out to hold true not only for \( v \in V_\infty \), but for any \( v \in V \). The same holds indeed for the limit of the Cauchy condition (4.6), which makes sense thanks, e.g., to (5.18).

5.3. Passage to the limit as \( \varepsilon \to 0 \). Let us observe that the bounds (5.10) and (5.11) are independent of \( \varepsilon \); hence, as \( \varepsilon \) tends to 0, they entail
\[ \chi_\varepsilon \to \chi \quad \text{weakly in } H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)), \] (5.24)
\[ \chi_\varepsilon \to \chi \quad \text{weakly–}^* \text{ in } L^\infty(0,T;V), \] (5.25)
\[ \nabla w_\varepsilon \to \eta \quad \text{weakly in } L^2(0,T;H^d), \] (5.26)
for suitable limit functions (\( \chi, \eta \)). On the contrary, we can no longer take advantage of (5.13–5.14). Hence, we have to perform a new estimate, relying on a variant of an argument devised by Kenmochi, Niezgodka, and Pawlow [17, Lemma 5.2, p. 345], that derives a bound of \( \beta_\varepsilon(\chi_\varepsilon) \) independent of \( \varepsilon \). For the sake of completeness, let us report this procedure.
Set \( \xi_\varepsilon(t) := \beta_\varepsilon(\chi_\varepsilon(t)) \) and \( x_\varepsilon(t) := (\xi_\varepsilon(t))_\Omega \). Then, test the \( n \)-limit of (4.5) by taking \( v = \xi_\varepsilon - x_\varepsilon \) and integrate over \( Q_1 \). Noting that \( \xi_\varepsilon - x_\varepsilon \in L^2(0,T;V_0) \subset L^2(0,T;V_0'), \) by Lemma 3.1 we can also take \( v = \nabla \chi_\varepsilon(\xi_\varepsilon - x_\varepsilon) \in L^2(0,T;V_0') \) in the \( n \)-limit of (4.4) and integrate over \( (0,t) \). Taking the
difference of the resulting relations and using (2.8) we see that two terms cancel and consequently infer
\[
\int_0^t \int_\Omega \left( \partial_t \chi_\varepsilon + a(\chi_\varepsilon) A \chi_\varepsilon + \xi_\varepsilon + g(\chi_\varepsilon) + f \right) (\xi_\varepsilon - x_\varepsilon) \, dx \, ds
= \int_0^t \langle \partial_t \chi_\varepsilon, N_{\chi_\varepsilon}(\xi_\varepsilon - x_\varepsilon) \rangle \, ds. \tag{5.27}
\]
Since \( x_\varepsilon \) is constant in space, we have
\[
\int_0^t \int_\Omega \xi_\varepsilon (\xi_\varepsilon - x_\varepsilon) \, dx \, ds = \int_0^t \int_\Omega (\xi_\varepsilon - x_\varepsilon)^2 \, dx \, ds. \tag{5.28}
\]
Moreover, by Lemma 3.1 and the continuous embedding \( H \hookrightarrow V' \), it follows
\[
\int_0^t \langle \partial_t \chi_\varepsilon, N_{\chi_\varepsilon}(\xi_\varepsilon - x_\varepsilon) \rangle \, ds \leq \frac{1}{4} \| \xi_\varepsilon - x_\varepsilon \|_{L^2(Q_t)}^2 + c \| \partial_\varepsilon \chi_{\varepsilon} \|_{L^2(0,T;V')}^2. \tag{5.29}
\]
Hence, recalling (3.1), (3.6), (5.10), and the boundedness of \( g \), the terms on the left hand side of (5.27) can be estimated independently of \( \varepsilon \). In particular, taking also (5.28) and (5.29) into account, we easily deduce
\[
\| \xi_\varepsilon - x_\varepsilon \|_{L^2(0,T;H)} \leq c. \tag{5.30}
\]
Now, in order to estimate the full \( L^2 \)-norm of \( \xi_\varepsilon \), we have to control the average \( x_\varepsilon \). Indeed, upon recalling (3.3) and the definition of \( \beta_\varepsilon \), we see that there exist two numbers \( m_1, m_2 \) such that
\[
-1 < m_1 < 0 < m_2 < 1, \quad m_1 < \chi_\Omega < m_2. \tag{5.31}
\]
Hence, for \( t \in [0,T] \), we set \( \Omega_{1,\varepsilon} = \Omega_{1,\varepsilon}(t) := \{ x \in \Omega : \chi_\varepsilon(x) < m_1 \} \), \( \Omega_{2,\varepsilon} = \Omega_{2,\varepsilon}(t) := \{ x \in \Omega : \chi_\varepsilon(x) > m_2 \} \), and \( \Omega_{0,\varepsilon} = \Omega_{0,\varepsilon}(t) := \{ x \in \Omega : m_1 < \chi_\varepsilon(x) < m_2 \} \). Moreover, we define \( m := \min \{ \chi_\Omega - m_1, m_2 - \chi_\Omega \} \). Since, \( \xi_\varepsilon = 0 \) a.e. in \( \Omega_{0,\varepsilon} \) by the definition of \( \beta_\varepsilon \), we have, a.e. in \( (0,T) \),
\[
\int_\Omega |\xi_\varepsilon| = \int_{\Omega_{1,\varepsilon}} (-\xi_\varepsilon) \, dx + \int_{\Omega_{2,\varepsilon}} \xi_\varepsilon \, dx = \int_{\Omega_{1,\varepsilon} \cup \Omega_{2,\varepsilon}} \xi_\varepsilon \frac{\chi_\varepsilon - \chi_\Omega}{|\chi_\varepsilon - \chi_\Omega|} \, dx
\leq \frac{1}{m} \int_{\Omega_{1,\varepsilon} \cup \Omega_{2,\varepsilon}} \xi_\varepsilon (\chi_\varepsilon - \chi_\Omega) \, dx = \frac{1}{m} \int_\Omega \xi_\varepsilon (\chi_\varepsilon - \chi_\Omega) \, dx
= \frac{1}{m} \int_\Omega (\xi_\varepsilon - x_\varepsilon) (\chi_\varepsilon - \chi_\Omega) \, dx, \tag{5.32}
\]
where the last passage is justified since $\chi_\varepsilon - \chi_\Omega$ has zero mean value. Then, it readily follows that
\[
\int_\Omega |\xi_\varepsilon| \leq \frac{1}{m} |\xi_\varepsilon - x_\varepsilon| |\chi_\varepsilon - \chi_\Omega|.
\] (5.33)

Now, let us estimate $x_\varepsilon$. Using Hölder’s inequality and the relation above, we have
\[
\|x_\varepsilon\|^2_{L^2(0,T;H)} = \frac{1}{|\Omega|} \int_0^t \left( \int_\Omega |\xi_\varepsilon| \, dx \right)^2 \, ds \\
\leq \frac{1}{|\Omega|^2 m^2} \int_0^t |(\xi_\varepsilon - x_\varepsilon)(s)|^2 |(\chi_\varepsilon - \chi_\Omega)(s)|^2 \, ds \\
\leq \frac{1}{|\Omega|^2 m^2} \|\xi_\varepsilon - x_\varepsilon\|^2_{L^2(0,T;H)} \| (\chi_\varepsilon - \chi_\Omega)(s) \|^2_{L^\infty(0,T;H)} \leq c. \quad (5.34)
\]
Indeed, $\chi_\Omega$ is a constant and we can rely on the bounds (5.10) and (5.30).

Now, using the above relation and (5.30) again, we see that $\xi_\varepsilon$ is bounded independently of $\varepsilon$ and consequently we can infer that
\[
\xi_\varepsilon \to \xi \quad \text{weakly in } L^2(0,T;H) \quad (5.35)
\]
for some $\xi \in L^2(Q)$. Furthermore, (5.24) and the Aubin compactness lemma entail
\[
\chi_\varepsilon \to \chi \quad \text{strongly in } L^2(0,T;V). \quad (5.36)
\]
Then, by the usual monotonicity argument of, e.g., [2, Prop. 1.1, p. 42], the above relations (5.35) and (5.36) readily imply (3.15). Test now the $n$-limit of (4.5) by $w$ and integrate over $(0,t)$. Proceeding as in the previous subsection and using in particular the $L^2$-boundedness of $\xi_\varepsilon$, we actually derive
\[
w_\varepsilon \to w \quad \text{weakly in } L^2(0,T;H), \quad (5.37)
\]
for some $w \in L^2(Q)$. Of course, we can assume that $\nabla w = \eta$ (cf. (5.26)). Moreover, (5.36) and the Lipschitz continuity of $a$, $b$, and $g$ allow us to get the analogous of (5.20) and (5.21), as well as
\[
g(\chi_\varepsilon) \to g(\chi) \quad \text{strongly in } L^2(0,T;H). \quad (5.38)
\]
Hence, we can take the limit of (4.5) and get back (3.14). The Cauchy conditions (4.6) pass to the limit as well, since (5.24) ensures strong convergence for $\chi_\varepsilon$ in $C^0([0,T];H)$. The same procedure used before permits us to recover (3.13) by taking the $\varepsilon$-limit in (4.4) and this concludes the proof of Theorem 3.2.
6. Singular limits and final remarks

We now study two singular limit problems related to (3.13–3.16). Hence, let us rewrite that system in view of emphasizing the parameters that are let to vanish,

\[ \partial_t \chi + B \chi w = 0 \quad \text{in} \quad L^2(0, T; V'), \quad (6.1) \]
\[ w = \varepsilon \lambda \partial_t \chi + \varepsilon' a_\varepsilon(\chi) A \chi + \xi + g(\chi) + f \quad \text{a.e. in} \quad Q, \quad (6.2) \]
\[ \xi \in \beta(\chi) \quad \text{a.e. in} \quad Q, \quad (6.3) \]
\[ \chi(0) = \chi_0 \quad \text{a.e. in} \quad \Omega. \quad (6.4) \]

In the above relation, \( \lambda, \nu \geq 0 \) are given and, for \( \varepsilon \in (0, 1) \), \( a_\varepsilon \) denotes a sequence of functions that are assumed to fulfill a condition slightly more general than (3.7), but more natural in this setting, namely,

\[ a_\varepsilon \in W^{1, \infty}([-1, 1]; \mathbb{R}), \quad \alpha \leq a_\varepsilon(r) \leq \mu, \quad \forall r. \quad (6.5) \]

It is clear that, for any fixed \( \varepsilon \), if \( a_\varepsilon \) satisfies the above condition, Theorem 3.2 still holds with almost no changes in the proof. Now, we state our first convergence result.

**Theorem 6.1.** Let us assume (3.2–3.5), take \( \lambda = 0, \nu = 1 \) in (6.2), and let \( a_\varepsilon \) fulfill (6.5) for any \( \varepsilon \in (0, 1) \). Moreover, suppose that

\[ f \in L^1(0, T; V) \cap L^2(0, T; H). \quad (6.6) \]

Then, denoting by \((\chi_\varepsilon, w_\varepsilon, \xi_\varepsilon)\) a family of solutions to system (6.1–6.4) satisfying (3.9–3.12), there exist functions \((\chi, w, \xi)\) such that, as \( \varepsilon \downarrow 0 \),

\[ \chi_\varepsilon \rightharpoonup \chi \quad \text{weakly}^* \text{ in} \quad H^1(0, T; H) \cap L^\infty(0, T; V), \quad (6.7) \]
\[ w_\varepsilon \rightharpoonup w \quad \text{weakly in} \quad L^2(0, T; V), \quad (6.8) \]
\[ \xi_\varepsilon \rightharpoonup \xi \quad \text{weakly in} \quad L^2(0, T; H), \quad (6.9) \]
\[ \varepsilon^{1/2} A \chi_\varepsilon \rightharpoonup 0 \quad \text{weakly in} \quad L^2(0, T; H). \quad (6.10) \]

Furthermore, the limit functions \((\chi, w, \xi)\) satisfy

\[ \partial_t \chi + B \chi w = 0 \quad \text{in} \quad L^2(0, T; V'), \quad (6.11) \]
\[ w = \partial_t \chi + \xi + g(\chi) + f \quad \text{a.e. in} \quad Q, \quad (6.12) \]
\[ \xi \in \beta(\chi) \quad \text{a.e. in} \quad Q, \quad (6.13) \]
\[ \chi(0) = \chi_0 \quad \text{a.e. in} \quad \Omega. \quad (6.14) \]
Remark 6.2. We notice once more that, from a physical point of view, the above result provides a sharp interface model at the limit, whereas the approximating systems account for a strictly positive thickness of the phase transition layer. Moreover, we recall that in the sharp interface model mushy regions should be expected as no surface tension effects are present. This is due to the fact that the gradient term is not present any longer.

The next theorem, instead, is concerned with the convergence of (6.1–6.4) to a variant of the Cahn-Hilliard system with nonconstant mobility, studied, e.g., in [13] (see also [14, 15] for an application to phase separation systems):

Theorem 6.3. Let us assume (3.2–3.5), let \( \lambda = 1 \), \( \mu = 0 \) in (6.2), and let \( a_\varepsilon \) fulfill (6.5) for any \( \varepsilon \in (0, 1) \). Moreover, suppose that there exist constants \( a \geq \mu > 0 \) and \( c_a > 0 \) such that
\[
a_\varepsilon \to a \quad \text{uniformly in } [-1, 1],
\]
\[
|a_\varepsilon'(r)| \leq c_a \varepsilon^{1/2} \quad \text{for all } \varepsilon \in (0, 1), \ r \in [-1, 1]
\]
and assume
\[
f \in W^{1,1}(0,T;L^1(\Omega)) \cap L^2(0,T;H).
\]
Then, if \( (\chi_\varepsilon, w_\varepsilon, \xi_\varepsilon) \) is a family of solutions to system (6.1–6.4) fulfilling (3.9–3.12), there exist functions \( (\chi, w, \xi) \) such that
\[
\chi_\varepsilon \to \chi \quad \text{weakly–* in } H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)),
\]
\[
w_\varepsilon \to w \quad \text{weakly in } L^2(0,T;V),
\]
\[
\xi_\varepsilon \to \xi \quad \text{weakly in } L^2(0,T;H),
\]
\[
\varepsilon^{1/2} \partial_t \chi_\varepsilon \to 0 \quad \text{weakly in } L^2(0,T;H),
\]
and fulfilling
\[
\partial_t \chi + B_\chi w = 0 \quad \text{in } L^2(0,T;V'),
\]
\[
w = aA\chi + \xi + g(\chi) + f \quad \text{a.e. in } Q,
\]
\[
\xi \in \beta(\chi) \quad \text{a.e. in } Q,
\]
\[
\chi(0) = \chi_0 \quad \text{a.e. in } \Omega.
\]

Remark 6.4. Let us point out that the regularity of the limit functions \( (\chi, w, \xi) \) can be read from (6.7–6.10) and (6.18–6.21), respectively. In particular, in both cases we deduce that \( \chi \in [-1, 1] \) a.e. in \( Q \), so that \( (\chi, w, \xi) \) is a physically meaningful solution to the limit systems (6.11–6.14) and (6.22–6.25), respectively.
Proof of Theorems 6.1 and 6.3. We carry out part of the proofs at one time, since both of them mainly rely on an energy estimate similar to that performed in Subsection 5.1. Actually, we test (6.1) by $w_\varepsilon$ and (6.2) by $\partial_t \chi_\varepsilon$. Taking the difference of the resulting relations, integrating over $(0, t)$, $t \in (0, T]$, and noting that two terms cancel, we readily derive a relation analogous to (5.1). Then, we test again (6.2) by $A\chi_\varepsilon$, integrate over $(0, t)$, multiply the result by $m > 0$ and add it to the expression obtained before. We also note that, although $\beta$ is non-smooth, the analogous of property (5.8) still holds thanks to Lemma 2.5, and the integration by parts of (5.5) can be mimicked by taking advantage of Lemma 2.4.

On the contrary, the estimation of the other terms, and in particular of $I_1 = I_1(\varepsilon)$ (that is defined as in (5.2)) is different in the two cases. Hence, let us start by detailing the proof of Theorem 6.1. We have

$$|I_1(\varepsilon)| \leq \frac{1}{4} \left\| \partial_t \chi_\varepsilon \right\|_{L^2(Q_t)}^2 + \mu^2 \varepsilon^2 \left\| A\chi_\varepsilon \right\|_{L^2(Q_t)}^2. \quad (6.26)$$

Moreover, the estimate (5.6) still holds as in Subsection 5.1 and we also have (cf. the second row of (5.7), (3.4), and (6.6)),

$$\int_0^t \int_\Omega \left( |g'(\chi_\varepsilon)| \right) A\chi_\varepsilon \, dx \, ds \leq \int_0^t \left( \max_{[-1,1]} |g'| |\nabla \chi_\varepsilon|^2 + |\nabla f| |\nabla \chi_\varepsilon| \right) \, ds. \quad (6.27)$$

We finally note that, by (5.3),

$$\int_0^t \int_\Omega w_\varepsilon A\chi_\varepsilon \, dx \, ds \leq \frac{\alpha}{2} \left\| \nabla w_\varepsilon \right\|_{L^2(Q_t)}^2 + \frac{1}{\alpha} \left\| \nabla \chi_\varepsilon \right\|_{L^2(Q_t)}^2. \quad (6.28)$$

Then, it is easy to deduce the following analogue of (5.9), written for $m = 1$,

$$\frac{\alpha}{2} \left\| \nabla w_\varepsilon \right\|_{L^2(Q_t)}^2 + \frac{1}{2} \left\| \nabla \chi_\varepsilon(t) \right\|_{Q_t}^2 + \frac{1}{2} \left\| \partial_t \chi_\varepsilon \right\|_{L^2(Q_t)}^2 \leq c + \frac{1}{\alpha} \left\| \nabla \chi_\varepsilon \right\|_{L^2(Q_t)}^2 + J(\chi_0).$$

Hence, provided $\varepsilon$ is sufficiently small (e.g. $\varepsilon < \alpha/\mu^2$), we can apply the generalized Gronwall’s lemma (cf., e.g., [1, Thm. 2.1]) to the function $t \mapsto |\nabla \chi_\varepsilon(t)|^2$ and prove the a priori estimates leading to convergence (6.7) and (6.10). In order to pass to the limit, we have to observe that the argument of [17] reported in Subsection 5.3 can be performed again with almost no changes. This clearly provides the weak convergence (6.9). The estimation of the $L^2$-norm of $w_\varepsilon$ can be obtained as before by testing (6.2) by $w_\varepsilon$ and
relying on the known bounds. This gives back (6.8). Finally, the passage to 
the limit for (6.1–6.2) can be achieved as in the previous section and in this 
way relations (6.11–6.14) are proved.

Let us come to the proof of Theorem 6.3 and consider again relation (5.1).
Now, the integral
\[ I_1(\varepsilon) \] (cf. (5.2)) has to be treated more carefully. We have
\[
\int_{\Omega} a_\varepsilon(\chi_\varepsilon) A\chi_\varepsilon \partial_t \chi_\varepsilon \, dx = \int_{\Omega} a'_\varepsilon(\chi_\varepsilon) |\nabla \chi_\varepsilon|^2 \partial_t \chi_\varepsilon \, dx + \int_{\Omega} a_\varepsilon(\chi_\varepsilon) \nabla \chi_\varepsilon \cdot \nabla \partial_t \chi_\varepsilon \, dx
\]
\[
= \frac{d}{dt} \int_{\Omega} a_\varepsilon(\chi_\varepsilon) \frac{|\nabla \chi_\varepsilon|^2}{2} \, dx + \frac{1}{2} \int_{\Omega} a'_\varepsilon(\chi_\varepsilon) |\nabla \chi_\varepsilon|^2 \partial_t \chi_\varepsilon \, dx.
\]
(6.29)

Hence, relying on the following Gagliardo-Nirenberg inequality (see, e.g., [24, 
Chap. 21, p. 287]),
\[
\|\nabla v\|_{L^4(\Omega)}^2 \leq C \|v\|_{L^\infty(\Omega)} \|v\|_{H^2(\Omega)},
\]
(6.30)
noting that the constraint (3.10) holds for any \( \varepsilon \), and recalling (6.16), by 
(6.29) we infer that
\[
I_1(\varepsilon) \geq \alpha \frac{1}{2} |\nabla \chi_\varepsilon(t)|^2 - \frac{\mu}{2} |\nabla \chi_0|^2 - \frac{1}{2} c_a \varepsilon^{1/2} \int_0^t \int_{\Omega} |\partial_t \chi_\varepsilon| |\nabla \chi_\varepsilon|^2 \, dx \, ds
\]
\[
\geq \alpha \frac{1}{2} \|\nabla \chi_\varepsilon(t)\|^2 - \frac{\mu}{2} |\nabla \chi_0|^2 - \frac{\varepsilon}{2} \|\partial_t \chi_\varepsilon\|^2_{L^2(Q_t)} - C \int_0^t \int_{\Omega} \|\chi_\varepsilon\|^2_{H^2(\Omega)} \, ds.
\]
(6.31)

Moreover, by the boundedness of \( g \) and (6.17), we have
\[
\left| \int_0^t \int_{\Omega} (g(\chi_\varepsilon) + f) A\chi_\varepsilon \, dx \, ds \right| \leq c + \frac{\alpha}{2} \|A\chi_\varepsilon\|^2_{L^2(Q_t)}.
\]
(6.32)

Then, setting \( \tilde{g}(r) := \int_0^r g(s) \, ds \) and recalling (3.4), there holds
\[
\left| \int_0^t \int_{\Omega} g(\chi_\varepsilon) \partial_t \chi_\varepsilon \, dx \, ds \right| \leq \int_{\Omega} |\tilde{g}(\chi_\varepsilon(t))| \, dx + \int_{\Omega} |\tilde{g}(\chi_0)| \, dx \leq c.
\]
(6.33)

Owing to (6.17) and (3.10), an integration by parts in time yields
\[
\left| \int_0^t \int_{\Omega} f \partial_t \chi_\varepsilon \, dx \, ds \right| = \left| - \int_0^t \int_{\Omega} \partial_t f \chi_\varepsilon \, dx \, ds + \int_\Omega \int_0^t f(t) \chi_\varepsilon(t) \, dx - \int_\Omega f(0) \chi_0 \, dx \right|
\]
\[
\leq \int_0^t \int |\partial_t f| \, dx \, ds + \int_{\Omega} |f(t)| \, dx + \int_{\Omega} |f(0)| \, dx \leq c
\]
(6.34)
Now, splitting the term with \( w_\varepsilon \) similarly as in (6.28) and collecting (6.29–6.34), we are able to infer
\[
\begin{align*}
\alpha \| \nabla w_\varepsilon \|_{L^2(Q_t)}^2 + \frac{\varepsilon}{2} \| \partial_t \chi_\varepsilon \|_{L^2(Q_t)}^2 + \frac{m\alpha}{2} \| A \chi_\varepsilon \|_{L^2(Q_t)}^2 \\
+ \frac{m\varepsilon + \alpha}{2} |\nabla \chi_\varepsilon(t) |^2 + J(\chi_\varepsilon(t)) \\
\leq c(f, g, m, \alpha) + \frac{m\varepsilon + \mu}{2} |\nabla \chi_0 |^2 + c^* \| \chi_\varepsilon \|_{L^2(0;H^2(\Omega))}^2 \\
+ m\sigma \| \nabla w_\varepsilon \|_{L^2(Q_t)}^2 + \frac{m}{4\sigma} \| \nabla \chi_\varepsilon \|_{L^2(Q_t)}^2 + J(\chi_0),
\end{align*}
\]
(6.35)
where \( c(f, g, m, \alpha) > 0 \) collects the terms depending on the known norms of \( f, g \) that result from (6.32–6.34), \( \sigma \) is as in (5.3), and \( c^* \) is some positive constant. Now, the standard elliptic regularity estimates entail that, a.e. in \((0, T)\),
\[
c^* \| \chi_\varepsilon \|_{H^2(\Omega)}^2 \leq c_\Omega (\| A \chi_\varepsilon \|_{H^2}^2 + \| \chi_\varepsilon \|_{H^2}^2) \leq c_\Omega (\| A \chi_\varepsilon \|_{H^2}^2 + c)
\]
(6.36)
for some constant \( c_\Omega > 0 \) only depending on \( \Omega \) and \( c^* \). Hence, choosing, e.g., \( m := 4c_\Omega / \alpha \) and \( \sigma := \alpha / 2m \), we can see that Gronwall’s lemma applies once more to \( t \mapsto |\nabla \chi_\varepsilon(t) |^2 \) and derive an a priori bound from (6.35). Anyway, in order to get (6.18–6.21), we still have to estimate \( w_\varepsilon, \xi_\varepsilon, \) and \( \partial_t \chi_\varepsilon \). To this aim, let us observe that (6.1), the first of (3.8), and the Poincaré-Wirtinger inequality (2.9) readily entail
\[
\| \partial_t \chi_\varepsilon \|_{V'} = \| B \chi_\varepsilon (w_\varepsilon - (w_\varepsilon)_\Omega) \|_{V'} \leq \mu \| w_\varepsilon - (w_\varepsilon)_\Omega \|_V \leq (1 + C_\Omega^{1/2}) \mu \| \nabla w_\varepsilon \|_H.
\]
(6.37)
Hence, the first convergence in (6.18) is achieved by relying on the \( L^2(Q) \)-bound for \( \nabla w_\varepsilon \) and this allows us to use again the argument of [17] to deduce (6.20) (indeed, the last term in (5.29) is still bounded). Observing that (6.19) is straightforward, since the \( L^2 \)-norm of \( w \) can be estimated as before by a comparison in (6.2), the limit system (6.22–6.23) is obtained as in the previous cases. Note indeed that (6.18) still entails
\[
\chi_\varepsilon \to \chi \quad \text{strongly in } L^2(0, T; V)
\]
and this estimate permits us to use once more [2, Prop. 1.1, p. 42] in order to recover (6.24). Finally, the remaining nonlinear terms pass to the limit as in the end of Subsection 5.2 and this concludes the proof of Theorem 6.3.

**Remark 6.5.** We notice that, as far as we know, no uniqueness result is known for any of the systems (3.13–3.16), (6.11–6.14), (6.22–6.25). Actually,
the main difficulty is due to the dependence on $\chi$ of the mobility matrix $M(\chi)$. For this reason a very strong regularity of the solution is needed in order to find an estimate for the difference of two solutions to $(3.13\text{--}3.16)$ (or to the other systems). In particular, the conditions $(3.9\text{--}3.12)$ seem not to be sufficient (cf. also [13, Sec. 5]); some uniqueness results, under stronger assumptions on the initial data, have been proved, e.g., in [3, 4] for some slightly different systems.

REFERENCES


