Abstract. In this paper we consider some integro-differential systems of two parabolic PDE’s coming from the Caginalp approach to phase transition models. The first (integro-differential) equation describes the evolution of the temperature and also accounts for memory effects through a memory kernel $k$. The latter equation, governing the evolution of the order parameter, is semilinear and of the fourth-order (in space). We prove some continuous dependence and regularity results for the solution of the Cauchy problem associated to the PDE’s. Taking advantage of these results, we prove a global in time conditional existence and uniqueness result for the identification problem consisting in recovering the memory kernel $k$ appearing in the first equation.
1 Introduction

In order to introduce our mathematical problem, let us consider a smooth bounded container \( \Omega \subset \mathbb{R}^d, \ 1 \leq d \leq 3, \) occupied by the substance undergoing the phase transition. Name \( \vartheta \) and \( \chi \) the basic state variables of the process, corresponding to the relative temperature and to the order parameter, respectively. Then, the energy balance equation, describing the evolution of \( \vartheta \), can be written in the form \([6, 24, 27, 28]\)

\[
\partial_t (\vartheta + \lambda(\chi)) - k_0 \Delta \vartheta - \Delta (k \ast \vartheta) = f,
\]

where \( \partial_t \) is the time derivative (in the remainder of the paper it will be indicated also by \((\cdot)_t\)), \( \ast \) stands for the standard time convolution product over \((0, t)\), \( k : (0, T) \to \mathbb{R} \) is the so-called heat conductivity relaxation kernel, \( f \) is a heat source also incorporating an additional term depending on the past history of \( \vartheta \) up to \( t = 0 \), which is assumed to be given, \( k_0 \) is the heat conductivity coefficient, and \( \lambda(\cdot) \) is a smooth function accounting for the latent heat.

In \([1, 13, 14]\) phase relaxation problems with memory have been considered, where, instead of the Fourier law, a Coleman-Gurtin law (cf. \([7]\)) has been assumed for the heat flux. This constitutive assumption leads to (1.1).

Then, this first equation ruling the evolution of \( \vartheta \) is coupled with the kinetic equation for the phase variable. We will consider the conserved case

\[
\begin{align*}
\partial_t \chi - \Delta w &= 0, \\
w &\in -\Delta \chi + \beta'(\chi) + \sigma'(\chi) - \lambda'(\chi) \vartheta.
\end{align*}
\]

In (1.2b) \( \beta \) is taken as a general maximal monotone graph, possibly multivalued, representing the derivative of the convex part of a double well free energy potential and \( \sigma \) is a smooth function accounting for its non-convex part. However, we have to underline that, in order to obtain the regularity results stated in this paper, we will need to consider only sufficiently regular functions \( \beta \) and \( \sigma \) (cf. Theorems 2.4, 2.9, 2.10, and 2.11).

Moreover, relations (1.1) and (1.2) are assumed to be complemented by homogeneous Neumann boundary condition both for \( \vartheta, \chi \), and for the auxiliary unknown \( w \), generally called chemical potential, and by the Cauchy conditions for \( \vartheta \) and \( \chi \). Actually, on account of the boundary condition on \( w \), it is straightforward to check that the average of \( \chi \) is constant in time.

Phase field models of Caginalp type (cf. \([6]\)), possibly accounting for memory effects, have been extensively investigated in recent years. From the viewpoint of the direct problem one can see, e.g., \([5, 8, 15, 20, 21, 22, 29]\). Let us also mention the paper \([12]\) where the authors studied the system (1.1+1.2) by assuming the Gurtin-Pipkin heat flux law (cf. \([24]\)), i.e. (1.1) with \( k_0 = 0 \) and \( k(0) > 0 \). Existence and uniqueness were proved in the case of a nonlinearity \( \lambda \) with at most a linear growth at infinity. In \([37]\) the author deals also with a quadratic \( \lambda \). Let us point out that this choice, which seems unusual in the classical framework of Stefan problem, becomes rather appropriate in other modelling contexts (see \([25, 26, 35, 36]\)). Moreover, we note that in the case of the Gurtin-Pipkin law, the authors of \([11]\) are able to show the existence of solutions to the initial-boundary value problem for (1.1+1.2), but the uniqueness remains an open question.
On the basis of the well-posedness result of [37] related to the direct problem, we will study here an identification problem related to system (1.1+1.2).

More precisely, our aim consists in recovering the unknown memory kernel $k$ in the space $L^2(0,T)$ ($T$ denoting the final time of the phase transition process) in (1.1+1.2) under an additional information involving some kind of measurement of the temperature of the system. We point out that the choice of the space $L^2(0,T)$ for $k$ allows us to deal with memory kernels with integrable singularities (cf. also [1]).

Let us note that in order to use a fixed point argument on $k$ (leading to the global existence for $k$), we need to recover the expression of $k$ by differentiating (in time) the equations (1.1) and (1.2) and working on the resulting expression. In order to perform rigorously such a procedure, however, we need to deal with very smooth solutions. Consequently, we first perform a preliminary analysis of our problem from the point of regularity and refine some existing results on the direct problem. Moreover, we also need to know a continuous dependence estimate of the solution $(\vartheta, \chi)$ (in the smaller spaces suggested by the regularity results) with respect to the initial data and also to $k$: also this investigation is carried out in the first part of the paper. We point out that these regularity and continuous dependence results for the direct problem, beyond acting as a basis for the subsequent analysis of the inverse problem, might have some independent interest.

In the second part of the paper, we pass to the analysis of the identification problem for $k$. Unfortunately, in order to get global (on the whole time interval $[0,T]$) existence of $k$, we need to impose (in case of a non-Lipschitz nonlinearity $\beta$ in (1.2b)) a condition on the $\chi$-component of the solution $(\vartheta, \chi)$ of (1.1+1.2), which is, in particular, satisfied if $\chi$ fulfills the physically meaningful constraint $\chi \in [0,1]$. Note that the fact that $\chi$ represents (in the physical representation of the model) the volume fraction of the substance subject to the phase transition gives a natural meaning to this condition. However we have to say that, unfortunately, we are not able to read it directly from the equations, due to the lack of maximum principle or comparison arguments for the fourth order (in space) system (1.1+1.2).

Let us remark that identifying memory kernels in systems of PDE’s coming from phase transition models is a quite new problem. A pioneering paper on this subject is [16] where the author uses analytic semigroup theory in order to study both the direct and the inverse problem (i.e. he determines also the convolution kernel locally in time) for a non-conserved system of PDE’s (cf. [2, 3, 18]) which couples (1.1) with a second order equation for the the phase variable, that is of the type of (1.2) but with the Identity operator instead of $-\Delta w$ in (1.2a).

Speaking instead of the conserved model considered in this note, let us quote the papers [17] and [32] where the authors (using semigroup techniques) study the local (in time) identification of a kernel for system (1.1+1.2). Moreover, in [23] the authors get the local (in time) existence of the kernel $k$ for a system of PDE’s analogous to our (1.1+1.2) where they have taken the Gurtin-Pipkin heat flux law into account, i.e., (1.1) with $k_0 = 0$. This gives rise to a hyperbolic dynamics in the analogous of equation (1.1). Let us point out once more that all these contributions provide local (in time) existence results for the inverse problem. Actually, the main novelty of the present investigation is that of addressing global existence, that we obtain at least in the conditional case discussed before.
Here is the plan of the paper. In the next section, we shall present some notation and state our precise mathematical results concerning both the direct and the inverse problem. In Section 3, we give the proofs of the regularity theorems for the direct problem stated in Section 2. In the subsequent Section 4, we show the continuous dependence estimate (for the direct problem) by working in the smaller space suggested by the regularity estimates. Finally, in the last Section 5 we give the proof of our main result dealing with the global existence and uniqueness of the triplet \((\theta, \chi, k)\) solving the identification problem under the additional condition provided by the measurement of temperature.

2 Main results

We begin this section by introducing some notation we will use throughout the paper.

First of all, we denote by \(\Omega \subset \mathbb{R}^d\) (1 \(\leq d \leq 3\)) a bounded, smooth, and connected domain, by \(\Gamma := \partial \Omega\) its smooth boundary, and, for \(T > 0\) (the final time of our phase transition process), by \(Q := \Omega \times (0, T)\) the cylindrical domain where the process takes place. Then, let us introduce the Hilbert spaces \(H := L^2(\Omega), V := H^1(\Omega)\), and endow them with their usual scalar products. We identify \(H\) and its dual, in order that the compact inclusion \(H \subset V'\) holds and \((V, H, V')\) forms a Hilbert triplet \([30, \text{p. 202}]\). We denote by \(|\cdot|_H\) the norms both in \(H\) and in \(H^d\) (actually both indicated by \(H\) for simplicity of notation), and by \(|\cdot|_X\) the norm in the generic Banach space \(X\). Finally, we indicate by \((\cdot, \cdot)\) the scalar product in \(H\) and by \(\langle \cdot, \cdot \rangle\) the duality pairing between \(V'\) and \(V\).

Next, for any \(\zeta \in V'\) we set
\[
\zeta_{\Omega} := \frac{1}{|\Omega|} \langle \zeta, 1 \rangle, \\
V'_0 := \{ \zeta \in V' : \zeta_{\Omega} = 0 \}, \\
V_0 := V \cap V'_0.
\]

The above notation \(V'_0\) is suggested by the sake of convenience. Indeed, \(V_0, V'_0\) will be viewed as (closed) subspaces of \(V, V'\), inheriting their norms, rather than as a couple of spaces in duality.

We introduce the realization of the Laplace operator with homogeneous Neumann boundary conditions as
\[
B : V \to V', \quad \langle Bu, v \rangle := \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad \text{for } u, v \in V.
\]

Clearly, \(B\) maps \(V\) onto \(V'_0\) and its restriction to \(V_0\) is an isomorphism of \(V_0\) onto \(V'_0\). We denote by \(\mathcal{N} : V'_0 \to V_0\) the (continuous) inverse of \(B\), so that for any \(u \in V\) and \(\zeta \in V'_0\) we have
\[
\langle Bu, \mathcal{N} \zeta \rangle = \langle BN \zeta, u \rangle = \langle \zeta, u \rangle.
\]

We also define
\[
W := \{ v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \Gamma \},
\]
\(\partial_n\) denoting the outward normal derivative of \(v\). Consequently, \(W\) turns out to be a closed subspace of \(H^2(\Omega)\) thanks to the continuity of the trace operator.
By using the Poincaré-Wirtinger inequality we easily conclude that the norm
\[
\left( \int_{\Omega} |\nabla (N\zeta)|^2 \right)^{1/2} = \langle \zeta, N\zeta \rangle^{1/2}
\]
for \( \zeta \in V_0' \) (2.6) is equivalent to the norm \( |\zeta|_* = |\zeta|_{V'} \) and we will use it, when it is convenient.

As mentioned in the Introduction, the first step required in order to address the identification problem for \( k \) is to improve the regularity and continuous dependence results for the direct problem. Hence, let us start just by this investigation.

### 2.1 The direct problem

We start by stating the precise mathematical formulation of system (1.1+1.2) and recalling some related well-posedness results.

Our basic assumptions on the data are:

1. \( f = f_1 + f_2 \) with \( f_1 \in L^2(0, T; V') \), \( f_2 \in L^1(0, T; H) \); (a1)
2. \( \lambda, \sigma \in C^2(\mathbb{R}) \) with \( \lambda', \sigma' \in C^{0,1}(\mathbb{R}) \); (a2)
3. \( \hat{\beta} : \mathbb{R} \to [0, +\infty] \) is convex, proper, and lower semicontinuous, \( \hat{\beta}(0) = 0, \beta = \partial \hat{\beta}, \beta(0) \ni 0 \); (a3)
4. \( k \in L^1(0, T), \ k_0 > 0 \); (a4)
5. \( \vartheta_0 \in H \); (a5)
6. \( \chi_0 \in V, \ \hat{\beta}(\chi_0) \in L^1(\Omega) \). (a6)

Letting now \( \omega \) be an arbitrary positive number, then we use the notation
\[
v_\omega(t) := \exp(-\omega t)v(t), \quad \forall t \in [0, T],
\]
where \( v \) is a generic function defined on \([0, T]\) with values in some Banach space \( X \).

Now, we can recall the following result related to the well-posedness of the direct problem associated to (1.1+1.2) and whose proof is given in [37, Thm. 2.1].

**Theorem 2.1.** Assume that (a1–a6) hold and that
\[
(\chi_0)_\Omega \in \text{int } D(\beta).
\]

Then, for any fixed \( T > 0 \) there exists a unique pair \((\vartheta, \chi)\) and some pair \((w, \xi)\) satisfying
\[
\vartheta \in H^1(0, T; V') \cap C^0([0, T]; H) \cap L^2(0, T; V),
\]
\[
\chi \in H^1(0, T; V') \cap C^0([0, T]; V) \cap L^2(0, T; W),
\]
\[
w \in L^2(0, T; V),
\]
\[
\xi \in L^2(0, T; H).
\]
The pairs \((\vartheta, \chi), (w, \xi)\) satisfy
\[
\begin{align*}
\partial_t (\vartheta + \lambda(\chi)) + k_0 B \vartheta + B (k \ast \vartheta) &= f \quad \text{in } V', \text{ a.e. in } (0, T), \\
\partial_t w + B x &= 0 \quad \text{in } V', \text{ a.e. in } (0, T), \\
w &= B x + \xi + \sigma'(\chi) - \lambda'(\chi) \vartheta \quad \text{in } V', \text{ a.e. in } (0, T), \\
\chi &\in D(\beta) \quad \text{and} \quad \xi \in \beta(\chi) \quad \text{a.e. in } Q, \\
\vartheta(0) &= \vartheta_0, \quad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega.
\end{align*}
\] (2.12)
Moreover (cf. (2.1)), we have that
\[
(2.13)
\]
defines \(w, \xi\). Assume that \((w, \xi)\) (cf. notation (2.7)) satisfies
\[
\frac{1}{|\Omega|} \int_{\Omega} \chi(t) \, dx = \chi_\Omega \quad \forall \, t \in [0, T].
\] (2.17)
Finally, assume that \(k\) and \(\omega\) satisfy the inequality
\[
\left(1 + \frac{2}{k_0}\right) |k_\omega|_{L^1(0, T)}^2 \leq \frac{k_0}{4}.
\] (2.18)
Then, there exists a positive constant \(C_1\) depending only on the norms of the data of the problem and on \(k_0\) such that the couple \((\vartheta, \chi)\) (cf. notation (2.7)) satisfies
\[
\begin{align*}
|\partial_\vartheta|_{H^1(0, T; V') \cap C^0([0, T]; H) \cap L^2(0, T; V')} + |\chi|_{H^1(0, T; V') \cap C^0([0, T]; V) \cap L^2(0, T; W)} + |w|_{L^2(0, T; V)} + |\xi|_{L^2(0, T; H)} &\leq C_1
\end{align*}
\] (2.19)
and consequently
\[
\begin{align*}
|\partial_\vartheta|_{H^1(0, T; V') \cap C^0([0, T]; H) \cap L^2(0, T; V')} + |\chi|_{H^1(0, T; V') \cap C^0([0, T]; V) \cap L^2(0, T; W)} + |w|_{L^2(0, T; V)} + |\xi|_{L^2(0, T; H)} &\leq C_1 \left(1 + e^{\omega T}\right).
\end{align*}
\] (2.20)
\textbf{Remark 2.2.} Assume that \((\vartheta, \chi)\) is the unique pair given by Theorem 2.1. Then, equation (2.13) defines \(w\) up to a function independent of \(x\), while (2.14) just leads to a known difference \(w - \xi\). Hence, uniqueness for \((w, \xi)\) is not ensured under our assumptions (cf., e.g., [12, Rmk. 2.3]). On the contrary, uniqueness even for \((w, \xi)\) surely holds if \(\beta\) is a single-valued function. Indeed, (2.15) yields a unique \(\xi\) in this case, and the above argument leads to a unique \(w\) as well.

\textbf{Remark 2.3.} In [37, Thm. 2.1] Theorem 2.1 is proved under the “thermodynamical consistence” assumption on the kernel \(k\), given by
\[
\exists \delta > 0 \text{ s.t. } \delta \int_0^t |v(s)|^2 \, ds \leq \int_0^t (k_0 v(s) + (k \ast v)(s)) \, v(s) \, ds
\]
and supposed to hold for all \(t \in (0, T)\) and \(v \in L^2(0, T)\). However, as it is plainly explained in [37, Rmk. 2.3], we could avoid taking this assumption on the memory kernel \(k\), provided we modify our proofs in this way: whenever an estimate for equation (2.12) is done, multiply it also by an exponential weight (such as \(\exp(-\omega t)\), with \(\omega\) suitably large, cf. (2.18)) and then use an equivalent norms argument (see also [1, 2, 10]). This procedure will be used also in this paper in the sequel.
Let us start now by stating our first regularity result whose proof will be detailed in the next Section 3.

**Theorem 2.4.** Suppose that, in addition to (a2–a4), (a7), the following assumptions hold:

\[ f \in L^2(0, T; H); \]  
\[ \vartheta_0 \in V; \]  
\[ \chi_0 \in H^3(\Omega) \cap W \quad \text{and} \quad \exists \xi^0 \in \beta(\chi_0) : \xi^0 \in V. \]  

Then, the unique solution \((\vartheta, \chi)\) to (2.12–2.16) given by Theorem 2.1 enjoys the further regularity

\[ \vartheta \in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W), \]  
\[ \chi \in W^{1, \infty}(0, T; V') \cap H^1(0, T; V) \cap L^\infty(0, T; W) \quad (\subset L^\infty(Q)). \]  

Moreover, under assumption (2.18) on \((k, \omega)\), there exists a positive constant \(C_2\) depending only on the norms of the data of the problem and on \(k_0\) such that the couple \((\vartheta, \chi)\) (cf. notation (2.7)) satisfies:

\[ |\vartheta'|_{H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W)} + |\chi'_{V'}|_{H^1(0, T; V) \cap L^\infty(0, T; W)} \leq C_2 \left(1 + e^{8\omega T}\right) \]  

and

\[ |\vartheta'|_{H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W)} + |\chi'_{V'}|_{H^1(0, T; V) \cap L^\infty(0, T; W)} \leq C_2 \left(1 + e^{9\omega T}\right). \]  

The multipliers \((1 + e^{8\omega T})\) and \((1 + e^{9\omega T})\) can be dropped out when \(\lambda\) is linear, i.e. the first of (r4) below holds.

Finally, in addition to (a2), (a4), and (r1), assume that

\[ \lambda' \equiv \ell \quad \text{with} \quad \ell \in \mathbb{R}, \quad \beta \in C^1(\mathbb{R}), \quad \beta' \geq 0 \quad \text{in} \quad \mathbb{R}; \]  
\[ \chi_0 \in H^6(\Omega) \cap W, \quad \vartheta_0 \in H^4(\Omega) \cap W, \]  

and take the following auxiliary variables (with \(t \in (0, T)\)):

\[ u := \vartheta - \vartheta_0, \quad v := \chi - \chi_0 - t\chi_\ell(0). \]  

Then, under assumption (2.18) for \((k, \omega)\), there exists a positive constant \(\overline{C}_2\) depending on the norms of the data of the problem and on \(k_0\) such that the following inequality holds:

\[ |u_\omega|^2_{H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W)} + |v_\omega|^2_{H^1(0, T; V) \cap L^2(0, T; W)} \]  
\[ + \beta' (\chi)^{1/2} (\vartheta_t) \omega^2_{L^2(0, T; H)} \leq \overline{C}_2 \left[|\chi_\ell(0)|^2_{L^\infty(\Omega)} + |\beta' (\chi)|_{2} L^1(Q) \right. \]  
\[ + \frac{1}{\omega}|\chi_\ell(0)|^2_V + \frac{1}{\omega}|B\vartheta_0|^2 + |f_\omega|^2_{L^2(0, T; H)} \]  

(2.26)
Remark 2.5. Note that the second part of Theorem 2.4 ensures that in case of more regular data (cf. (r4–r5)) we get an estimate whose right hand side becomes smaller as \( \omega \) gets larger (cf. (2.26)). We also point out that, thanks to (r4–r5), we are allowed to work with very regular solutions (cf. Theorem 2.11 below, with \( s = 0 \)). In particular, taking the value \( \chi_t(0) \) in (2.25) and (2.26) makes sense in this setting.

Remark 2.6. Thanks to relation (2.22) and continuity of the embeddings of \( H^1(0, T; V) \) into \( C^{0,1/2}([0, T]; V) \), we have that \( \chi \in L^\infty(0, T; V) \cap C^{0,1/2}([0, T]; V) \). Then, easy interpolation tools (see, e.g., [34, Prop. 1.1.4, p. 13]) and Sobolev’s embeddings permits to see that \( \chi \) is globally Hölder continuous in \( \bar{\Omega} \times [0, T] \).

Remark 2.7. Assume that \( \beta \) satisfies
\[
|\beta'(r)| \leq b(1 + |r|^p), \quad \forall r \in \mathbb{R} \text{ and some } p \in (1, 2). \tag{m1}
\]
Then, the first term on the right hand side of (2.26) can be estimated using (2.19) and Hölder’s inequality as follows:
\[
\int_0^t e^{-2\omega s} |\beta'(\chi)(s)|_{L^1(\Omega)} \, ds \leq b \int_0^t e^{-p\omega s} [1 + |\chi(s)|_{L^p(\Omega)}^p] e^{-(2-p)\omega s} \, ds \\
\leq c \int_0^t [e^{-2\omega s} + [m(\Omega)]^{(2-p)/2} e^{-\omega s} |\chi(s)|_{L^p(\Omega)}^p] e^{-(2-p)\omega s} \, ds \\
\leq c(1 + |\chi|_{L^\infty(0,T;H^2)}^p) \left( \int_0^t e^{-2\omega s} \, ds \right)^{2/(2-p)} \leq c\omega^{-2/(2-p)}.
\]

Remark 2.8. Assume that \( \chi \) satisfies
\[
|\beta' \circ \chi|_{L^p(0,T;L^1(\Omega))} \leq b' \quad \text{for some given constant } b' > 0 \text{ and } p \in (1, +\infty). \tag{m2}
\]
Then,
\[
||\beta' \circ \chi||_{L^p(Q)} \leq b'(2p')^{-1/p'} \omega^{-1/p'}.
\]

In the following theorems we will introduce some positive constants \( K_1(\omega), K_2(\omega), \ldots \) depending on the data and additionally on \( \omega \). Note that (even in the case \( \lambda' \equiv \ell \)) they will be of the form \( \mu(1 + e^{\mu \omega T}) \) where \( (k, \omega) \) satisfy (2.18) and \( \mu \) is a positive constant allowed to depend on the data, on \( k_0 \), on \( T \), but not on \( \omega \). We continue now with the second regularity result holding for a smaller class of nonlinearities \( \beta \), that is for sufficiently regular monotone functions defined on the whole real line.

Theorem 2.9. In addition to (a2–a4) and (a6), let the following assumptions hold:
\[
f = f_1 + f_2, \quad f_1 \in L^2(0, T; V) \quad \text{and} \quad f_2 \in L^1(0, T; H^2(\Omega)); \tag{r6}
\]
\[
\vartheta_0 \in W. \tag{r7}
\]
Then the \( \vartheta \)-component of the solution \((\vartheta, \chi)\) to (2.12–2.16) has the further regularity
\[
\vartheta \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)) \tag{2.27}
\]
and, under assumption (2.18) for \((k, \omega)\), there exists a positive constant \(K_1(\omega)\) such that
\[
|\vartheta_\omega|_{L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega))} \leq K_1(\omega).
\] (2.28)

Next, if we also suppose that \((r4 \rightarrow r5)\) hold and assume that either \((m1)\) or \((m2)\) is satisfied, then there exists a positive constant \(C_3\) depending only on the norms of the data of the problem, on \(k_0\), on \(p\), and on \(b\) (or \(b'\)) such that the \(\vartheta\)-component of the solution \((\vartheta, \chi)\) to (2.12–2.16) satisfies
\[
|\vartheta_\omega|_{L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega))} \leq C_3 \left[1 + \frac{1}{\omega^{1/p'}}\right],
\] (2.29)
where, in the case of \((m2)\), we have set \(p' := 2/2 - p\) in order to have uniformity of notation. Finally, if, in addition to \((a4)\) and \((r6 \rightarrow r7)\), we assume that
\[
\lambda \in C^4(\mathbb{R});
\] (r8)
\[
\beta \in C^3(\mathbb{R}), \quad \sigma \in C^4(\Omega), \quad \beta' \geq 0 \quad \text{in } \mathbb{R},
\] (r9)
then the \(\chi\)-component of the solution \((\vartheta, \chi)\) to (2.12–2.16) has the further regularity
\[
\chi \in L^\infty(0, T; H^3(\Omega)) \cap L^2(0, T; H^5(\Omega))
\] (2.30)
and, under assumption (2.18) for \((k, \omega)\), it satisfies
\[
|\chi_\omega|_{L^\infty(0, T; H^3(\Omega)) \cap L^2(0, T; H^5(\Omega))} \leq K_2(\omega)
\] (2.31)
for some positive constant \(K_2(\omega)\).

Let us continue with the following

**Theorem 2.10.** In addition to \((a2), (a4)\), assume that
\[
f = f_1 + f_2, \quad f_1 \in H^1(0, T; V') \cap L^2(0, T; H^3(\Omega))
\] and \(f_2 \in W^{1,1}(0, T; H) \cap L^1(0, T; H^4(\Omega));\) (r10)
\[
\lambda'' \equiv L \quad \text{with } L \in \mathbb{R};
\] (r11)
\[
\beta \in C^5(\mathbb{R}), \quad \sigma \in C^6(\mathbb{R}), \quad \beta' \geq 0 \quad \text{in } \mathbb{R};
\] (r12)
\[
\vartheta_0 \in H^4(\Omega) \cap W;
\] (r13)
\[
\chi_0 \in H^5(\Omega) \cap W,
\] (r14)
then the solution \((\vartheta, \chi)\) to (2.12–2.16) has the further regularity
\[
\vartheta \in W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; H^4(\Omega))
\] \(\cap L^2(0, T; H^5(\Omega)),\) (2.32)
\[
\chi \in H^2(0, T; V') \cap W^{1,\infty}(0, T; V) \cap H^1(0, T; H^3(\Omega))
\] \(\cap L^\infty(0, T; H^5(\Omega)) \cap L^2(0, T; H^7(\Omega)).\) (2.33)

Furthermore, under assumption (2.18) on \((k, \omega)\), there exists a positive constant \(K_3(\omega)\) such that the couple \((\vartheta, \chi)\) (cf. notation (2.7)) satisfies
\[
|\vartheta_\omega|_{W^{1,\infty}(0, T; H) \cap H^1(0, T; V) \cap L^\infty(0, T; H^4(\Omega)) \cap L^2(0, T; H^5(\Omega))}
\] \(+ |\chi_\omega|_{H^2(0, T; V') \cap W^{1,\infty}(0, T; V) \cap H^1(0, T; H^3(\Omega)) \cap L^\infty(0, T; H^5(\Omega)) \cap L^2(0, T; H^7(\Omega))} \leq K_3(\omega).
\] (2.34)
The last regularity regularity result, holding only in case of an affine latent heat function $\lambda$, is the following

**Theorem 2.11.** Suppose that (a2), (a4), (r4) hold and take for $s \in \mathbb{N}$ (with the convention $H^0 \equiv H$),

$$
\begin{align*}
  f = f_1 + f_2, & \quad f_1 \in H^1(0, T; H^s(\Omega)) \quad \text{and} \quad f_2 \in W^{1,1}(0, T; H^{s+1}(\Omega)); \\
  \beta \in C^{s+3}(\mathbb{R}), & \quad \sigma \in C^{s+4}(\mathbb{R}), \quad \beta' \geq 0 \quad \text{in} \ \mathbb{R}; \\
  \vartheta_0 \in H^{s+4}(\Omega) \cap W; & \quad (r15) \\
  \chi_0 \in H^{s+6}(\Omega) \cap W. & \quad (r16)
\end{align*}
$$

Then, the solution $(\vartheta, \chi)$ to (2.12–2.16) satisfies

$$
\vartheta \in W^{1,\infty}(0, T; H^{s+1}(\Omega)) \cap H^1(0, T; H^{s+2}(\Omega)),
$$

$$
\chi \in H^2(0, T; H^s(\Omega)) \cap W^{1,\infty}(0, T; H^{s+2}(\Omega))
$$

and, under assumption (2.18) for $(k, \omega)$, there exists a positive constant $K_4(\omega)$ such that the couple $(\vartheta, \chi)$ (cf. notation (2.7)) fulfills

$$
|\vartheta_\omega|_{W^{1,\infty}(0,T;H^{s+1}(\Omega)) \cap H^1(0,T;H^{s+2}(\Omega))} + |\chi_\omega|_{H^2(0,T;H^s(\Omega)) \cap W^{1,\infty}(0,T;H^{s+2}(\Omega))} \leq K_4(\omega).
$$

(2.37)

The last result concerning the direct problem is the following continuous dependence theorem for the regular solution obtained in Theorem 2.11 (with $s = 2$).

**Theorem 2.12.** Suppose that assumptions of Theorem 2.11 with $s = 2$ and $f_1 = 0$ hold. Let $(\vartheta_1, \chi_1)$ be a solution to (2.12–2.16) corresponding to data $f^1$, $k_1$, $\vartheta_{01}$, $\chi_{01}$ and let $(\vartheta_2, \chi_2)$ be a solution to (2.12–2.16) corresponding to data $f^2$, $k_2$, $\vartheta_{02}$, $\chi_{02}$, both enjoying the regularity properties (r15–r18) stated in Theorem 2.11, together with

$$
\begin{align*}
  k_j & \in L^2(0, T), \quad j = 1, 2; \\
  |(k_j)_\omega|_{L^1(0, T)} & \leq \frac{k_0}{2(2 + k_0)^{1/2}}, \quad |k_j|_{L^2(0, T)} \leq m, \quad j = 1, 2;
\end{align*}
$$

(2.38)

for some $m > 0$. Then, there exists a positive constant $K = K(\omega, T, \delta, m)$ depending on $\omega$ and the data of the problem, on $T$, and on $|k_1|_{L^2(0, T)}$, $|k_2|_{L^2(0, T)}$, and $m$ such that, for any $\tau \in (0, T]$, we have the estimate

$$
\begin{align*}
  &|\partial_t \vartheta_2 - \partial_t \vartheta_1|_{L^2(0,\tau;H)} + |\partial_t \chi_2 - \partial_t \chi_1|_{L^2(0,\tau;H)} + |B \vartheta_2 - B \vartheta_1|_{L^2(0,\tau;H)} \\
  &+ |B^2 \chi_2 - B^2 \chi_1|_{L^2(0,\tau;H)} \leq K\left[|f^2 - f^1|_{L^2(0,\tau;H)} + |k_2 - k_1|_{L^1(0,\tau)}ight] + |\vartheta_{02} - \vartheta_{01}|_{H^1(\Omega)} + |\chi_{02} - \chi_{01}|_{H^2(\Omega)}.
\end{align*}
$$

(2.40)

### 2.2 The inverse problem

In this subsection we assume that the memory kernel $k : (0, T) \to \mathbb{R}$ in equation (2.12) is unknown and we are given an additional condition in order to recover it. Moreover,
to find out a suitable equation for $k$ we make use of assumptions (r4–r5) and rewrite the direct problem (2.12–2.16) in the following more explicit form:

$$
\begin{align*}
\partial_t (\vartheta + \ell \chi) + k_0 B \vartheta + B (k \ast \vartheta) &= f, \quad \text{in } Q, \\
\partial_t \chi + B[B \chi + \beta(\chi) + \sigma'(\chi) - \ell \vartheta] &= 0, \quad \text{in } Q, \\
\vartheta(\cdot, 0) &= \vartheta_0, \quad \chi(\cdot, 0) = \chi_0, \quad \text{in } \Omega, \\
\partial_n(k_0 \vartheta + k \ast \vartheta) &= \partial_n \chi = \partial_n[B \chi + \beta(\chi) + \sigma'(\chi) - \ell \vartheta] = 0, \quad \text{on } \partial \Omega \times (0, T).
\end{align*}
$$

(2.41)
(2.42)
(2.43)
(2.44)

To recover the kernel $k$ we prescribe the additional information

$$
\Phi[\vartheta(t)] := (\varphi, \vartheta(t)) = g(t), \quad t \in [0, T].
$$

(ir1)

We make the following assumptions:

$$
\begin{align*}
f &\in W^{1,p}(0, T; H), \quad g \in W^{2,p}(0, T), \quad p \in (2, +\infty), \quad \text{(ir2)} \\
\varphi &\in W^{4,\infty}(\Omega), \quad \partial_n \Delta^j \varphi = 0 \text{ on } \partial \Omega, \quad j = 0, 1.
\end{align*}
$$

(ir3)

We point out that the assumptions above fit with the framework of Theorem 2.11, e.g. for $s = 1$ (however, we will take $s = 2$ in the sequel), save for some additional regularity in time which is required to $f$. From (2.43) and (ir1) we immediately deduce the consistency condition

$$
\Phi[\vartheta_0] = g(0).
$$

(ir4)

From now on we will assume that the unknown kernel $k$ is searched for in $L^2(0, T)$.

**Remark 2.13.** From the equality $\Phi[\partial_t \vartheta(0, \cdot)] = g'(0)$ and equations (2.41–ir1) we easily deduce the further consistency condition

$$
g'(0) = \Phi[k_0 \Delta \vartheta_0 + f(0, \cdot) + \ell \{\Delta^2 \chi_0 - \Delta[\beta(\chi_0) + \sigma'(\chi_0) - \ell \vartheta_0]\}].
$$

(ir5)

We conclude this section by stating our *conditioned* existence, uniqueness and continuous dependence result.

**Theorem 2.14.** Let assumptions listed in Theorem 2.11 (with $s = 2$ and $f_1 = 0$) be satisfied. Assume also that (ir2) and (ir3) hold along with the consistency conditions (ir4), (ir5), and

$$
\Phi[B \vartheta_0] \neq 0, \quad |\beta' \circ \chi|_{L^2(0, T; L^1(\Omega))} \leq M,
$$

(ir6)

for some a priori given constant $M > 0$, then the identification problem (2.41–ir1) admits a unique solution $(\vartheta, \chi, k) \in [W^{1,\infty}(0, T; H^3(\Omega)) \cap H^1(0, T; H^1(\Omega))] \times [H^2(0, T; H^2(\Omega)) \cap W^{1,\infty}(0, T; H^4(\Omega))] \times L^2(0, T)$. Further, let $(\vartheta_1, \chi_1, k_1)$ and $(\vartheta_2, \chi_2, k_2)$ be the solutions to (2.41–ir1) corresponding, respectively, to the data $(f^1, g_1, \vartheta_{01}, \chi_{01})$ and $(f^2, g_2, \vartheta_{02}, \chi_{02})$. Then, the following continuous dependence estimate holds:

$$
|\partial_t \vartheta_2 - \partial_t \vartheta_1|_{L^2(0, T; H)} + |\partial_t \chi_2 - \partial_t \chi_1|_{L^2(0, T; H)} + |B \vartheta_2 - B \vartheta_1|_{L^2(0, T; H)} + |B^2 \chi_2 - B^2 \chi_1|_{L^2(0, T; H)} + |k_2 - k_1|_{L^2(0, T)}
\leq K(\omega, T, \delta, M) \left\{|f^2 - f^1|_{W^{1,p}(0, T; H)} + |\vartheta_{02} - \vartheta_{01}|_{H^4(\Omega)}
\right.
\left.\quad + |\chi_{02} - \chi_{01}|_{H^6(\Omega)} + |g_2 - g_1|_{W^{2,p}(0, T)} \right\},
$$

(2.45)
where the positive constant $K(\omega, T, \delta, M)$ depends continuously and increasingly on the norms of the data.

**Remark 2.15.** The latter condition in (ir6) should be trivially satisfied if we might show that the $\chi$-component of the solution to the direct problem, with the prescribed regularity, took its values in $[0, 1]$. Unfortunately this is, at present, an open question depending on the fact that no maximum principle is known for our fourth order Caginalp-type system.

**Remark 2.16.** Comparing the present continuous dependence result related to the inverse problem with the one related to the direct problem, it should be noted that the result for the inverse problem requires a much stronger regularity in the data $(f, \vartheta, \chi, g)$. But this is a classical occurrence when dealing with inverse problems.

**Remark 2.17.** In the case when $\beta$ is Lipschitz-continuous, the latter condition in (ir6) is trivially satisfied. In this case we have an unconditioned existence, uniqueness and continuous dependence result. Unfortunately this assumption on $\beta$ disagrees with the usual (physical) assumptions related to Caginalp’s model, where $\beta$ is a polynomial of degree 3.

### 3 Proofs of the regularity results

In this section we deal with the proofs of the regularity results stated in Section 2.1.

From now on the symbol $c$ will be used to indicate possibly different strictly positive constants appearing in the computations and assumed to depend only on $\Omega, \ell, L, T$, and, in particular, not on the kernel $k$. When we need to denote some constant (depending on the same parameters as above) that plays a specific role, a notation like $c_1, c_2, \ldots$ will be used, instead. A constant which depends only from one of the data of the problem $(\cdot)$ (e.g. from the latent heat function $\lambda$ or from $k_0$) will be denoted by $c(\cdot)$ (e.g., by $c(\lambda)$) or $c(\cdot)$ (e.g., by $c_0$). A constant noted instead, e.g., as $c_\eta$ will be allowed to depend by an additional (small) positive parameter (here $\eta$). For instance, in the sequel we will repeatedly use the elementary Young inequality, holding for all positive $\eta$, in the form

$$ab \leq \eta a^2 + c_\eta b^2 \quad \forall a, b \in \mathbb{R}.$$  \hfill (3.1)

Moreover we will use the well-known Young theorem stating that

$$|a * b|_{L^p(0,T;X)} \leq |a|_{L^p(0,T)} |b|_{L^q(0,T;X)},$$  \hfill (3.2)

$X$ being a real Banach space, $1 \leq p, q, r \leq \infty$, $1/r = (1/p) + (1/q) - 1$. Finally, we note that from the compact embedding of $V$ into $L^4(\Omega)$ and $H$, it follows (see, e.g., [30, Thm. 16.4, p. 102])

$$|v|^2 + |v|^2_{L^4(\Omega)} \leq \delta |v|^2 + c_\delta |v|^2_{\ast} \quad \forall v \in V,$$  \hfill (3.3)

for any $\delta > 0$ and some positive constant $c_\delta$. More precisely, since $H^{3/4}(\Omega) \hookrightarrow L^4(\Omega)$ and $H^{3/4}(\Omega)$ is the interpolation space of place $1/8$ between $V$ and $V'$, from the well-known interpolation inequality (cf. [30, Prop. 2.3, p. 19]) we get

$$|v|^2_{L^4(\Omega)} \leq c |v|^{7/4}_{V'} |v|^{1/4}_{\ast}.$$  \hfill (3.4)
3.1 Proof of Theorem 2.4

Let us begin with the proof of the first regularity result on the solution \((\vartheta, \chi)\) of (2.12–2.16) given by the existence Theorem 2.1. We will perform some a priori estimates, pointing out that our procedure is formal and that a rigorous argument should rely, e.g., on a Faedo-Galerkin approximation and a regularization of (2.12–2.16) including a Lipschitz approximation of the maximal monotone graph \(\beta\) in (2.14) (e.g., its Yosida approximation, cf. [8, Section 5], [9, Appendix], and [4]).

First of all, let us write down (for the reader’s convenience) the time derivatives of (2.12–2.16) (this is also formal). In particular, in order to differentiate equation (2.14) we should consider, instead of the maximal monotone operator \(\beta\), its Yosida approximation \(\beta_{\epsilon}\) (cf. [4]), and then pass to the limit as \(\epsilon \downarrow 0\). However, for simplicity, let us just perform here formal computations and refer again to [8, Section 5] for further details (other comments were given in the Introduction).

\[
\begin{array}{l}
\partial_t \vartheta + \lambda''(\chi_0) \chi_t^2 + \lambda'(\chi_0) \partial_t \chi + B(k_0 \partial_t \vartheta + k \ast \partial_t \vartheta + k \vartheta) = \partial_t f \\
in V', \ a.e. \ in \ [0,T]; \\
\partial_t \chi + B(\partial_t w) = 0 \quad \text{in} \ V', \ a.e. \ in \ [0,T]; \\
\partial_t w = B(\partial_t \chi) + \sigma''(\chi) \partial_t \chi + \beta'(\chi) \partial_t x - \lambda''(\chi_0) \partial_t \vartheta - \lambda'(\chi) \partial_t \vartheta \\
in V', \ a.e. \ in \ [0,T]; \\
\partial_t \vartheta(0) = -\lambda'(\chi_0) \partial_t x(0) - B(k_0 \partial_0 + k \ast \vartheta_0) + f(0) \quad \text{in} \ V'; \\
\partial_t \chi(0) = -B(B \chi_0 + \sigma'(\chi_0) + \xi_0 - \lambda'(\chi_0) \partial_0) \quad \text{in} \ V',
\end{array}
\]

with \(\vartheta_0, \chi_0, \xi_0\) as in (2.16) and (3).

**First estimate.** Recalling notation (2.7), let us perform the scalar product of (2.12) with \(\exp(-2\omega s) (\partial_t \vartheta)\) with \(0 \leq s \leq t \leq T\). Then, multiply (3.6) by \(\mathcal{N}(\exp(-2\omega s) \chi_t)\) and sum to (3.7) multiplied by \(-\exp(-2\omega s) \chi_t\). Taking the sum of the two resulting equations and integrating it over \((0, t)\), due to a cancellation of two terms we obtain

\[
\begin{align*}
|\partial_t \vartheta|_{L^2(0,t;H)}^2 + & \frac{k_0}{2} \|\nabla \vartheta(t)\|_H^2 + \frac{1}{2} \|\chi_t(t)\|_{V'}^2 + \|\chi_t(t)\|_{H^1}^2 \\
+ \int_0^t \int_\Omega \beta'(\chi(s)) \chi_t^2(s) \exp(-2\omega s) \, ds \leq & \frac{k_0}{2} \|\nabla \vartheta_0\|_H^2 + \frac{1}{2} \|\chi_0(0)\|_{V'}^2 \\
& + \sum_{j=1}^3 I_j(t) + |f_\omega|_{L^2(0,t;H)}^2 + \frac{1}{4} \|\vartheta(t)\|_{L^2(0,t;H)}^2,
\end{align*}
\]

where

\[
\begin{align*}
I_1(t) & := \int_0^t ds \int_\Omega (k \ast \Delta \vartheta)(s) \vartheta_t(s) \exp(-2\omega s) \, dx; \\
I_2(t) & := \int_0^t ds \int_\Omega \chi''(\chi)(s) \chi_t^2(s) \vartheta(s) \exp(-2\omega s) \, dx; \\
I_3(t) & := - \int_0^t ds \int_\Omega \sigma''(\chi)(s) \chi_t^2(s) \vartheta(s) \exp(-2\omega s) \, dx.
\end{align*}
\]
We multiply now (2.12) by $e^{-2\omega t}B\theta$ obtaining
\[
\frac{1}{2}|(\nabla \vartheta)\omega(t)|_{H}^{2} + k_{0}|(B\vartheta)\omega|_{L^{2}(0,t;H)}^{2} \leq \frac{1}{2}|(\nabla \vartheta)\omega|_{H}^{2}
\]
\[
+ \frac{k_{0}}{4}|(B\vartheta)\omega|_{L^{2}(0,t;H)}^{2} + \sum_{j=4}^{5} I_{j}(t) + c_{k_{0}}\int_{0}^{t} |\omega|_{L^{2}(0,t,H)}^{2}
\]
where
\[
I_{4}(t) := - \int_{0}^{t} ds \int_{\Omega} (k * \Delta \vartheta)(s)\Delta \vartheta(s) \exp(-2\omega s) dx;
\]
\[
I_{5}(t) := \int_{0}^{t} ds \int_{\Omega} \gamma(s) \chi(s) \Delta \vartheta(s) \exp(-2\omega s) dx.
\]
We estimate now $I_{1}$ and $I_{4}$ by using (3.1), (3.2), and assumption (2.18) on $(k,\omega)$ as follows
\[
I_{1}(t) + I_{4}(t) \leq \frac{1}{4} \left(|(\vartheta_{t})\omega|_{L^{2}(0,t;H)}^{2} + k_{0}|(B\vartheta)\omega|_{L^{2}(0,t;H)}^{2}\right)
\]
\[
+ \left(1 + \frac{1}{k_{0}}\right)|\omega|_{L^{1}(0,T)}^{2} |(B\vartheta)\omega|_{L^{2}(0,t,H)}^{2}
\]
\[
\leq \frac{1}{4} |(\vartheta_{t})\omega|_{L^{2}(0,t;H)}^{2} + \frac{k_{0}}{2}|(B\vartheta)\omega|_{L^{2}(0,t;H)}^{2}.
\]

Regarding $I_{2}$ and $I_{3}$, we use (a2), the well known H"older inequality, together with (3.3–3.4), (2.19–2.20), and (3.1), in the following way:
\[
I_{2}(t) + I_{3}(t) \leq c \int_{0}^{t} ds \int_{\Omega} |(\chi_{t})\omega(s)|_{L^{4}(\Omega)}^{2} |\vartheta(s)|_{H}^{2} dx + \frac{1}{4} |(\chi_{t})\omega|_{L^{2}(0,t,V)}^{2} + c |(\chi_{t})\omega|_{L^{2}(0,t,V')}^{2}
\]
\[
\leq c \int_{0}^{t} ds \int_{\Omega} |(\chi_{t})\omega(s)|_{L^{4}(\Omega)}^{2} (1 + e^{2\omega s}) dx + \frac{1}{4} |(\chi_{t})\omega|_{L^{2}(0,t,V)}^{2} + c |(\chi_{t})\omega|_{L^{2}(0,t,V')}^{2}
\]
\[
\leq c \int_{0}^{t} ds \int_{\Omega} |(\chi_{t})\omega(s)|_{V}^{7/4} |(\chi_{t})\omega(s)|_{V}^{1/4} (1 + e^{2\omega s}) dx + \frac{1}{4} |(\chi_{t})\omega|_{L^{2}(0,t,V)}^{2} + c |(\chi_{t})\omega|_{L^{2}(0,t,V')}^{2}
\]
\[
\leq \frac{1}{2} |(\chi_{t})\omega|_{L^{2}(0,t,V)}^{2} + c |(\chi_{t})\omega|_{L^{2}(0,t,V')}^{2} (1 + e^{16\omega T})
\]
\[
\leq \frac{1}{2} |(\chi_{t})\omega|_{L^{2}(0,t,V)}^{2} + c (1 + e^{16\omega T}).
\]

Finally, in order to estimate $I_{5}$ we need again to use (a2) together with (2.19–2.20), the well-known H"older inequality, (3.1), and (3.3–3.4) to get
\[
I_{5}(t) \leq c \int_{0}^{t} ds \int_{\Omega} (1 + |\chi(s)||\chi_{t}(s)|\Delta \vartheta(s)\exp(-2\omega s) dx
\]
\[
\leq c (1 + e^{\omega T}) \int_{0}^{t} |(\chi_{t})\omega(s)|_{V}^{7/8} |(\chi_{t})\omega(s)|_{V}^{7/8} |(B\vartheta)\omega(s)|_{H} ds
\]
\[
\leq \frac{k_{0}}{4} \int_{0}^{t} |(B\vartheta)\omega(s)|_{H}^{2} ds + c c_{k_{0}} \int_{0}^{t} |(\chi_{t})\omega(s)|_{V}^{5/4} |(\chi_{t})\omega(s)|_{V}^{3/4} (1 + e^{2\omega T}) ds
\]
\[
\leq \frac{k_{0}}{4} \int_{0}^{t} |(B\vartheta)\omega(s)|_{H}^{2} ds + \frac{1}{4} \int_{0}^{t} |(\chi_{t})\omega(s)|_{V}^{2} ds + c c_{k_{0}} (1 + e^{16\omega T}).
\]
Now, summing up (3.10) and (3.11), collecting (3.12–3.14), and using (a3–a4) with (r1–
Hence, thanks to the obvious fact that $\exp(-2\omega T) \leq \exp(-2\omega t)$ for all $t \in [0, T]$, using
and (3.6) using the auxiliary variables (2.25). For any
Second estimate. In order to obtain the $L^\infty(0, T; W)$ bound on $\chi$, we multiply (2.13)
by $\chi$ and use the definition of $u$ given in (2.14) along with the previous regularity estimate
which implies
This concludes the proof of estimates (2.23–2.24) of Theorem 2.4.
Proof of (2.26). In order to conclude the proof of Theorem 2.4 it remains to prove
inequality (2.26). Hence, let us consider assumptions (r4–r5) and rewrite equations (2.12)
and (3.6) using the auxiliary variables (2.25). For any $t \in (0, T)$ we get:
\[
\partial_t u + \ell \partial_v + k_0 B u = f_0 - k \ast B u \quad \text{in } V';
\]  
(3.18)
\[
\partial_t v + B^2 \partial_v + B \left( \left[ (\beta' + \sigma')(\chi_t + t\chi_t(0) + v) \right] (\chi_t(0) + \partial_t v) \right) = -B^2 \chi_t(0) + \ell B (\partial_t u) \quad \text{in } V',
\]  
(3.19)
where
\[
f_0 := f - \ell \chi_t(0) - k_0 B \partial_0 - (1 \ast k) B \partial_0.
\]  
(3.20)
Test (3.19) with $-\exp(-2\omega t)\mathcal{N}(\partial_t v)$ and sum it up to (3.18) multiplied by $\exp(-2\omega t)\partial_t u$
and integrate the resulting equation on $(0, t)$, $t \in (0, T)$. Notice that, for $t \in (0, T)$,
applying (3.1) and (3.3), it follows:

\[
(\beta'(\chi(t))(\chi_t(0) + \partial_t v(t)), \partial_t v(t)) = (\beta'(\chi(t))\partial_t v(t), \partial_t v(t)) + \int_\Omega \beta'(\chi(t))^{1/2} \chi_t(0)[\beta'(\chi(t))]^{1/2} \partial_t v(t) \, dx
\]

\[
\geq \frac{1}{2}[\beta'(\chi(t))]^{1/2} \partial_t v \bigg| \frac{1}{2} \| \chi_t(0) \|_{L^\infty(\Omega)}^2 \| \beta'(\chi(t)) \|_{L^1(\Omega)}; \tag{3.21}
\]

\[
\| (B^2(\chi_t(0)), N(\partial_t v(t))) \| \leq \| \nabla \chi_t(0) \|_H^2 + \frac{1}{4} \| \nabla \partial_t v(t) \|_H^2; \tag{3.22}
\]

\[
|\sigma''(\chi) \chi_t(0) + \partial_t v(t), \partial_t v(t) | \leq c(\| \chi_t(0) \|_H^2 + \| \partial_t v(t) \|_V^2) + \frac{1}{4} \| \partial_t v(t) \|_V^2. \tag{3.23}
\]

Since \( u(0) = 0 \) and \( \partial_t u(0) = 0 \), we obtain

\[
\frac{k_0}{2} |\nabla u(\omega(t))|_H^2 + \frac{1}{2} |(\partial_t v(\omega(t)))|_H^2 + \int_0^t \left[ |(\partial_t u(\omega))|_H^2 + \frac{1}{2} |(\nabla \partial_t v(\omega))|_H^2 \right]
+ \frac{1}{2} \int_0^t \left[ |(\beta'(\chi))^{1/2} (\partial_t v(\omega))|_H^2 = \frac{1}{2} |\chi_t(0)|_{L^\infty(\Omega)}^2 \int_0^t e^{-2\omega s} |\beta'(\chi)|_{L^1(\Omega)} \, ds + \int_0^t |(f_0)\omega|_H^2 + \frac{1}{2} \int_0^t |(\partial_t v(\omega))|_V^2. \tag{3.24}
\]

Now, multiply (3.18) by \( e^{-2\omega t}Bu \) and note that, for \( t \in (0, T) \), we have

\[
e^{-2\omega t}(\partial_t u(t), B(u(t))) = \frac{d}{dt} \int e^{-2\omega t}|\nabla u(t)|_H^2 + \omega e^{-2\omega t}|\nabla u(t)|_H^2. \tag{3.25}
\]

Hence, integrating over \((0, t)\), we get

\[
\frac{1}{2} |\nabla u(\omega(t))|_H^2 + \int_0^t \left[ \omega |(\nabla u(\omega))|_H^2 + \frac{k_0}{2} |(Bu(\omega))|_H^2 \right] \leq \frac{2}{k_0} \int_0^t |(f_0)\omega|_H^2 + \frac{2}{k_0} \int_0^t |(\partial_t v(\omega))|_H^2 + \frac{1}{4} \int_0^t |(\nabla \partial_t v(\omega))|_H^2. \tag{3.26}
\]

Let us note that thanks to (3.20), we can estimate the term containing \( f_0 \) in (3.26) using also (2.18) as follows:

\[
\frac{2}{k_0} \int_0^t |(f_0)\omega|_H^2 \leq c \int \omega |(\chi_0)|_H^2 + c \int |B(\omega)|_H^2 + \int \omega |(B\partial_0)|_H^2 \leq c \int \omega |(\chi_0)|_H^2 + \frac{1}{4} \int_0^t |(\nabla \partial_t v(\omega))|_H^2 + \frac{1}{4} \int_0^t |(\nabla \partial_t v(\omega))|_H^2. \tag{3.27}
\]

Analogously, one can estimate the term containing \( f_0 \) in (3.24). Now, summing up (3.24) and (3.26), treating as in (3.27) the convolution terms, and using (2.18) and Gronwall’s lemma, we obtain

\[
|\nabla u(\omega)|_H^2 \leq \| (\partial_t u(\omega))_H^2 \|_L^2(0,T;H) + \| (Bu(\omega))_H^2 \|_L^2(0,T;H) + \| (\partial_t v(\omega))_H^2 \|_L^2(0,T;V)
+ \| (\partial_t v(\omega))_H^2 \|_L^2(0,T;V') + \| (\beta'(\chi))^{1/2} \partial_t v(\omega)|_H^2 \|_L^2(0,T;H) \leq c \left[ |\chi_t(0)|_{L^\infty(\Omega)}^2 |[\beta'(\chi)]_{2\omega} \|_{L^1(Q)} + \omega^{-2} \|\chi_t(0)\|_V^2 + \omega^{-2} |B\partial_0|^2 + \|f_0\|_{L^2(0,T;H)}^2 \right]. \tag{3.28}
\]
Hence, in that case, we get immediately

\[ \mu \text{ vary from line to line) of the form } \mu(1 + e^{\mu \omega T}), \text{ where } \omega \text{ is as in } (2.18) \text{ and } \mu > 0 \text{ may depend on the data, on } k_0, \text{ on } T, \text{ but not on } \omega \text{ and } k. \]

### 3.2 Proof of Theorem 2.9

Let us denote from now on by \( \kappa(\omega) \) a positive constant (whose value is still allowed to vary from line to line) of the form \( \mu(1 + e^{\mu \omega T}) \), where \( \omega \) is as in (2.18) and \( \mu > 0 \) may depend on the data, on \( k_0 \), on \( T \), but not on \( \omega \) and \( k \).

**First estimate.** In order to get the regularity properties (2.27) we begin multiplying (2.12) by \( \exp(-2\omega s)B^2 \theta \) (with \( (k, \omega) \) as in (2.18)) obtaining (thanks also to (3.1) and (3.2), and after an integration over \((0, t)\) with \(0 \leq s \leq t \leq T\))

\[
\frac{1}{2} |\theta_\omega(t)|_{H^2(\Omega)}^2 + k_0 \int_0^t |\theta_\omega(s)|_{H^3(\Omega)}^2 ds \leq \frac{1}{2} |(\theta_0)_\omega|_{H^2(\Omega)}^2 + c_k_\omega |(f_1)_\omega|_{L^2(0,T;V)}^2 \\
+ \int_0^t |(f_2)_\omega(s)|_{H^2(\Omega)} |\theta_\omega(s)|_{H^2(\Omega)} ds + \frac{k_0}{8} |\theta_\omega|_{L^2(0,t;H^3(\Omega))^2}^2 \\
+ c_k \int_0^t \exp(-2\omega s) |\lambda'(\chi(s))\chi_t(s)|_{H^1(\Omega)}^2 ds. \tag{3.29}
\]

Now, we can directly estimate the last integral in (3.29), using (2.23) and (2.24), and Sobolev’s embeddings in the following way:

\[
c_k \int_0^t \exp(-2\omega s) |\lambda'(\chi(s))\chi_t(s)|_{H^1(\Omega)}^2 ds \\
\leq c_k \int_0^t ds \int_\Omega \exp(-2\omega s) |\lambda''(\chi(s))| \nabla \chi(s)^2 dx \\
+ c_k \int_0^t ds \int_\Omega \exp(-2\omega s) |\lambda'(\chi(s))| \nabla \chi_t(s)^2 dx + c_k \int_0^t \lambda'(\chi(s))^2 \chi_t^2(s) ds \\
\leq c c_k_\omega |(\lambda_t)_\omega|_{L^2(0,T;V)}^2 (1 + |\chi|^2_{L^{\infty}(0,T;W^1)}) \leq \kappa(\omega). \tag{3.30}
\]

Finally, using the regularity properties of the data (r6) and (r7), putting (3.30) and (3.29) together, and applying the Gronwall lemma (cf. [4, Appendix]), we get

\[
|\theta_\omega|_{L^{\infty}(0,T;H^2(\Omega))} + c_k_\omega |\theta_\omega|_{L^2(0,T;H^3(\Omega))^2} \leq \kappa(\omega) \tag{3.31}
\]

and hence also (2.27). Note that in case (r4–r5) and (m1) or (m2) hold, then we can employ estimate (2.26) and rewrite (3.30) as

\[
c_k \int_0^t \exp(-2\omega s) |\lambda'(\chi(s))\chi_t(s)|_{H^1(\Omega)}^2 ds = c_k \int_0^t \exp(-2\omega s) \ell^2 |\chi_t(s)|_{H^1(\Omega)}^2 ds \\
\leq c |(\chi_t)_\omega|_{L^2(0,T;V)}^2 \leq c \left[ 1 + \frac{1}{\omega^{1/p'}} \right].
\]

Hence, in that case, we get immediately

\[
|\theta_\omega|_{L^{\infty}(0,T;H^2(\Omega))\cap L^2(0,T;H^3(\Omega))} \leq c \left[ 1 + \frac{1}{\omega^{1/p'}} \right].
\]
Second estimate. We proceed now in order to recover (2.30). Hence, we test (2.13) by $B^3x$, integrate between 0 and $t$ with $t \in (0, T)$, and use (2.14) along with Hölder’s inequality and (3.1) obtaining

$$\frac{1}{2}|\chi(t)|_{H^3(\Omega)}^2 + \frac{1}{4}|\chi|^2|_{L^2(0,t;H^5(\Omega))} \leq \frac{1}{2}|\chi_0|_{H^3(\Omega)}^2$$

$$+ \int_0^t \left( |\beta(\chi(s)) + \sigma'(\chi(s))|^2|_{H^3(\Omega)} + |\lambda'(\chi(s))\vartheta(s)|^2|_{H^3(\Omega)} \right) ds.$$

Then, we can employ Lemmas 6.2 and 6.3 (with $\nu = 3$, $\gamma = \beta + \sigma'$, $\alpha = \lambda$, $\nu = \chi$, $w = \chi$, and $z = \vartheta$) and the regularity assumptions (r3), (r8–r9) along with the estimates (2.24) and (2.27), obtaining

$$\frac{1}{2}|\chi(t)|_{H^3(\Omega)}^2 + \frac{1}{4}|\chi|^2|_{L^2(0,t;H^5(\Omega))} \leq \kappa(\omega)\left(1 + |\chi|_{L^2(0,t;H^3(\Omega))}^2\right).$$

Finally, applying the standard Gronwall lemma, we get the bound:

$$|\chi|_{L^\infty(0,T;H^3(\Omega))}^2 + |\chi|_{L^2(0,T;H^5(\Omega))}^2 \leq \kappa(\omega). \tag{3.32}$$

This concludes the proof of Theorem 2.9.

\section{3.3 Proof of Theorem 2.10}

First estimate. We proceed now to prove (2.32–2.33). Hence, we take (3.5–3.9) with $\lambda'' \equiv L$ (cf. (r11)) and test (3.5) by $-2\omega s \vartheta_t$ (with $\omega$ as in (2.18)), sum it up to (3.6) multiplied by $\mathcal{N}(\exp(2\omega s)\chi_{tt})$, integrate over $(0, t)$ with $0 \leq s \leq t \leq T$, and use (3.7) obtaining

$$\frac{1}{2}|(\vartheta_t)_{\omega}(t)|_{H^3}^2 + k_0|(|\nabla \vartheta_t)_{\omega}|_{L^2(0,t;H)}^2 + |(\chi_{tt})_{\omega}|_{L^2(0,t;V')}^2 + \frac{1}{2}|(|\nabla \chi_t)_{\omega}(t)|_{H}^2$$

$$\leq \frac{1}{2}|\vartheta_t(0)|_{H}^2 + \frac{1}{2}|\nabla \chi_t(0)|_{H}^2 + \frac{k_0}{4}|(|\vartheta_t)_{\omega}|_{L^2(0,T;V')}^2 + c_k_0|(|f(s)_{\omega})_{\omega}|_{L^2(0,T;V')}^2 + \sum_{i=1}^4 I_i(t)$$

$$+ \int_0^t \left[ |(f_{ix})(s)_{\omega}|_{H}|(\vartheta_t)_{\omega}(s)|_{H} \right] ds + c_k_0 \int_0^t |k_{\omega}(s)||((\vartheta_t)_{\omega}(s))_{H}B\vartheta_0|_{H} ds, \tag{3.33}$$

where

$$I_1(t) := -\int_0^t ds \int_\Omega \left[ (k \ast \nabla \vartheta_t(s)) \nabla \vartheta_t(s) \exp(-2\omega s) \right] dx;$$

$$I_2(t) := -L \int_0^t ds \int_\Omega \left[ \chi_{tt}(s) \vartheta_t(s) \exp(-2\omega s) \right] dx;$$

$$I_3(t) := -\int_0^t ds \int_\Omega \left[ (\beta'(\chi(s)) + \sigma''(\chi(s))) \chi_t(s) \chi_{tt}(s) \exp(-2\omega s) \right] dx;$$

$$I_4(t) := L \int_0^t ds \int_\Omega \left[ \chi_t(s) \chi_{tt}(s) \vartheta(s) \exp(-2\omega s) \right] dx.$$
Now, we use Hölder’s inequality and (2.18) obtaining
\[
I_1(t) \leq \frac{k_0}{4} \| \nabla \partial_t \omega \|_{L^2(0,T;H)}^2.
\] (3.34)

Using again Hölder’s inequality and the continuous embedding of $V$ into $L^4(\Omega)$, we get
\[
I_2(t) \leq c_L \int_0^t \left[ \| \partial_t \omega \|_H \| \chi_t \omega \|_V^2 e^\omega t \right] ds.
\] (3.35)

Note that the function $|\chi_t \omega|^2$ is in $L^4(0,T)$ due to (2.23). Now, integrating by parts in time and using assumptions (r13–r14), (r11), along with (3.3) and (3.17), we get
\[
I_3(t) = \frac{1}{2} \int_0^t ds \int_{\Omega} \left( [\beta''(\chi(s)) + \sigma''(\chi(s))] \chi_t^2(s) \exp(-2\omega s) \right) dx
- \omega \int_0^t ds \int_{\Omega} \left( [\beta'(\chi(s)) + \sigma''(\chi(s))] \chi_t^2(s) \exp(-2\omega s) \right) dx
- \frac{1}{2} \int_0^t \int_{\Omega} \left( [\beta'(\chi(t)) + \sigma''(\chi(t))] \chi_t^2(t) \exp(-2\omega t) \right) dx
+ \frac{1}{2} \int_0^t \int_{\Omega} \left( [\beta'(\chi_0) + \sigma''(\chi_0)] \chi_t(0)^2 \right) dx
\leq \kappa(\omega) \left[ \int_0^t \left( |\chi_t \omega(s)|_H |(\chi_t \omega(s))_H + |(\chi_t \omega(s))|_{L^4(\Omega)} \cdot |\chi_t(s)|_{L^4(\Omega)} \right) ds
+ 1 + c_\delta |\chi_t \omega(t)|_V^2 + \delta |\chi_t \omega(t)|_V^2 \right]
\] (3.36)

for all $\delta > 0$ and some constant $c_\delta > 0$.

Note that the function $t \mapsto |\chi_t(t)|_{L^4(\Omega)}$ lies in $L^4(0,T)$ due to (2.23–2.24). Finally, integrating again by parts in time and using (2.20) and the continuous and compact embedding of $V$ into $L^4(\Omega)$ (cf. (3.3)), we get
\[
I_4(t) = -\frac{L}{2} \left( \int_0^t ds \int_{\Omega} \chi_t^2(s) \partial_t(s) \exp(-2\omega s) dx - \int_0^t \chi_t^2(t) \partial_t(t) \exp(-2\omega t) dt \right)
+ \int_0^t \chi_t^2(0) \partial_t(0) dx + \omega L \int_0^t ds \int_{\Omega} \chi_t^2(s) \partial_t(s) \exp(-2\omega s) dx
\leq c_L \left( 1 + \int_0^t |(\chi_t \omega(s))_V |(\chi_t \omega(s))_V |(\partial_t(s))_H + \omega |\partial(s)|_H \right) ds
+ \eta |\chi_t \omega(t)|_V^2 + c_\eta |\chi_t \omega(t)|_V^2
\] (3.37)

for all $\eta > 0$ and some positive constant $c_\eta$.

Note that the function $t \mapsto |(\chi_t \omega(t))_V |(\partial_t(t))_H + \omega |(\partial(t))_H$ belongs to $L^4(0,T)$ due to (2.23–2.24). Now, using (3.34–3.37) in (3.33) (with suitable $\delta$ and $\eta$) and the Gronwall’s lemma [4, Lemme A.5., p. 156], we get the desired bound
\[
|\partial_t \omega|^2_{L^2(0,T;V')} + |(\chi_t \omega)|^2_{L^2(0,T;V')} + |(\chi_t \omega)|^2_{L^\infty(0,T;V)} \leq \kappa(\omega).
\] (3.38)

and hence also
\[
|\partial_t|^2_{L^2(0,T;V')} + |\chi_t|^2_{L^2(0,T;V')} + |\chi_t|^2_{L^\infty(0,T;V)} \leq \kappa(\omega).
\] (3.39)
Second estimate. We multiply (3.6) (with $\lambda'' \equiv L$ (cf. (r10))) by $\exp (-2\omega s) B \chi_t$, with $(k, \omega)$ as in (2.18), and use (3.7). After an integration over $(0, t)$ with $0 \leq s \leq t \leq T$ and using (3.2), (r12–r14), and the estimates (3.17), (3.38), we get

$$
\frac{1}{2} |(\chi_t)_\omega(t)|_V^2 + \int_0^t |(\chi_t)_\omega(s)|_{H^3(\Omega)}^2 ds \leq \frac{1}{2} |\chi_t(0)|_V^2 + \frac{1}{2} |(\chi_t)_\omega|_{L^2(0,T; H^3(\Omega))}^2
$$

$$
+ \int_0^t \exp (-2\omega s) [\sigma''(\chi(s)) + \beta'(\chi(s))] \chi_t(s)|_V^2 ds
$$

$$
+ c_\lambda \int_0^t \left[ |\vartheta(s)|_{L^\infty(\Omega)}^2 + |\vartheta(s)|_{H^2(\Omega)}^2 \right] |(\chi_t)_\omega(s)|_V |(\chi_t)_\omega(s)|_V ds
$$

$$
+ c_\lambda \int_0^t (1 + |\chi(s)|_{W^1}^2 + |\chi(s)|_{L^\infty(\Omega)}^2) |(\chi_t)_\omega(s)|_V^2 ds
$$

$$
\leq c_\lambda \left( \int_0^t \left[ |\chi(s)|_W^2 + |\vartheta(s)|_{L^\infty(\Omega)}^2 + |\vartheta(s)|_{H^2(\Omega)}^2 \right] |(\chi_t)_\omega(s)|_V |(\chi_t)_\omega(s)|_V ds \right)
$$

$$
+ \kappa(\omega) + \frac{1}{2} |(\chi_t)_\omega|_{L^2(0,T; H^3(\Omega))}^2.
$$

Doing this computation, we have used the estimate (with $\alpha := \beta' + \sigma''$)

$$
|\alpha'(\chi)\chi_t|_V^2 \leq c |\alpha'(\chi)|_{L^\infty(\Omega)} |\chi_t|_V^2 + |\alpha''(\chi)\nabla \chi \chi_t|_H^2 \leq c_\chi |\chi|_W |\chi_t|_V^2.
$$

Note that, due to estimates (3.15), (3.17), and (3.31), the function

$$
s \mapsto \left[ |\chi(s)|_W^2 + |\vartheta(s)|_{L^\infty(\Omega)}^2 + |\vartheta(s)|_{H^2(\Omega)}^2 \right]
$$

is bounded in $L^1(0, T)$. Finally, applying the standard Gronwall lemma and using (3.38), we get the bound

$$
|\chi_\omega|_{W^{1, \infty}(0,T; V) \cap H^2(0,T; H^3(\Omega))} \leq \kappa(\omega).
$$

(3.40)

Third estimate. Multiply (2.12) by $\exp (-2\omega s) B^4 \vartheta$ (with $\omega$ as in (2.18)) obtaining, thanks also to (3.1) and (3.2) and after an integration over $(0, t)$ ($0 \leq s \leq t \leq T$),

$$
\frac{1}{2} |\vartheta(t)|_{H^4(\Omega)}^2 + \frac{k_0}{4} \int_0^t |\vartheta(s)|_{H^4(\Omega)}^2 ds \leq \frac{1}{2} |(\vartheta_0)_t|_{H^4(\Omega)}^2 + \frac{k_0}{8} |\vartheta_0|_{L^2(0,T; H^3(\Omega))}^2
$$

$$
+ c_\lambda \int_0^t \exp (-2\omega s) |\lambda'(\chi(s))\chi_t(s)|_{H^3(\Omega)}^2 ds
$$

$$
+ c_\lambda \int_0^t \exp (-2\omega s) |\lambda'(\chi(s))\chi_t(s)|_{H^3(\Omega)}^2 ds,
$$

(3.41)

where we have used the notation (2.7). We can employ now Lemma 6.3 (in particular (6.7) with the choices $\nu = 3$, $\alpha \equiv \lambda$, $w = \chi$, $z = \chi_t$) in order to estimate the last integral in (3.29):

$$
c_\lambda \int_0^t \exp (-2\omega s) |\lambda'(\chi(s))\chi_t(s)|_{H^3(\Omega)}^2 ds \leq \kappa(\omega)(1 + |(\chi_t)_\omega|_{L^2(0,T; H^3(\Omega))}^2) \leq \kappa(\omega).
$$

(3.42)
Let us proceed by induction. Hence, let us first prove the property for 3.4 Proof of Theorem 2.11

Note that in order to obtain the last inequality in (3.42) we have used (3.40). Finally, using (r12), (r13), and (3.41), an application of Gronwall’s lemma in (3.42) gives

\[ |\vartheta|_{L^\infty(0,T;H^s(\Omega))}^2 + c_{\lambda_0} |\vartheta|_{L^2(0,T;H^5(\Omega))}^2 \leq \kappa(\omega) \]  

and hence also

\[ |\vartheta|_{L^\infty(0,T;H^4(\Omega))}^2 + c_{\lambda_0} |\vartheta|_{L^2(0,T;H^6(\Omega))}^2 \leq \kappa(\omega). \]  

Fourth estimate. In order to conclude the proof of Theorem 2.10, we test (2.13) with \( B^m x \) (\( m = 4, 5 \)), integrate between 0 and \( t \) with \( t \in (0, T) \), and use (2.14) along with Hölder’s inequality and (3.1), obtaining

\[ \frac{1}{2} |\chi(t)|_{H^m(\Omega)}^2 + \frac{1}{4} |\chi|_{L^2(0,t;H^{m+2}(\Omega))}^2 \leq \frac{1}{2} |\chi_0|_{H^m(\Omega)}^2 \]

\[ + \int_0^t \left( |\beta(\chi(s)) + \sigma'(\chi(s))|_{H^m(\Omega)}^2 + |\lambda'(\chi(s))\vartheta(s)|_{H^m(\Omega)}^2 \right) ds. \]  

(3.45)

Then, we can employ Lemmas 6.2 and 6.3 (with \( \nu = m \) and \( r = \beta + \sigma', \alpha = \lambda, v = \chi, w = \chi, z = \vartheta \)). Taking advantage of (r9–r13) and (3.32) and setting \( m = 4 \), we get

\[ |\chi|_{C^0([0,T];H^4(\Omega))}^2 + |\chi|_{L^2([0,T];H^5(\Omega))}^2 \leq \kappa(\omega). \]  

(3.46)

Next, repeating the procedure with \( m = 5 \) and using (3.46), we deduce

\[ |\chi|_{C^0([0,T];H^5(\Omega))}^2 + |\chi|_{L^2([0,T];H^6(\Omega))}^2 \leq \kappa(\omega). \]  

(3.47)

This concludes the proof of Theorem 2.10. \( \blacksquare \)

### 3.4 Proof of Theorem 2.11

Let us proceed by induction. Hence, let us first prove the property for \( s = 0 \). Then, let us consider (3.5–3.9) with \( \chi \equiv \ell \), multiply (3.5) by \( B(\vartheta_0 \exp(-2\omega t)) \) and sum it up to (3.6) tested by \( \chi_\ell \exp(-2\omega t) \) (with \((k, \omega) as in (2.18)). Using also (3.7), we get

\[ \frac{1}{2} \frac{d}{dt} \left( |(\vartheta_\ell(t)|_{L^2}^2 + |(B\chi_\ell(t)|_{H^s}^2 \right) + k_0 |(B\vartheta_\ell(t)|_{H^s}^2 + |(\chi_\ell(t)|_{H^s}^2 \right) \]

\[ \leq - \exp(-2\omega t) \langle B(\cdot \vartheta_\ell(t)), B\vartheta_\ell(t) \rangle \]

\[ -k(t) \exp(-2\omega t) \langle B\vartheta_\ell(t), B\vartheta_\ell(t) \rangle + \exp(-2\omega t) \langle B\vartheta_\ell(t), f_\ell(t) \rangle \]

\[ - \exp(-2\omega t) \langle B[\sigma''(\chi(t))\chi_\ell(t) + \beta'(\chi(t))\vartheta_\ell(t), \chi_\ell(t) \rangle. \]  

(3.48)

It is now easy to check that, thanks to the assumptions on \( k \) (2.18), \( f \) (r15), \( \vartheta_\ell \) (r17), and \( \beta \) and \( \sigma \) (r16), all terms on the right hand side are uniformly controlled by the quantities on the left hand side plus \( \kappa(\omega) \). Actually, performing a direct calculation and using (2.34) in Theorem 2.9, it is easy to see that

\[ |B[\sigma''(\chi(t))\chi_\ell(t) + \beta'(\chi(t))\vartheta_\ell(t)|_{H^s} \leq \kappa(\omega) \langle 1 + |\chi(t)|_{H^s(\Omega)} \rangle |\chi_\ell(t)|_{L^2} \leq \kappa(\omega) |\chi_\ell(t)|_{L^2}. \]
Thus, (3.48) readily gives

\[ |(\partial_t)^2 \omega|_{L^2(0,T;V)} + |(\partial_t) \omega|_{H^1(0,T;H^2(\Omega))} + |(\chi_t) \omega|_{L^2(0,T;H)} + |(\chi) \omega|_{W^{1,\infty}(0,T;H^2(\Omega))} \leq \kappa(\omega) \]

and hence also

\[ |\partial_t|^2 \omega|_{L^2(0,T;V)} + |\partial_t|^2 \omega|_{H^1(0,T;H^2(\Omega))} + |\chi_t|^2 |_{L^2(0,T;H^2(\Omega))} + |\chi|^2 |_{W^{1,\infty}(0,T;H^2(\Omega))} \leq \kappa(\omega). \]

Next, to complete the induction argument and conclude the proof, let us suppose our property true for \( s - 1, \ s \geq 1 \), and show it at the step \( s \). Consider \( (3.5–3.9) \) with \( \lambda' \equiv \ell \), multiply \( (3.5) \) by \( B^{s+1}(\partial_t) \exp(-2\omega t) \), and sum the result to \( (3.6) \) tested by \( B^s \chi_t \exp(-2\omega t) \) (with \( \omega \) as in (2.18)). This gives

\[
\frac{1}{2} \frac{d}{dt} |(\partial_t) \omega|_{H^{s+1}(\Omega)} + |(\chi_t) \omega|_{H^{s+2}(\Omega)} + (k_0 - |k_\omega|^2_{L^1(0,T)}) |(\partial_t) \omega|_{H^{s+2}(\Omega)} + |(\chi_t) \omega|_{H^{s+1}(\Omega)} \\
\leq c_s |(f_{1t}) \omega|_{H^{s+1}(\Omega)} + \delta |(\partial_t) \omega|_{H^{s+2}(\Omega)} + |(f_{2t}) \omega|_{H^{s+1}(\Omega)} |(\partial_t) \omega|_{H^{s+1}(\Omega)} \\
+ |k_\omega| |(\partial_t) \omega|_{H^{s+1}(\Omega)} |\partial_0^2 |_{H^{s+3}(\Omega)} - \int_\Omega \exp(-2\omega t)[\sigma''(\chi) \chi_t + \beta'(\chi) \chi_t] B^{s+1} \chi_t \, dx.
\]

Using now Lemma 6.3, with \( \alpha \equiv \sigma' + \beta, \nu = s + 2, \ w = \chi, z = \chi_t \), we have

\[
- \int_\Omega \exp(-2\omega t)[\sigma''(\chi) \chi_t + \beta'(\chi) \chi_t] B^{s+1} \chi_t \, dx \\
\leq \frac{1}{2} |(\chi_t) \omega|_{H^s(\Omega)} + c(|\chi_t|_{H^{s+1}(\Omega)}, |\chi_t|_{H^{s+2}(\Omega)}) (|\chi_t|^2_{H^{s+2}(\Omega)} + |\chi_t|^2_{H^{s+1}(\Omega)}).
\]

Then, thanks to (r14–r17), using the induction hypothesis and Gronwall’s lemma we get

\[ |\partial_t \omega|_{W^{1,\infty}(0,T;H^{s+1}(\Omega)) \cap H^1(0,T;H^{s+2}(\Omega))} + |\chi_t|_{H^2(0,T;H^s(\Omega)) \cap W^{1,\infty}(0,T;H^{s+2}(\Omega))} \leq \kappa(\omega) \]

and hence

\[ |\partial_t \omega|_{W^{1,\infty}(0,T;H^{s+1}(\Omega)) \cap H^1(0,T;H^{s+2}(\Omega))} + |\chi_t|_{H^2(0,T;H^s(\Omega)) \cap W^{1,\infty}(0,T;H^{s+2}(\Omega))} \leq \kappa(\omega). \]

### 4 Proof of Theorem 2.12

Assume that \((\varphi_j, \chi_j), \ j = 1, 2\), are two solutions to system (2.12–2.16) corresponding to the data \((f^j, k_j, \varphi_{0j}, \chi_{0j})\).

Consider then the identities, where \( \gamma = \beta + \sigma' \),

\[
\Delta [\gamma(\chi_2) - \gamma(\chi_1)] = \Delta \left\{ \int_0^1 \gamma'(s \chi_2 + (1 - s) \chi_1) \, ds (\chi_2 - \chi_1) \right\} \\
= \int_0^1 \Delta \{\gamma'(s \chi_2 + (1 - s) \chi_1) \} \, ds (\chi_2 - \chi_1) \\
+ 2 \int_0^1 \nabla \{\gamma'(s \chi_2 + (1 - s) \chi_1) \} \cdot \nabla (\chi_2 - \chi_1) \\
+ \int_0^1 \gamma'(s \chi_2 + (1 - s) \chi_1) \, ds \Delta (\chi_2 - \chi_1); \quad (4.1)
\]
\[ \nabla \{ \gamma'(s\chi_2 + (1-s)\chi_1) \} = \gamma''(s\chi_2 + (1-s)\chi_1)(s\nabla\chi_2 + (1-s)\nabla\chi_1); \quad (4.2) \]
\[ \Delta \{ \gamma'(s\chi_2 + (1-s)\chi_1) \} = \gamma''(s\chi_2 + (1-s)\chi_1)(s\Delta\chi_2 + (1-s)\Delta\chi_1) \\
+ \gamma'''(s\chi_2 + (1-s)\chi_1)|s\nabla\chi_2 + (1-s)\nabla\chi_1|^2. \quad (4.3) \]

Introduce now the auxiliary functions
\[ \tilde{\vartheta} := \vartheta_2 - \vartheta_1, \quad \tilde{\chi} := \chi_2 - \chi_1, \quad \tilde{f} := f^2 - f^1, \quad \tilde{k} := k_2 - k_1, \quad (4.4) \]
\[ \tilde{\vartheta}_0 := \vartheta_{0,2} - \vartheta_{0,1}, \quad \tilde{\chi}_0 := \chi_{0,2} - \chi_{0,1}. \quad (4.5) \]

Then, from equations (2.12)–(2.16) we easily deduce that \((\tilde{\vartheta}, \tilde{\chi})\) solves the problem
\[ \partial_t(\tilde{\vartheta} + t\tilde{\chi}) + k_0B\tilde{\vartheta} = -k_2 * B\tilde{\vartheta} - \tilde{k} * B\partial_1 + \tilde{f} \quad \text{in} \ Q, \quad (4.6) \]
\[ \partial_t\tilde{\chi} + B^2\tilde{\chi} = -\alpha_0\tilde{\chi} - \alpha_1 \cdot \nabla\tilde{\chi} - \alpha_2 B\tilde{\chi} + \ell B\tilde{\vartheta} \quad \text{in} \ Q, \quad (4.7) \]
\[ \tilde{\vartheta}(0) = \tilde{\vartheta}_0, \quad \tilde{\chi}(0) = \tilde{\chi}_0 \quad \text{in} \ \Omega, \quad (4.8) \]

where
\[ \alpha_0 := \int_0^1 \Delta \{(\beta' + \sigma'')(s\chi_2 + (1-s)\chi_1)\} \, ds, \quad (4.9) \]
\[ \alpha_1 := 2 \int_0^1 \nabla \{(\beta' + \sigma'')(s\chi_2 + (1-s)\chi_1)\} \, ds, \quad (4.10) \]
\[ \alpha_{2+j} := \int_0^1 (\beta^{(1+j)} + \sigma^{(2+j)})(s\chi_2 + (1-s)\chi_1) \, ds, \quad j = 0, 1, 2. \quad (4.11) \]

Introduce now the new unknown
\[ \tilde{\eta} = \tilde{\vartheta} + t\tilde{\chi} \iff \tilde{\vartheta} = \tilde{\eta} - t\tilde{\chi}. \quad (4.12) \]

Then, problem (4.6–4.8) is equivalent to the following one, which is in normal form:
\[ \partial_t\tilde{\eta} + k_0B\tilde{\eta} = \tilde{f} - \tilde{k} * B\partial_1 - \tilde{k} * B\tilde{\vartheta} + k_0 \ell B\tilde{\chi} + \ell k_2 * B\tilde{\chi} \]
\[ =: L_1(\tilde{f}, \tilde{k}, \tilde{\vartheta}) + L_2(\tilde{\eta}, \tilde{\chi}) \quad \text{in} \ Q, \quad (4.13) \]
\[ \partial_t\tilde{\chi} + B^2\tilde{\chi} = -\alpha_0\tilde{\chi} - \alpha_1 \cdot \nabla\tilde{\chi} - \alpha_2 B\tilde{\chi} + \ell B\tilde{\vartheta} - \ell^2 B\tilde{\chi} =: L_3(\tilde{\eta}, \tilde{\chi}) \quad \text{in} \ Q, \quad (4.14) \]
\[ \tilde{\eta}(0) = \tilde{\vartheta}_0 + t\tilde{\chi}_0 =: \tilde{\eta}_0, \quad \tilde{\chi}(0) = \tilde{\chi}_0 \quad \text{in} \ \Omega. \quad (4.15) \]

Observe now that according to formulae (4.2), (4.3), (4.9–4.11), and Theorem 2.11 with \(s = 2\) we deduce that all functions \(\alpha_j, j = 0, 1, 2\), belong to \(W^{1,\infty}(0, T; C(\Omega))\) and can be estimated by a constant \(K(\omega)\) depending on \(\omega\).

Since it is well-known that \(-B\) and \(-B^2\) generate two semigroups of linear bounded operators in \(H\) (cf. [34]), we immediately derive that \((\tilde{\vartheta}, \tilde{\chi})\) is a solution to the following Volterra integral system
\[ \tilde{\eta}(t) = e^{-(t-s)k_0B}\tilde{\eta}_0 + \int_0^t e^{-(t-s)k_0B}L_1(\tilde{f}, \tilde{k})(s) \, ds + \int_0^t e^{-(t-s)k_0B}L_2(\tilde{\eta}, \tilde{\chi})(s) \, ds, \quad (4.16) \]
\[ \tilde{\chi}(t) = e^{-tB^2}\tilde{\chi}_0 + \int_0^t e^{-(t-s)B^2}L_3(\tilde{\eta}, \tilde{\chi})(s) \, ds, \quad (4.17) \]
provided $L_1(\tilde{f}, k), L_2(\tilde{\eta}, \tilde{x}), L_3(\tilde{\eta}, \tilde{x})$ all belong to $L^2(0, T; H)$ and
\begin{align}
\partial_0 &\in H^2(\Omega) \cap W, \quad \chi_0 \in H^4(\Omega) \cap W, \\
B(\partial_0 + \ell \chi_0) &\in H^1_{2,0_n}(\Omega), \quad \tilde{\chi}_0 \in H^2_{2,\{0_n, \partial_n\Delta\}}(\Omega).
\end{align}
(4.18) (4.19)
where $H^1_{2,\partial_n}(\Omega)$ and $H^2_{2,\{0_n, \partial_n\Delta\}}(\Omega)$ are the Besov spaces related to $(L^2(\Omega), B, \partial_n)$ and $(L^2(\Omega), B, \partial_n, \partial_n\Delta)$, respectively.

According to the results in [39, p. 320], we easily deduce the following representations
\begin{align}
H^1_{2,\partial_n}(\Omega) &= H^1(\Omega), \\
H^2_{2,\{0_n, \partial_n\Delta\}}(\Omega) &= \{ u \in H^2(\Omega) : \partial_n u = 0 \text{ on } \partial \Omega \}.
\end{align}
We note that properties (4.18) and (4.19) are easy consequences of our assumptions (r17), (r18).

Next, let us observe that, taking (2.39) into account, the convolutions $k * f$ can be estimated in $L^2(0, T; H)$ by using [33, Lemma 2.4] with $p = 2$:
\begin{align}
|k * f|_{L^2(0, T; H)} &\leq |k|_{L^1(0, T)}^{1/2} s^{-1/2} |f|_{L^2(0, T; s; H)} \int_0^T ds \\
&\leq (Tm)^{1/2} \left( \int_0^T |k(s)||f|_{L^2(0, T; s; H)}^2 ds \right)^{1/2}.
\end{align}
(4.20)
Then, from the well-known estimate
\begin{align}
|B e^{-tB^2}|_{L(H)} &\leq C t^{-1/2}, \quad \forall t \in \mathbb{R}_+, \quad \text{(4.21)}
\end{align}
we easily deduce the following integral inequality holding for any $\tau \in (0, T]$ (cf. [38]):
\begin{align}
|B \tilde{\chi}|_{L^2(0, T; H)} &\leq C |B \tilde{\chi}|_{L^2(0, T; H)} + C(T) \int_0^\tau s^{-1/2} |L_3(\tilde{\eta}, \tilde{x})|_{L^2(0, T; H)} ds.
\end{align}
(4.22)
From (4.22) and the obvious estimate
\begin{align}
|L_3(\tilde{\eta}, \tilde{x})|_{L^2(0, T; H)} &\leq \kappa(\omega, T) [ |B \tilde{\eta}|_{L^2(0, T; H)} + |B \tilde{\chi}|_{L^2(0, T; H)} ], \quad \forall \tau \in (0, T],
\end{align}
(4.23)
we immediately get
\begin{align}
|B \tilde{\chi}|_{L^2(0, T; H)}^2 &\leq C |B \tilde{\chi}|_{L^2(0, T; H)} + \kappa(\omega, T) \left( \int_0^\tau s^{-1/4} |B \tilde{\eta}|_{L^2(0, T; H)}^2 + |B \tilde{\chi}|_{L^2(0, T; H)}^2 ds \right)^2 \\
&\leq C |B \tilde{\chi}|_{L^2(0, T; H)}^2 + 4T \kappa(\omega, T)^2 \int_0^\tau s^{-1/2} |B \tilde{\eta}|_{L^2(0, T; H)}^2 + |B \tilde{\chi}|_{L^2(0, T; H)}^2 ds, \quad \forall \tau \in (0, T].
\end{align}
(4.24)
Moreover, from the estimate (cf. [38]), for $j = 1, 2$,
\begin{align}
|\partial_t [e^{-tB^{j-1}B^j} f]|_{L^2(0, T; H)}^2 + |B^j [e^{-tB^{j-1}B^j} f]|_{L^2(0, T; H)}^2 &\leq C(T) |f|_{L^2(0, T; H)}^2 + |B^j f|_{L^2(0, T; H)}^2, \quad \forall \tau \in (0, T], \quad \forall f \in L^2(0, T; H), \quad j = 1, 2,
\end{align}
(4.25)
and from (4.24) and (4.20), for all $\tau \in (0, T]$ we deduce the estimates

\[
|B[e^{-tk_0B} * L_2(\tilde{q}, \tilde{x})]|_{L^2(0, \tau; H)}^2 \leq C(T) \left\{ |B\tilde{x}|_{L^2(0, \tau; H)}^2 + Tm \int_0^\tau |k_2(s)||B\tilde{u}|_{L^2(0, \tau-s; H)}^2 \, ds \right\}
\]

\[
\leq \kappa(\omega, m, T) \left\{ \int_0^\tau |s^{-1/2} + |k_2(s)||B\tilde{u}|_{L^2(0, \tau-s; H)}^2 \, ds \right\}.
\]

Likewise we get

\[
|B[e^{-tk_0B} * L_1(f, \tilde{k})]|_{L^2(0, \tau; H)}^2 \leq K(\omega, m, T)[f]_{L^2(0, \tau; H)}^2 + \tilde{k}_{L^1(0, \tau)}^2.
\]

Set now

\[
p_1(\tau) := |\tilde{f}|_{L^2(0, \tau; H)}^2, \quad q_1(\tau) := |B\tilde{u}|_{L^2(0, \tau; H)}^2, \quad q_2(\tau) := |B\tilde{x}|_{L^2(0, \tau; H)}^2.
\]

Then, from (4.16), (4.25), (4.26), (4.27) we deduce the following integral inequalities

\[
q_1(\tau) \leq C|B\eta_0|_{H}^2 + \kappa(\omega, m, T)p_1(\tau)^2 + K(\omega, m, T)[\tilde{f}]_{L^2(0, \tau; H)}^2 + \tilde{k}_{L^1(0, \tau)}^2
\]

\[
+ \kappa(\omega, m, T) \left\{ T|B\tilde{x}|_{H}^2 + \int_0^\tau [(\tau - s)^{-1/2} + |k_2(\tau - s)||q_1(s) + q_2(s)] \, ds \right\}.
\]

\[
q_2(\tau) \leq C|B\tilde{x}|_{L^2(\Omega)}^2 + 4T\kappa(\omega, m, T)^2 \int_0^\tau (\tau - s)^{-1/2}[q_1(s) + q_2(s)] \, ds, \quad \forall \tau \in (0, T).
\]

Consequently, from (4.28) and (4.29) we obtain the integral inequality

\[
q_1(\tau) + q_2(\tau) \leq q_0(\tau) + \kappa(\omega, m, T) \int_0^\tau [(\tau - s)^{-1/2} + |k_2(\tau - s)||q_1(s) + q_2(s)] \, ds,
\]

where

\[
q_0(\tau) \leq C|B\eta_0|_{H}^2 + \kappa(\omega, m, T)[p_1(\tau)^2 + |B\tilde{x}|_{H^2(\Omega)}^2]
\]

\[
+ K(\omega, m, T)[\tilde{f}]_{L^2(0, \tau; H)}^2 + \tilde{k}_{L^1(0, \tau)}^2 + \kappa(\omega, m, T)T|B\tilde{x}|_{H}^2 + C|B\tilde{x}|_{L^2(\Omega)}^2.
\]

We note that inequality (4.30) can be rewritten in the simplified form

\[
r(\tau) \leq q_0(\tau) + \int_0^\tau h(\tau - s)r(s) \, ds,
\]

where

\[
r(\tau) = q_1(\tau) + q_2(\tau), \quad h(\tau) = \kappa(\omega, m, T)[\tau^{-1/2} + |k_2(\tau)|].
\]

Observe that $h$ is a function in $L^p(0, T)$ for any $p \in [1, 2)$, since, by assumption, $k_2 \in L^2(0, T)$.

According to [32, Lemma 1.1] we easily deduce the pointwise inequality

\[
r(\tau) \leq q_0(\tau) + \sum_{n=1}^{\infty} (h*)^n q_0(\tau)
\]
which implies
\[
|r|_{L^1(0,T)} \leq |q_0|_{L^1(0,T)} + \sum_{n=2}^{+\infty} |(h^*)^{n-1}h|_{L^1(0,T)}.
\] (4.35)

Then, from (4.35) and [33, Lemma 2.5] we easily deduce the estimate
\[
|r|_{L^1(0,T)} \leq \rho(t^{1/p'}|h|_{L^p(0,T)})|q_0|_{L^1(0,T)}, \quad \forall \tau \in (0,T], \; p \in (1,2),
\] (4.36)

where
\[
\rho(z) = \sum_{n=1}^{+\infty} (n!)^{-1/p'}z^n, \quad z \in \mathbb{C}.
\] (4.37)

Finally, from (2.39), (4.27), (4.33), (4.34), and (4.36) we easily deduce the desired estimate holding for any \( \tau \in (0,T] \):
\[
|B\tilde{\eta}|_{L^2(0,\tau;H)}^2 + |B\tilde{\chi}|_{L^2(0,\tau;H)}^2 \leq \kappa(\omega, m, T)(q_0(\tau) + |\bar{f}|_{L^2(0,\tau;H)}^2 + |\tilde{k}|_{L^1(0,\tau)}^2).
\] (4.38)

Consequently, from (4.13), (4.14), (4.38), and well-known results (cf. [38]) we deduce
\[
\sum_{j=1}^{2} |L_{j+1}\tilde{\eta}|_{L^2(0,\tau;H)} \leq \kappa(\omega, m, T)(|\bar{f}|_{L^2(0,\tau;H)} + |\tilde{k}|_{L^1(0,\tau)}).
\] (4.39)

Due to (4.16) and (4.20) the previous inequality implies
\[
|\partial_t \tilde{\eta}|_{L^2(0,\tau;H)} + |\partial_t \tilde{\chi}|_{L^2(0,\tau;H)} \leq \kappa(\omega, m, T, \delta)\left\{|B\bar{\vartheta}_0|_H + |\bar{f}|_{L^2(0,\tau;H)} + |\tilde{k}|_{L^1(0,\tau)} \right\}.
\] (4.40)

Moreover, from (4.16), (4.24), and (4.39), for any \( \tau \in (0,T] \) we easily deduce
\[
|\tilde{\eta}|_{L^2(0,\tau;H)} \leq C|\tilde{\eta}_0|_H + \kappa(\omega, m, T)\int_0^\tau |q_0(s) + |\bar{f}|_{L^2(0,\tau;H)} + |\tilde{k}|_{L^1(0,\tau)}| ds,
\] (4.41)
\[
|B\tilde{\chi}|_{L^2(0,\tau;H)} \leq C|B\tilde{\chi}_0|_H + \kappa(\omega, m, T)\int_0^\tau (\tau - s)^{-1/2}|q_0(s) + |\bar{f}|_{L^2(0,\tau;H)}
\]
\[
+|\tilde{k}|_{L^1(0,s)}| ds.
\] (4.42)

Finally, from (4.40) and (4.4), (4.5) we derive the desired estimate holding for any \( \tau \in (0,T] \):
\[
|\partial_\tau \vartheta_2 - \partial_\tau \vartheta_1|_{L^2(0,\tau;H)} + |\partial_\tau \chi_2 - \partial_\tau \chi_1|_{L^2(0,\tau;H)}
\]
\[
\leq \kappa(\omega, m, T)\left\{|\vartheta_0,2 - \vartheta_0,1 + \ell(\chi_0,2 - \chi_0,1)|_{H^1(\Omega)} + |\chi_0,2 - \chi_0,1|_{H^2(\Omega)}
\]
\[
+|f^2 - f^1|_{L^2(0,\tau;H)} + |k_2 - k_1|_{L^1(0,\tau)} \right\}.
\] (4.43)

Summing up, from (4.40) and (4.12) we obtain that the following estimates for \((\vartheta, \chi)\) hold for any \( \tau \in (0,T] \):
\[
|\partial_\tau \vartheta_2 - \partial_\tau \vartheta_1|_{L^2(0,\tau;H)} + |\partial_\tau \chi_2 - \partial_\tau \chi_1|_{L^2(0,\tau;H)} + |B\vartheta_2 - B\vartheta_1|_{L^2(0,\tau;H)}
\]
\[
+ |B^2\chi_2 - B^2\chi_1|_{L^2(0,\tau;H)} \leq \kappa(\omega, m, T)\left\{|\vartheta_0,2 - \vartheta_0,1|_{H^1(\Omega)}
\]
\[
+ |\chi_0,2 - \chi_0,1|_{H^2(\Omega)} + |f^2 - f^1|_{L^2(0,\tau;H)} + |k_2 - k_1|_{L^1(0,\tau)} \right\}.
\] (4.44)
Remark 4.1. From (4.40) and (4.12) we obtain also that the following estimates for \((\vartheta, \chi)\) hold for any \(\tau \in (0, T)\):

\[
|\vartheta_2 - \vartheta_1|_{L^2(0, \tau; H)} \leq C[|\vartheta_{0,2} - \vartheta_{0,1}|_H + |\chi_{0,2} - \chi_{0,1}|_H]
+ K(\omega, m, T) \int_0^\tau ([|f^2 - f^1|_{L^2(0, s; H)} + |k_2 - k_1|_{L^1(0, s)}] ds,
(4.45)
\]

\[
|B\chi_2 - B\chi_1|_{L^2(0, \tau; H)} \leq C|B(\chi_{0,2} - \chi_{0,1})|_H
+ K(\omega, m, T) \int_0^\tau (\tau - s)^{-1/2}[|f^2 - f^1|_{L^2(0, s; H)} + |k_2 - k_1|_{L^1(0, s)}] ds.
(4.46)
\]

We stress that these latter two estimates differ from (4.44) in that they, beside implying continuous dependence, have a contracting character with respect to \(k_2 - k_1\) due to the presence of the convolution integrals.

5 Proof of the identification result

Deriving a uniform bound for \(k\). First apply to (2.41) the functional \(\Phi(u) = \langle \varphi, u \rangle\), where \(\varphi \in W^{4, \infty}(\Omega)\), with \(\partial_{\mu} J^j \varphi = 0\) on \(\partial \Omega, j = 0, 1\). We obtain:

\[
g'(t) = \Phi[\partial \vartheta(t)] = \Phi[\partial \vartheta(t) + \ell \chi(t)] - \ell \Phi[\partial \chi(t)] = -k_0 \Phi[g(t)] - k * \Phi[g(t)]
+ \Phi[f(t)] + \ell \Phi[\beta(\chi(t))] - \ell^2 \Phi[B \vartheta(t)].
(5.1)
\]

Differentiating equation (5.1) with respect to \(t\), we get:

\[
g''(t) = -k_0 \Phi[B \vartheta(t)] - k(t) \Phi[\partial \vartheta] - k * \Phi[B \vartheta(t)] + \Phi[\partial f(t)]
+ \ell \Phi[B^2 \chi(t)] - \ell \Phi[B(\beta'(\chi(t)) + \sigma''(\chi(t))) \partial \chi(t)] - \ell^2 \Phi[B \vartheta(t)].
(5.2)
\]

Assume now that \((\vartheta, \chi) = (\Theta(k), X(k))\) denotes the solution to system (2.41–2.44) under the assumption in Theorem 2.14. Then, thanks to the fact that \(\rho^{-1} := \Phi[B \partial_0] \neq 0\) and our assumption on \(\varphi\), we can rewrite (5.2) in the following way:

\[
k(t) = K(f, g)(t) - \rho k \Phi[\partial \vartheta(t)] - \rho(k_0 + \ell^2) \Phi[\partial f(t)]
+ \rho \ell \Phi_2[\partial \chi(t)] + \rho \ell \Phi_1[(\beta'(\chi(t)) + \sigma''(\chi(t))) \partial \chi(t)],
(5.3)
\]

where

\[
\Phi_j[u] := (-1)^j(\Delta^j \varphi, u), j = 0, 1, 2,
K(f, g) := \rho \{ -g''(t) + \Phi_0[\partial f(t)] \}.
(5.4)
\]

From (5.3) the equivalent equation follows (recall notation (2.7)):

\[
k_\omega(t) = K(f, g)_\omega(t) - \rho k_\omega \Phi_1[(\partial \vartheta)_\omega(t)] - \rho(k_0 + \ell^2) \Phi_1[(\partial \vartheta)_\omega(t)]
+ \rho \ell \Phi_2[\partial \chi_\omega(t)] + \rho \ell \Phi_1[(\beta' + \sigma''(\chi(t))) \partial \chi_\omega(t)].
(5.5)
\]

Note that equation (5.5) can be obviously equivalently rewritten in terms of the auxiliary variables

\[
u = U(k) = \Theta(k) - \vartheta_0, \quad v = V(k) = X(k) - \chi_0 - t\chi_1(0),
\]
(cf. (2.25)) as:

\[ k_\omega(t) = K(f, g)(t) - \rho k_\omega \Phi_1[(\partial_t u)(\omega(t))] - \rho(k_0 + \ell)\Phi_1[(\partial_t u)(\omega(t))]
+ \rho \ell \Phi_2[\chi_t(0)]e^{-\omega t} + \rho \ell \Phi_2[(\partial_t v)(\omega(t))] + \rho \ell e^{-\omega t}\Phi_1[(\beta' + \sigma')(\chi(t))\chi_t(0)]
+ \rho \ell \Phi_1[(\beta'(\chi(t)))^{1/2}(\beta'(\chi(t)))^{1/2}(\partial_t v)(\omega(t)) + \sigma''(\chi(t))(\partial_t v)(\omega(t))]
\]

\[ = N_0(\partial_0 + U(k), \chi_0 + t\chi_t(0) + V(k), k)(t) = N_0(\Theta(k), X(k), k)(t) =: N_1(k) \]  

(5.6)

Hence, thanks to estimate (2.26) in Theorem 2.4 and to assumption \( \varphi \in W^{4, \infty}(\Omega) \), we derive the inequalities:

\[ |e^{-t\omega}N_1(k)|_{L^2(0,T)} \leq c_1|k_\omega|_{L^1(0,T)}(\partial_t u)_{\omega}|_{L^2(0,T;H)} + \left\{ |K(f, g)_{\omega}|_{L^2(0,T)}
+ c_2\left( |(\partial_t u)_{\omega}|_{L^2(0,T;H)} + |(\partial_t v)_{\omega}|_{L^2(0,T;H)}\right)
+ \frac{c_2}{\omega}|\chi_t(0)| + c_4(1 + M)\omega^{-1/2}|\chi_t(0)|_{L^\infty(\Omega)}
+ c_5M^{1/2}[(\beta'(\chi))^{1/2}(\partial_t v)_{\omega}|_{L^2(0,T;H)}\right\}
\leq c_1T^{1/2}I_2(\omega)|k_\omega|_{L^2(0,T)} + I_3(\omega). \]  

(5.7)

Here we have set

\[ I_2(\omega) = C_2\left[ \frac{1}{\omega}|\chi_t(0)|_{L^\infty(\Omega)} + \frac{1}{\omega}|\chi_t(0)|_{H} + \frac{1}{\omega}|B\partial_0| + \omega^{-(p-2)/p}|f|_{L^p(\Omega)} \right], \]  

(5.8)

\[ I_3(M, \omega) = \omega^{-(p-2)/p}|K(f, g)|_{L^p(\Omega)} + (c_2 + c_5)I_2(\omega) + \frac{c_2}{\omega}|\chi_t(0)|
+ c_4(1 + M)\omega^{-1/2}|\chi_t(0)|_{L^\infty(\Omega)}. \]  

(5.9)

Since the right-hand side in (5.8) tends to 0 as \( \omega \to +\infty \), we can choose \( \omega \in \mathbb{R}_+ \) so large as to satisfy

\[ I_2(\omega) \leq \frac{1}{2c_1T^{1/2}}. \]  

(5.10)

As a consequence, from (5.7), under conditions (2.18), (5.8), (5.9), (5.10), we deduce the estimate

\[ |e^{-t\omega}N_1(k)|_{L^2(0,T)} \leq \frac{1}{2}|k_\omega|_{L^2(0,T)} + I_3(M, \omega). \]  

(5.11)

Therefore, under the same conditions, all the solutions to the fixed-point equation

\[ k = N_1(k) \]  

(5.12)

satisfy the estimate

\[ |e^{-t\omega}k|_{L^2(0,T)} = |k_\omega|_{L^2(0,T)} \leq 2I_3(M, \omega) =: 2I_3(f, g, \vartheta_0, \chi_0, M, \omega), \]  

(5.13)

where \( I_3(f, g, \vartheta_0, \chi_0, \omega) \to 0 \) as \( \omega \to +\infty \). Note that such a limit is uniform with respect to the data varying in every (fixed) closed ball.
Summing up, we have proved that, under conditions (2.18) and (5.10), operator $N_0$ necessarily satisfies estimate (5.11), while each solution to the fixed-point equation (5.12) satisfies (5.13).

Conversely, choose any $\omega_0$ satisfying the system of inequalities:

$$\begin{cases}
T^{1/2}I_3(f, g, \vartheta_0, \chi_0, M, \omega_0) \leq \frac{k_0}{2(2+k_0)^{1/2}}, \\
I_2(f, g, \vartheta_0, \chi_0, \omega_0) \leq 1/(2c_1T^{1/2}).
\end{cases}$$

(5.14)

Of course, this $\omega_0$ is independent of $k$, since $I_2$ and $I_3$ are.

Moreover, let us choose

$$X = \{k \in L^2(0, T) : |k_\omega|_{L^2(0, T)} \leq 2I_3(f, g, \vartheta_0, \chi_0, M, \omega_0)\}.$$  

(5.15)

Then, for any $k \in X$, we get

$$|k_\omega|_{L^1(0, T)} \leq T^{1/2}|k_\omega|_{L^2(0, T)} \leq T^{1/2}I_3(f, g, \vartheta_0, \chi_0, M, \omega_0) \leq \frac{k_0}{2(2+k_0)^{1/2}}.$$  

(5.16)

Consequently, according to Theorem 2.11 with $s = 2$, for any kernel $k \in L^2(0, T)$ and any $\omega = \omega_0$ satisfying (5.14), the solution $(\vartheta, \chi) = (\Theta(k), X(k))$ to the direct problem satisfies the estimates (cf. (2.37))

$$|\Theta(k)|_{W^{1, \infty}(0, T; H^3(\Omega)) \cap H^1(0, T; H^4(\Omega))} + |X(k)|_{W^{1, \infty}(0, T; H^4(\Omega)) \cap H^2(0, T; H^2(\Omega))} \leq K_4(\omega_0),$$

(5.17)

$K_4$ being the positive constant (appearing in estimate (2.34)) depending only on the norms of the data and on $\omega_0$.

Finally, observe that, according to (5.7), the nonlinear operator $N_1$ maps the closed ball

$$X = \{k \in L^2(0, T) : |e^{-\omega_0 k}|_{L^2(0, T)} \leq 2I_3(f, g, \vartheta_0, \chi_0, M, \omega_0)\}$$

(5.18)

into itself.

**An equivalent fixed-point problem.** In this paragraph we would like to rewrite equations (3.34–3.35) in a way suitable for using the required regularity on our test function $\varphi$ (cf. (ir3)). Indeed, we aim to apply the (weak but with the right-hand side decreasing as $\omega \nearrow \infty$) estimate (2.26) on the solution components $\vartheta$ and $\chi$ to our problem. Hence, first we restate equations (2.41), (2.42) in the following form:

$$\begin{align}
\partial_t \chi &= -B^2 \chi - B[\beta(\chi) + \sigma'(\chi)] + \ell B \vartheta \quad \text{in } Q, \\
\partial_t \vartheta &= -\ell \partial_t \chi - k_0 B \vartheta - B(k \ast \vartheta) + f \\
&= \ell B^2 \chi + \ell B[\beta(\chi) + \sigma'(\chi)] - (\ell^2 + k_0)B \vartheta - B(k \ast \vartheta) + f \quad \text{in } Q,
\end{align}$$

(5.19, 5.20)
Then, we observe that from (5.19) and (5.20), after some integrations by parts, we deduce the following identities, where \( \varphi_1 := \Delta \varphi \):

\[
\begin{align*}
\Phi_1[\partial_t \vartheta(t)] &= \Phi_2[\ell B X + \ell \beta(\chi) + \ell \sigma'(\chi) - (\ell^2 + k_0) \vartheta - k \ast \vartheta] + \Phi_1[f] \quad \text{[5.21]} \\
\Phi_j[\partial_t \chi(t)] &= -\Phi_{j+1}[B X + (\beta + \sigma')(\chi) - \ell \vartheta], \quad j = 1, 2; \quad \text{[5.22]} \\
\Phi_1[(\beta(\chi) + \sigma'(\chi)) \partial_t \chi(t)] &= -\Phi_1[(\beta(\chi) + \sigma'(\chi)) B(B X + \beta(\chi) + \sigma'(\chi) - \ell \vartheta)] \\
&= -((\beta(\chi) + \sigma'(\chi)) \Delta \varphi_1, B X + \beta(\chi) + \sigma'(\chi) - \ell \vartheta)_H \\
&\quad - 2((\beta'(\chi) + \sigma''(\chi)) \nabla \chi \cdot \nabla \varphi_1, B X + \beta(\chi) + \sigma'(\chi) - \ell \vartheta)_H \\
&\quad - (\varphi_1(\beta'(\chi) + \sigma''(\chi)) \Delta \chi, B X + \beta(\chi) + \sigma'(\chi) - \ell \vartheta)_H \\
&\quad - (\varphi_1(\beta''(\chi) + \sigma'''(\chi)) |\nabla \chi|^2, B X + \beta(\chi) + \sigma'(\chi) - \ell \vartheta)_H. \quad \text{[5.23]}
\end{align*}
\]

As a consequence, we can rewrite (5.3) in the following (equivalent) more appropriate form, where \((\vartheta, \chi) = (\Theta(k), X(k))\),

\[
k = N_0(\Theta(k), X(k), k) = K(f, g) - \rho k \ast \Phi_1[\partial_t \vartheta] \\
&\quad - \rho(\ell^2)\{\Phi_2[B X + \ell \beta(\chi) + \ell \sigma'(\chi) - (\ell^2 + k_0) \vartheta - k \ast \vartheta] + \Phi_1[f]\} \\
&\quad - \rho(\beta(\chi) + \sigma'(\chi)) \Delta \varphi_1, B X + (\beta(\chi) + \sigma'(\chi) - \ell \vartheta)_H \\
&\quad + 2\rho(\beta'(\chi) + \sigma''(\chi)) \nabla \chi \cdot \nabla \varphi_1, B X + (\beta(\chi) + \sigma'(\chi) - \ell \vartheta)_H \\
&\quad + \rho(\varphi_1(\beta'(\chi) + \sigma''(\chi)) \Delta \chi, B X + (\beta(\chi) + \sigma'(\chi) - \ell \vartheta)_H \\
&\quad + \rho(\varphi_1(\beta''(\chi) + \sigma'''(\chi)) |\nabla \chi|^2, B X + (\beta(\chi) + \sigma'(\chi) - \ell \vartheta)_H \\
&=: N_2(k). \quad \text{[5.24]}
\]

**Remark 5.1.** In the preceding computation we have used the regularity properties for \( \varphi \) stated in (ir3).

We now observe that equation (4.11) can be rewritten, in terms of the nonlinear operator defined by (5.12), in the fixed-point form

\[
k = N_1(k) = N_2(k). \quad \text{[5.25]}
\]

**A fixed-point argument.** First of all, let us observe that, due to (5.11) and (5.25), \( N_2 \) maps \( X \) into itself.

Setting \( \gamma := \beta + \sigma', \alpha_j = \alpha_j(V(k_1), V(k_2)) \) (cf. (4.11)), and

\[
\Theta(k_1, k_2) := \Theta(k_2) - \Theta(k_1), \quad X(k_1, k_2) := X(k_2) - X(k_1), \quad \text{[5.26]}
\]
we get
\[
N_2(k_2) - N_2(k_1) = K(f_2 - f_1, g_2 - g_1) - \rho k_2 * \Phi_1[\partial_\tau \tilde{\Theta}(k_1, k_2)] \\
- \rho \Phi_1[\partial_\tau \Theta(k_1)] - \rho (k_0 + \ell^2) \epsilon \Phi_2[B \tilde{X}(k_1, k_2) + \alpha \tilde{X}(k_1, k_2)] \\
+ \rho (k_0 - l) \Phi_2[(k_0 + \ell^2) \tilde{\Theta}(k_1, k_2) + k_2 * \tilde{\Theta}(k_1, k_2) + (k_2 - k_1) * \Theta(k_1))] \\
+ \Phi_1[f_2 - f_1] - \rho \Phi_3[B \tilde{X}(k_1, k_2) + \alpha \tilde{X}(k_1, k_2) - \ell \Theta(k_1, k_2)] \\
+ \rho (\alpha \tilde{X}(k_1, k_2) \Delta \varphi_1, B \tilde{X}(k_2) + \gamma(X(k_2)) - \ell \Theta(k_2))_H \\
+ \rho (\gamma(X(k_1)) \Delta \varphi_1, B \tilde{X}(k_1, k_2) + \alpha \tilde{X}(k_1, k_2) - \ell \Theta(k_1, k_2))_H \\
+ 2\rho (\alpha_3 \tilde{X}(k_1, k_2) \nabla X(k_2) \cdot \nabla \varphi_1 + \gamma'(X(k_2)) \nabla \tilde{X}(k_1, k_2) \cdot \nabla \varphi_1, \\
B X(k_2) + \gamma(X(k_2)) - \ell \Theta(k_2))_H \\
+ 2\rho (\gamma'(X(k_1)) \nabla X(k_1) \cdot \nabla \varphi_1, B \tilde{X}(k_1, k_2) + \alpha \tilde{X}(k_1, k_2) - \ell \Theta(k_1, k_2))_H \\
+ \rho (\varphi_1(\alpha_3 \tilde{X}(k_1, k_2) B X(k_2) + \gamma'(X(k_2)) \Delta \tilde{X}(k_1, k_2)), \\
B X(k_2) + \gamma(X(k_2)) - \ell \Theta(k_2))_H \\
+ \rho (\varphi_1 \gamma'(X(k_1)) B X(k_1), B \tilde{X}(k_1, k_2) + \alpha \tilde{X}(k_1, k_2) - \ell \Theta(k_1, k_2))_H \\
+ \rho (\varphi_1(\alpha_4 \tilde{X}(k_1, k_2) \nabla X(k_2))^2 + \gamma''(X(k_2)) \nabla(X(k_1) + X(k_2)) \cdot \nabla \tilde{X}(k_1, k_2), \\
B X(k_2) + \gamma(X(k_2)) - \ell \Theta(k_2))_H \\
+ \rho (\varphi_1 \gamma''(X(k_1)) \nabla X(k_1))^2, B \tilde{X}(k_1, k_2) + \alpha \tilde{X}(k_1, k_2) - \ell \Theta(k_1, k_2))_H. \\
(5.27)
\]

Next, before estimating the left-hand side in (5.27), we recall the following estimates holding for any \( \tau \in (0, T) \):
\[
|\tilde{X}(k_1, k_2)(\tau, \cdot)|_V \leq C|B \tilde{X}(k_1, k_2)(\tau, \cdot)|, \\
|k_2 \ast \partial_\tau \tilde{\Theta}(k_1, k_2)|^2_{L^2(0, \tau, H)} \leq |k_2|_{L^2(0, \tau)} \int_0^\tau |k_2(s)||\partial_\tau \tilde{\Theta}(k_1, k_2)(s, \cdot)|^2_{L^2(0, \tau, H)} ds, \\
|(k_2 - k_1) \ast \partial_\tau \Theta(k_1)|^2_{L^2(0, \tau, H)} \leq |\partial_\tau \Theta(k_1)|^2_{L^\infty(0, \tau, H)} |1 * (k_2 - k_1)|^2_{L^2(0, \tau)} \\
\leq |\partial_\tau \Theta(k_1)|^2_{L^\infty(0, \tau, H)} T \int_0^\tau |k_2 - k_1|^2_{L^2(0, s)} ds. \\
(5.28)
\]

Then, we observe that \( \alpha_j, j = 2, 3, 4 \) belong to \( L^\infty((0, T) \times \Omega) \) according to Theorem 2.11 with \( s = 2 \) and \( \omega = \omega_0 \) and can be estimated independently of \( k \) due to (ir2) and to (2.24) with \( \omega = \omega_0 \).

Consequently, from (5.26–5.28), (5.12), and (4.44–4.46) with \( \vartheta_j = U(k_j), \chi_j = V(k_j) \) and \( f_1 = f_2, g_1 = g_2, \vartheta_{0,1} = \vartheta_{0,2}, \chi_{0,1} = \chi_{0,2} \), we easily deduce the following estimate for any \( \tau \in (0, T) \):
\[
|N_2(k_2) - N_2(k_1)|^2_{L^2(0, \tau)} \leq \int_0^\tau h(\tau - s)|k_2 - k_1|^2_{L^2(0, s)} ds, \\
(5.29)
\]
where
\[
h(\tau) = K(\omega_0, M, T)[\tau^{-1/2} + |k_2(\tau)|], \quad \forall \tau \in (0, T]. \\
(5.30)
\]
As a consequence, the sequence of the iterates \( \{N_2^m\}_{m=1}^{\infty} \) of \( N_2 \) satisfies the following recurrence integral inequality for any \( \tau \in (0, T] \) and any \( m \in \mathbb{N} \setminus \{0\} \):

\[
|N_2^m(k_2) - N_2^m(k_1)|_{L^2(0, \tau)}^2 \leq \int_0^\tau h(\tau - s)|N_1^{m-1}(k_2) - N_1^{m-1}(k_1)|_{L^2(0, s)}^2 \, ds.
\]

Then, from the proof of [31, Lemma 1.1] we obtain

\[
|N_2^m(k_2) - N_2^m(k_1)|_{L^2(0, \tau)}^2 \leq \int_0^\tau \psi_m(\tau - s)|k_2 - k_1|_{L^2(0, s)}^2 \, ds,
\]

where

\[
\psi_m(\tau) := (h^*)^{m-1}h(\tau), \quad \forall \tau \in (0, T].
\]  

Finally, from (5.31), (5.32), and [33, Lemma 2.5] we deduce the desired estimate

\[
|N_2^m(k_2) - N_2^m(k_1)|_{L^2(0, \tau)}^2 \leq |k_2 - k_1|_{L^2(0, \tau)}^2 \int_0^\tau \psi_m(s) \, ds \\
\leq (m!)^{-1/p'}[\tau^{1/p'}|h|_{L^p(0, \tau)}]^{m}|k_2 - k_1|_{L^2(0, \tau)}^2.
\]

Setting \( \tau = T \), from (5.33) we conclude that there exists \( m_0 \in \mathbb{N} \) such that, for any \( m \geq m_0 \), \( N_2^m \) is a contracting operator in \( L^2(0, T) \) endowed with the usual metrics. To show that \( N_2^m \) is a contracting mapping also with respect to the weighted metrics of \( X \) (cf. (5.15)) we simply use the embeddings

\[
e^{-T\omega_0}|k|_{L^2(0, T)} \leq |e^{-T\omega_0}k|_{L^2(0, T)} \leq |k|_{L^2(0, T)}, \quad \forall k \in L^2(0, T).
\]

Consequently, the fixed-point equation \( k = N_2(k) \) admits a unique solution \( k \in L^2(0, T) \).

**A continuous dependence result.** Our last task consists of showing that the solution \( k \) to the equation \( k = N_1(k) \) continuously depends on the data \( (f, \vartheta_0, \chi_0, g) \) belonging to a closed ball in \( W^{1,p}(0, T; H) \times H^4(\Omega) \times H^6(\Omega) \times W^{2,p}(0, T), \ p \in (2, +\infty) \) with radius, say, \( r > 0 \).

From equation (5.27), with \( N_1(k_j) = k_j, \ j = 1, 2, \) after some standard computations, we deduce the estimate (cf. (5.31))

\[
|k_2 - k_1|_{L^2(0, \tau)}^2 \leq \int_0^\tau h(\tau - s)|k_2 - k_1|_{L^2(0, s)}^2 \, ds + C_2(T, M, r)\left[|f^2 - f_1|_{W^{1,p}(0, T; H)}^2 \right. \\
+ |\vartheta_2 - \vartheta_1|_{H^4(\Omega)}^2 + |\chi_2 - \chi_1|_{H^6(\Omega)}^2 + |g_2 - g_1|_{W^{2,p}(0, T)}^2 \left. \right], \quad \forall \tau \in (0, T].
\]

Finally, from [31, Lemma 1.1] we easily deduce the desired estimate

\[
|k_2 - k_1|_{L^2(0, T)}^2 \leq C_3(T, M, r)\left[|f^2 - f_1|_{W^{1,p}(0, T; H)}^2 \right. \\
+ |\vartheta_{0,2} - \vartheta_{0,1}|_{H^4(\Omega)}^2 + |\chi_{0,2} - \chi_{0,1}|_{H^6(\Omega)}^2 + |g_2 - g_1|_{W^{2,p}(0, T)}^2 \left. \right].
\]

Finally, from (2.40) and (5.35) we easily derive the desired estimate (2.45).
6 Appendix

We present here some technical lemmas which were used in the proofs of the regularity results.

Lemma 6.1. Let $\nu \geq 2$, $u, v \in H^\nu(\Omega)$. Then, $uv \in H^\nu(\Omega)$ and

$$|uv|_{H^\nu(\Omega)} \leq c\left(|u|_{H^\nu(\Omega)} + |v|_{H^\nu(\Omega)}\right),$$

(6.1)

where the constant $c$ only depends only on $\Omega$ and, for $\nu \geq 3$, on the norms of $u$ and $v$ in $H^{\nu-1}(\Omega)$, while for $\nu = 2$ it is also be allowed to depend on the norms of $u$ and $v$ in $H^s(\Omega)$, where $s$ is any integer in $(3/2, 2]$.

Proof. First of all, let us observe that the case of $\nu = 2$ can be easily treated by a direct computation and relying on well-known embeddings between Sobolev spaces. For a generic $\nu \geq 3$, let then $i_1, i_2, \ldots, i_\nu \in \{1, \ldots, N\}$ be non necessarily distinct integer numbers. Let also

$$D := D_{i_1} \circ \cdots \circ D_{i_\nu}$$

be the differential operator of order $\nu$ associated to the given $\nu$-tuple. Then, assuming that $u, v$ are regular enough (if not, we can proceed by a standard regularization-density argument) and setting $K := \{1, \ldots, \nu\}$, the following formula holds:

$$D(uv) = \sum_{\sigma \in 2^K} D_\sigma u D_{K \setminus \sigma} v,$$

(6.2)

where, for simplicity of notation, we have indicated by $D_\sigma$ the differential operator associated to $D$ and identified by the subset $\sigma$. For example,

$$\sigma = \{1, 3, 6\} \implies D_\sigma = D_{i_1} \circ D_{i_3} \circ D_{i_6}.$$

It is clear that it suffices to estimate the $L^2$-norm of the expression in (6.2). Then, let us collect the various terms on the right hand side according to $\#(\sigma)$. For $\#(\sigma) = 0$, we have

$$\|D_\sigma u D_{K \setminus \sigma} v\|_{L^2} \leq \|u\|_{L^\infty} \|D_{K \setminus \sigma} v\|_H \leq \kappa \|u\|_{H^{\nu-1}} \|v\|_{H^\nu}.$$  

(6.3)

Note that, here and below, $\kappa > 0$ is an embedding constant only depending on $\Omega$. As $\#(\sigma) = 1$, we have

$$\|D_\sigma u D_{K \setminus \sigma} v\|_H \leq \|D_\sigma u\|_{L^1} \|D_{K \setminus \sigma} v\|_{L^1} \leq \kappa \|u\|_{H^{\nu-1}} \|v\|_{H^\nu}.$$  

(6.4)

Finally, for $\#(\sigma) \geq 2$ (and, indeed, $\#(K \setminus \sigma) \geq 2$ - otherwise we exchange the roles of $u$ and $v$) it is

$$\|D_\sigma u D_{K \setminus \sigma} v\|_H \leq \|D_\sigma u\|_{L^2} \|D_{K \setminus \sigma} v\|_{L^\infty} \leq \kappa \|u\|_{H^{\nu-1}} \|v\|_{H^\nu}.$$  

(6.5)

This actually concludes the proof. 

Lemma 6.2. Let $\nu \in \mathbb{N}$, $\nu \geq 2$, $\gamma \in C^\nu(\mathbb{R})$, and $v \in H^\nu(\Omega)$. Then, $\gamma(v) \in H^\nu(\Omega)$ and the following estimate

$$|\gamma(v)|_{H^\nu(\Omega)} \leq c|v|_{H^\nu(\Omega)}$$

(6.6)

holds with $c$ is a positive constant depending only on $\Omega$, $\gamma$, and $|v|_{H^{\nu-1}(\Omega)}$ if $\nu \geq 3$, while it might also depend on $|v|_{H^s(\Omega)}$, $s \in (3/2, 2]$ if $\nu = 2$. 

Proof. We just give the highlights; actually, the procedure just relies on a number of direct computations which do not present any conceptual difficulty and are therefore left to the reader. As a first step, the cases of $\nu = 2, 3$ can be treated directly by estimating the norms of the derivatives of $\gamma(v)$ up to the second (or third, respectively) order. We actually notice that, as $\varphi \in C^0(\mathbb{R})$ and $u \in L^\infty(\mathbb{R})$, it is clearly

$$|\varphi(u)|_{L^\infty(\Omega)} \leq c(\varphi, |u|_{L^\infty(\Omega)}).$$

This also explains why the constant $c$ will depend also on the $H^s$ norm as $\nu = 2$; indeed, we have to use the continuous embedding $H^s(\Omega) \subset L^\infty(\Omega)$.

Then, as $\nu \geq 4$, it is sufficient to proceed by induction. Actually, assuming the assertion true for $\nu - 1$, it is enough to compute, for some $i \in \{1, \ldots, N\}$,

$$D_i \gamma(v) = \gamma'(v) D_i v$$

and use the induction hypothesis and the preceding Lemma.

The last Lemma in a sense puts together the previous two results.

**Lemma 6.3.** Let $\nu \in \mathbb{N}$, $\nu \geq 2$, $\alpha \in C^{\nu+1}(\mathbb{R})$, $w, z \in H^\nu(\Omega)$. Then, $\alpha'(w)z \in H^\nu(\Omega)$ and the following estimate

$$|\alpha'(w)z|_{H^\nu(\Omega)} \leq c(|w|_{H^{\nu-1}(\Omega)} + |z|_{H^{\nu-1}(\Omega)})$$

holds, where $c$ is a positive constant depending only on $\Omega$, $\alpha$, $|w|_{H^{\nu-1}(\Omega)}$, and $|z|_{H^{\nu-1}(\Omega)}$ (with, as before, $\nu - 1$ replaced by $s$, $s \in (3/2, 2]$, when $\nu = 2$).

**Proof.** The case $\nu = 2$ is again treated by a simple and direct computation. For $\nu \geq 3$ the assertion follows instead from the previous two Lemmas.

**References**


