ON A DOUBLY NONLINEAR
CAHN-HILLIARD-GURTIN SYSTEM

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Dedicated to Peter E. Kloeden on the occasion of his 60th birthday

Abstract. Our aim in this paper is to study a doubly nonlinear Cahn-Hilliard-type system. In particular, we prove existence and uniqueness results and the existence of global attractors.

1. Introduction. We consider in this paper the following system, in a bounded domain $\Omega \subset \mathbb{R}^3$:

\begin{align}
  u_t - \Delta z &= 0, \\
  z &= \alpha(u_t) - \Delta u + W'(u) - g,
\end{align}

associated with homogeneous Neumann boundary conditions. Here, $\alpha$ is a nonlinear monotone function with a controlled growth rate at infinity, $W$ is a possibly nonconvex configuration potential, and $g$ is an external source. The additional unknown $z$, usually called chemical potential, is introduced mainly for the sake of convenience; indeed, it is clear that (1.1)-(1.2) could be rewritten as a single equation.

Such equations arise, e.g., in the context of generalizations of the Cahn-Hilliard equation proposed by M. Gurtin in [19]. More precisely, M. Gurtin obtained, by considering a mechanical version of the second law of thermodynamics and by introducing a new balance law for interactions at a microscopic level, the following equations:

\begin{align}
  u_t - \Delta z &= 0, \\
  z &= a(u, \nabla u, u_t) u_t - \Delta u + W'(u) - g.
\end{align}

In particular, when $a = a(u_t)$, we obtain a system as above.

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In the past decades, many papers have been devoted to the study of doubly nonlinear equations. In particular, second order problems like

\[ \alpha(u)_t - \Delta u + W'(u) = g \]  

and

\[ \alpha(u_t) - \Delta u + W'(u) = g \]  

have been analyzed in a consistent number of works. Such equations arise, e.g., in the context of the Allen-Cahn equation; one actually again considers generalizations of the Allen-Cahn equation proposed by M. Gurtin in [19]. In particular, one now has satisfactory results on the existence and uniqueness of solutions and on the asymptotic behavior of the solutions (existence of (finite dimensional) attractors and convergence of single trajectories to steady states), both when \( \alpha \) does not degenerate and degenerates. We refer the reader to, e.g., [15], [14], [17], [18], [22], [25], [28], [31], [33], [34], [35], [36], and the references therein.

In [28], the authors considered the doubly nonlinear Cahn-Hilliard system

\[ u_t - \Delta z = 0, \]  

\[ z = \alpha(u)_t - \Delta u + W'(u) - g, \]  

corresponding to \( a = a(u) \) in (1.4), and proved the existence and uniqueness of solutions and the existence of finite dimensional attractors (they also studied the limit \( \alpha \to 0 \) in (1.8), corresponding to the usual Cahn-Hilliard equation).

Our aim in this paper is to address similar issues for the doubly nonlinear Cahn-Hilliard model (1.1)-(1.2). More precisely, in the first part of the paper we study the existence and uniqueness properties under several different conditions on the nonlinear functions \( \alpha \) and \( W \) and on the initial datum \( u_0 \). In general, we find that the validity of uniqueness requires more restrictive conditions (especially concerning the growth of \( W \)) than those needed for existence. Then, we pass to the the long-time behavior of the system and prove two different results on the existence of attractors. In the first of these, we address a situation where uniqueness does not seem to hold and, for this reason, we have to use a suitable generalized theory. To be more precise, the setting is the one of “limiting semiflows” proposed in [30] as a generalization of the theory of attractors for generalized (multivalued) semiflows due to Babin and Vishik (see [2], cf. also [4] and [5]).

This paper is organized as follows. In Section 2, we state our assumptions and main results. Then, Section 3 is devoted to the proofs of existence and uniqueness results, while Section 4 contains the proofs of existence of global attractors.

### 2. Main results.

Let \( \Omega \subset \mathbb{R}^3 \) be a smooth bounded domain. Let us set \( H := L^2(\Omega) \) and denote by \( \langle \cdot, \cdot \rangle \) the scalar product in \( H \) and by \( \| \cdot \| \) the related norm. The same symbols are also used for \( H^1 \) and its scalar product and norm. The symbol \( \| \cdot \|_X \) will indicate the norm in the generic Banach space \( X \). Next, we set \( V := H^1(\Omega) \), endowed with its standard scalar product and norm. We also set

\[ B : V \to V', \quad \langle Bv, z \rangle = \int_{\Omega} \nabla v \cdot \nabla z, \]  

the notation \( \langle \cdot, \cdot \rangle \) standing for the duality between \( V' \) and \( V \). We will write, in what follows,

\[ \zeta_\Omega := \| \Omega \|^{-1} \langle \zeta, 1 \rangle \]  

(2.2)
for $\zeta \in V'$, where $|\Omega|$ denotes the Lebesgue measure of $\Omega$. Let us also set
\begin{equation}
V_0' := \{ \zeta \in V' : \zeta_\Omega = 0 \}, \quad H_0 := H \cap V_0', \quad V_0 := V \cap V_0',
\end{equation}
and observe that, if $v \in V$ and $\zeta \in V'$, then
\begin{equation}
\langle \zeta - \zeta_\Omega, v \rangle = \langle \zeta - \zeta_\Omega, v - v_\Omega \rangle = \langle \zeta, v - v_\Omega \rangle.
\end{equation}
Moreover, $B$ is bijective from $V_0$ to $V_0'$, so that we can define the inverse $N := (B|_{V_0})^{-1}$ which fulfills, for all $v \in V$, $\zeta \in V'$,
\begin{equation}
\langle Bv, N(\zeta - \zeta_\Omega) \rangle = \langle \zeta - \zeta_\Omega, v \rangle.
\end{equation}
Finally, for $\zeta \in V_0'$, we can define
\begin{equation}
\| \zeta \|_*^2 := \langle N\zeta, \zeta \rangle
\end{equation}
and it is clear that the norm $\| \cdot \|_*$ is a norm on $V_0'$ which is equivalent to the standard one that $V_0'$ inherits as a closed subspace of $V'$.

We now present our basic assumptions on the data and the nonlinear terms. Further hypotheses will be specified in the statements of the various theorems. First, we assume that $\alpha$ is a nonlinear function such that
\begin{equation}
\alpha \in C^0_{\text{loc}}(\mathbb{R}; \mathbb{R}) \text{ is increasingly monotone and } \alpha(0) = 0.
\end{equation}
Next, we come to the potential $W$. We let $I$ be an open interval of $\mathbb{R}$ containing 0 (possibly unbounded or even coinciding with the whole real line) and assume that, for some $\lambda \geq 0$,
\begin{equation}
W \in C^2(I; \mathbb{R}), \quad W'(0) = 0,
\end{equation}
\begin{equation}
W''(r) \geq -\lambda \quad \forall r \in I,
\end{equation}
\begin{equation}
\lim_{r \to \partial I} W'(r) \text{ sign}(r) = +\infty.
\end{equation}
We also require that
\begin{equation}
g \in H.
\end{equation}
Finally, to present our hypotheses on the initial datum $u_0$, we first have to introduce the following energy functional which will be shown to act as a Lyapunov functional for our system:
\begin{equation}
\mathcal{E} : H \to \mathbb{R}, \quad \mathcal{E}(u) := \frac{1}{2} \| \nabla u \|^2 + \int_{\Omega} W(u) - (g, u).
\end{equation}
We denote by $D(\mathcal{E})$ the subset of $H$ on which $\mathcal{E}$ takes finite values. Note that $D(\mathcal{E}) \subset V$ and that $\mathcal{E}$ is clearly bounded from below on the whole space $H$. We then take
\begin{equation}
u_0 \in D(\mathcal{E}).
\end{equation}
We can now present our first result, stating the existence of at least one global (i.e., defined for a.e. $t$ in some reference interval $[0, T]$, $T > 0$ being arbitrary) solution to a suitable variational formulation of system (1.1)-(1.2) in the case when $\alpha$ has a linear growth at infinity.

**Theorem 2.1.** Let (2.7), (2.8)-(2.10), (2.11), and (2.13) hold. Let us also assume that
\begin{equation}
\exists \kappa_\alpha, K_\alpha > 0 : \quad \alpha(r)r \geq \kappa_\alpha r^2 - K_\alpha \quad \forall r \in \mathbb{R}.
\end{equation}
Then, there exists at least one pair of functions \((u, z)\) with
\[ u \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)), \tag{2.16} \]
\[ \alpha(u_t) \in L^2(0, T; H), \tag{2.17} \]
\[ z \in L^2(0, T; V), \tag{2.18} \]
\[ W'(u) \in L^2(0, T; H), \tag{2.19} \]
solving
\[ u_t + Bz = 0, \quad \text{in } V', \tag{2.20} \]
\[ z = \alpha(u_t) + Bu + W'(u) - g, \quad \text{in } V', \tag{2.21} \]
a.e. in \((0, T)\), together with the initial condition
\[ u|_{t=0} = u_0, \quad \text{a.e. in } \Omega. \tag{2.22} \]
In what follows, the pair \((u, z)\) (or, sometimes, the sole function \(u\)) as in the theorem (or one of the theorems below) will be generically called a “solution”. We notice that, if \(u\) is a solution, testing (1.1) by 1 we obtain
\[ (u(t))_\Omega = (u_0)_\Omega := u_\Omega, \tag{2.23} \]
i.e., the spatial average of \(u\) is conserved in time. Moreover, (2.14) entails that
\[ \alpha(r) r \geq \sigma_\alpha \alpha^2(r) - C_\alpha \quad \forall r \in \mathbb{R} \tag{2.24} \]
and for some \(\sigma_\alpha, C_\alpha > 0\). Note also that (2.15) allows the graph of \(\alpha\) to contain horizontal segments (e.g., \(\alpha\) can vanish in a neighborhood of 0).

As a next result (compare with [15, Thm. 3]), when the “coercivity” assumption (2.15) does not necessarily hold, we can still prove existence, but just under stronger assumptions on the initial data.

**Theorem 2.2.** Let (2.7), (2.8)-(2.10), (2.11), (2.13), and (2.14) hold. Let us also assume that
\[ Bu_0, W'(u_0), g \in V. \tag{2.25} \]
Then, there exists at least one pair of functions \((u, z)\) with
\[ u \in W^{1,\infty}(0, T; V') \cap H^1(0, T; V) \cap L^\infty(0, T; H^2(\Omega)), \tag{2.26} \]
\[ \alpha(u_t) \in L^\infty(0, T; V) \cap L^\infty(0, T; H^2(\Omega)), \tag{2.27} \]
\[ z \in L^\infty(0, T; V'), \tag{2.28} \]
\[ W'(u) \in L^\infty(0, T; H), \tag{2.29} \]
solving (2.20)-(2.21) a.e. in \((0, T)\) and satisfying (2.22).

**Remark 2.3.** It will be clear from the proof that, in case the coercivity assumption (2.15) holds, then (2.25) can be replaced by the slightly weaker assumption
\[ Bu_0 + W'(u_0) - g \in H. \tag{2.30} \]
In that case, Theorem 2.2 can be considered as a regularity property of the solutions given by Theorem 2.1 holding under more restrictive conditions on the initial data. We also notice that, by standard regularity results for elliptic equations with monotone nonlinear terms, (2.30) is equivalent to asking that each summand belongs to \(H\). The same may not always hold in the case of (2.25) and, actually, in the framework of our approximation, we will need \(Bu_0, W'(u_0), g\) to lie separately in \(V\) (and not just their sum).
Theorem 2.2 allows to consider, e.g., the situation where \( \alpha(r) = r \) for \( r \geq 0 \) and \( \alpha(0) = 0 \) otherwise, which does not seem easy to deal with within the weaker setting of Theorem 2.1.

Our next result is a further existence theorem which holds under very weak assumptions on \( \alpha \), but requires a growth restriction on the potential \( W \) and yields a weaker concept of a solution. To introduce this, assuming (2.7), we let us define the analogue of \( \alpha \) in the duality \( V' - V \). We say that \( \xi \in \alpha_w(v) \) if and only if \( \xi \in V' \), \( v \in V \), and

\[
\langle \xi, \eta - v \rangle \leq \int_{\Omega} \tilde{\alpha}(\eta) - \int_{\Omega} \tilde{\alpha}(v), \quad \forall \eta \in V, \tag{2.31}
\]

where \( \tilde{\alpha} \) is the antiderivative of \( \alpha \) which vanishes at 0. It is clear that, if \( v \in V \), \( \xi \in H \), and \( \xi = \alpha(v) \) a.e. in \( \Omega \), then \( \xi \in \alpha_w(v) \). However, the maximal monotone operator \( \alpha_w : V \to V' \) might be a proper extension of the restriction of \( \alpha \) to \( V \). A precise characterization of operators like \( \alpha_w \) is given in the papers [11] and [20] (see also [7]) in the (slightly different) situation where \( V = H^1_{0}(\Omega) \). Actually, since here \( \alpha \) is defined on the whole of \( \mathbb{R} \), one would expect (cf. [20, Thm. 3]) that, also in the case \( V = H^1_{0}(\Omega) \), \( \xi \in \alpha_w(v) \) if and only if \( \xi \in L^1(\Omega) \), \( \xi(x) = \alpha(v(x)) \) for a.a. \( x \in \Omega \), and \( \xi v \in L^1(\Omega) \).

**Theorem 2.4.** Let (2.7), (2.8)-(2.10), (2.11), (2.13), and (2.25) hold. Let also

\[
I = \mathbb{R}, \quad |W'(r)| \leq c_W(1 + |r|^5) \quad \forall r \in \mathbb{R}, \tag{2.32}
\]

and for some \( c_W > 0 \). Then, there exists at least one triplet of functions \( (u, z, \xi) \) with

\[
u \in W^{1,\infty}(0, T; V') \cap H^1(0, T; V), \tag{2.33}
\]

\[
\xi \in L^\infty(0, T; V'), \tag{2.34}
\]

\[
z \in L^\infty(0, T; V), \tag{2.35}
\]

\[
W'(u) \in L^\infty(0, T; L^{6/5}(\Omega)), \tag{2.36}
\]

solving, a.e. in \( (0, T) \), (2.20), together with

\[
z = \xi + Bu + W'(u) - g, \quad \text{in } V', \tag{2.37}
\]

\[
\xi = \alpha_w(u), \quad \text{in } V', \tag{2.38}
\]

and satisfying (2.22).

Let us now pass to uniqueness. We are able to prove it only in a class of sufficiently regular solutions and under more restrictive assumptions on the potential \( W \), which in particular has to be defined on the whole real line. Hence, uniqueness remains an open issue if the sole assumptions of the existence Theorem 2.1 are taken. Note that, anyway, the following result applies also to the “generalized” solutions yielded by Theorem 2.4.

**Theorem 2.5.** Let (2.7) and (2.8)-(2.10) hold.

(i) Let us also assume that \( I = \mathbb{R} \) and

\[
W \in C^{{\infty,}1}_{\text{loc}}(\mathbb{R}; \mathbb{R}), \tag{2.39}
\]

Then, the solution \( (u, z) \) is unique in the regularity class (2.26)-(2.29).

(ii) If, in addition to (2.39), we also have

\[
|W''(r)| \leq c(1 + |r|^3) \quad \text{for a.e. } r \in \mathbb{R}, \tag{2.40}
\]

then uniqueness also holds in the (wider) regularity class (2.33)-(2.36).
Finally, let us turn to the longtime analysis of the system from the point of view of global attractors. First, we notice that the set $D(E)$ can be endowed with the metric
\[ d_E(v, w) := \|v - w\|_V + \int_\Omega |W(v) - W(w)|. \] (2.41)
It is a standard matter to prove that $D(E)$, associated with this metric, is a complete metric space (see, e.g., [29, Lemma 3.8]). Then, accounting for (2.23), given
\[ m_1, m_2 \in I \quad \text{such that} \quad m_1 < 0 < m_2, \] (2.42)
we introduce the phase space
\[ V = V(m_1, m_2) := \{ u \in D(E) : m_1 \leq u_\Omega \leq m_2 \}. \] (2.43)
It is easy to see that $V$ is positively invariant under the flow generated by our system. Moreover, $V$ clearly is a closed subset of $D(E)$.

Our first result covers the case when $\alpha$ grows linearly at infinity as in Theorem 2.1.

**Theorem 2.6.** Let (2.7), (2.8)-(2.10), (2.11), and (2.15) hold. Let also, in place of (2.14), the following slightly different condition hold:
\[ \exists c_\alpha > 0 : |\alpha'(r)| \leq c_\alpha \quad \text{for a.e. } r \in \mathbb{R}. \] (2.44)
Then, for given $m_1, m_2$ as in (2.42), the semiflow on $V$ associated with system (2.20)-(2.21) possesses a global attractor $A$ such that, for some $c_A > 0$,
\[ \|u\|_{H^2(\Omega)} + \|W'(u)\| \leq c_A \quad \forall u \in A. \] (2.45)

**Remark 2.7.** The theorem will be shown by observing that the system possesses a uniformly absorbing set $B$ which is bounded in the sense of (2.45) and hence compact in $V$. In particular, we then have the $H^2$-regularity for $t > 0$ (but, of course, not for $t = 0$). Consequently, uniqueness is proved to hold for strictly positive times and provided that we assume (2.39), i.e., $W$ is a “regular” potential. Indeed, under these conditions Theorem 2.5 can be applied for $t > 0$. As a consequence, the attractor can be intended in the sense of semiflows with unique continuation; in other words, from the initial datum may depart also “spurious” trajectories (possibly) not approaching the attractor.

**Remark 2.8.** In the case when (2.39) is not satisfied, i.e., we have a “singular” potential, the situation is even more intricated and some further explanations are needed. Actually, the argument that we use to prove Theorem 2.6 (i.e., the construction of a compact absorbing set) still works, but assumes just a “formal” character. Actually, we could now have genuine nonuniqueness for all positive times, and the construction of the absorbing set, as well as the family of solutions absorbed by it, would then depend on the chosen approximation argument (a time-discretization scheme in our case). In this situation, we still have existence of an attractor in a suitably generalized sense, but its precise characterization would also depend on the approximation scheme. Moreover, the attractor would be just quasi-invariant rather than fully invariant. We do not want to enter more details on this case, since it is marginal with respect to the aims of the present paper. Nevertheless, it is worth noting that proper analytical tools which are suitable for describing the long-time behavior of systems with lack of uniqueness have been presented in several papers (see [2], [4, 5], see also the recent [30]), to which we refer for more details.
Finally, we move to the generalized solutions given by Theorem 2.4. In this setting, we can still prove the existence of the global attractor $\mathcal{A}$, provided that (2.39) and (2.40) hold and $\alpha$ is coercive in the sense of (2.15). In this situation (cf. Remark 2.3), we have existence and uniqueness (by Theorem 2.5) of a generalized solution, provided that $u_0$ lies in $H^2(\Omega)$, which is then the correct phase space (indeed, the term $W'(u_0)$ in (2.30) is then automatically in $H^2$).

**Theorem 2.9.** Let (2.7), (2.8)-(2.10), (2.11), and (2.15) hold and let $m_1, m_2$ be as in (2.42). Let also assume both (2.39) and (2.40). Then, introducing the phase space

$$\mathcal{W} = \mathcal{W}(m_1, m_2) := \mathcal{V}(m_1, m_2) \cap H^2(\Omega),$$

the semiflow on $\mathcal{W}$ associated with system (2.20)-(2.21) possesses a global attractor $\mathcal{A}$. More precisely, $\mathcal{A}$ is a bounded (and thus weakly compact) and fully invariant set of $H^2(\Omega)$ with respect to the weak topology of $H^2(\Omega)$.

The main argument in the proof of the theorem will be to show that any solution starting from initial data in $\mathcal{W}$ never leaves the phase space (and, more precisely, its $H^2$-norm remains uniformly bounded with respect to time). Indeed, due to the doubly nonlinear character of the equation, this apparently intuitive property will require some effort.

**Remark 2.10.** In fact, we will see at the end of the proof that, if $g$ and $\alpha$ are a bit smoother, then it should be possible, although not straightforward, to prove that the attractor can be intended in a stronger sense (i.e., bounded in $H^3(\Omega)$ and attracting with respect to the strong $H^2$-metric).

3. **Proofs.**

3.1. **Proof of Theorem 2.1.** We will use an approximation and time discretization procedure. To introduce it, let us first set $\beta := W' + 3\lambda \text{Id}$, so that, by (2.9), $\beta'(r) \geq 2\lambda$ for all $r \in I$. Moreover, let us take $\delta > 0$ and introduce the Yosida approximation $\beta_\delta$ of $\beta$. It is then easy to show that, for a suitable $\delta_0 > 0$, there holds $\beta'_\delta(r) \geq \lambda$ for all $r \in \mathbb{R}$ and $\delta \in (0, \delta_0]$.

Given $\overline{g} \in H$ and $\overline{\tau} \in V_0$, we then take a time discretization parameter $\tau > 0$ and replace (2.20)-(2.21) by an approximated and discretized system. More precisely, by subtracting from $u$ the average value $u_\Omega$, which is constant in time, and setting $v := u - u_\Omega$, we obtain

$$\frac{v - \overline{\tau}}{\tau} + Bz = 0,$$

$$z = \alpha\left(\frac{v - \overline{\tau}}{\tau}\right) + Bv + \beta_\delta(v + u_\Omega) - 3\lambda(\overline{\tau} + u_\Omega),$$

where $\overline{\tau}$ denotes the (known) value of the discrete solution at the preceding time step (see below). Of course, both $\tau$ and $\delta$ will be intended to go to 0 in the limit. Note that, if $\overline{g} = g$, then (3.1)-(3.2) correspond to a semiimplicit discretization of (2.20)-(2.21).

To prove an existence theorem for the above scheme, it is useful to rewrite it as a single equation. In view of a precise statement, let us then set

$$\overline{h} := \overline{g} + 3\lambda(\overline{\tau} + u_\Omega), \quad \overline{h}_0 := \overline{h} - \overline{h}_\Omega.$$


Notice that \( \overline{h} \) is a known quantity when the data \( \overline{g}, \overline{v}, u_\Omega \) are given and that \( \overline{h}_0 \in H_0 \). Finally, let us define the family of (nonlinear and unbounded) operators \( \{Z\} \) (depending on the values of \( \tau, \delta \) and on the choice of the "data" \( \overline{\tau} \) and \( u_\Omega \)) by setting

\[
Z: D(Z) \to H, \quad Z(v):= \alpha \left( \frac{v - \overline{\tau}}{\tau} \right) + Bv + \beta_\delta(v + u_\Omega) + \mathcal{N} \left( \frac{v - \overline{\tau}}{\tau} \right). \tag{3.4}
\]

Note that, thanks to the growth condition (2.14) and the Lipschitz regularity of \( \beta_\delta \), the right-hand side of (3.4) is an element of \( V' \) whenever \( v \in V_0 \). However, since we need \( Z \) to be an operator on \( H \), the domain \( D(Z) \) will in fact be smaller. Actually, a standard application of the regularity theory for second-order elliptic problems shows that

\[
D(Z) = \{ v \in V_0 \cap H^2(\Omega) : \partial_n v = 0 \text{ on } \partial \Omega \}, \tag{3.5}
\]

\( \partial_n \) standing for the outer normal derivative on \( \partial \Omega \) (recall that \( \Omega \) is always assumed smooth enough for our purposes). However, \( Z \) need not take values in \( H_0 \). For this reason, we also set

\[
Z_0: D(Z_0) \to H_0,
\]

\[
Z_0(v):= Z(v) - (Z(v))_\Omega = Z(v) - \frac{1}{|\Omega|} \int_\Omega \left[ \alpha \left( \frac{v - \overline{\tau}}{\tau} \right) + \beta_\delta(v + u_\Omega) \right], \tag{3.6}
\]

so that \( Z_0 \) is now a nonlinear operator from \( H_0 \) onto itself with \( D(Z_0) = D(Z) \) still given by (3.5). We can now state and prove the

**Lemma 3.1.** For all \( \overline{g} \in H, \overline{\tau} \in V_0, u_\Omega \in I, \delta \in (0, \delta_0], \) and \( \tau > 0 \), there exists a unique function \( v \in D(Z_0) \) such that

\[
Z_0(v) = \overline{h}_0, \tag{3.7}
\]

where \( \overline{h}_0 \in H_0 \) is given by (3.3). Moreover, setting

\[
z:= \frac{1}{|\Omega|} \int_\Omega \left[ \alpha \left( \frac{v - \overline{\tau}}{\tau} \right) + \beta_\delta(v + u_\Omega) \right] - \overline{h}_0 - \mathcal{N} \left( \frac{v - \overline{\tau}}{\tau} \right), \tag{3.8}
\]

the pair \( (v, z) \in V_0 \times V \) also solves

\[
\frac{v - \overline{\tau}}{\tau} + Bz = 0, \tag{3.9}
\]

\[
z = \alpha \left( \frac{v - \overline{\tau}}{\tau} \right) + Bv + \beta_\delta(v + u_\Omega) - \overline{h}, \tag{3.10}
\]

and it is the unique pair in \( V_0 \times V \) which solves (3.9)-(3.10).

**Proof.** We will use, in the proof, some more or less standard tools from the theory of maximal monotone operators in Hilbert spaces, for which we refer to the monographs [6] and [12]. Actually, it is clear that \( Z_0 \) is monotone as an operator on \( H_0 \). More precisely, given \( v_1, v_2 \in D(Z_0) \), it is a standard matter to check that

\[
(Z_0(v_1) - Z_0(v_2), v_1 - v_2) \geq \| \nabla (v_1 - v_2) \|^2 + \lambda \| v_1 - v_2 \|^2 + \tau^{-1} \| v_1 - v_2 \|^2. \tag{3.11}
\]

Then, the standard theory of monotone operators (see, e.g., [12, Sec. 2.5]) shows that \( Z_0 \) is maximal monotone and surjective. Thus, (3.7) possesses at least one solution \( v \). Moreover, an adaptation of formula (3.11) allows to see that this solution is in fact unique.

Next, given \( v \) and defining \( z \) by (3.8), (3.9) directly follows from (3.8), while (3.10) follows from the combination of (3.8), (3.4), (3.6), and (3.7). Finally, given two pairs \((z_1, v_1)\) and \((z_2, v_2)\) in \( V_0 \times V \), both solving (3.9)-(3.10) with the same "data" \( \overline{\tau}, u_\Omega, \) and \( \overline{\tau}, \) to show that \( (z_1, v_1) = (z_2, v_2) \), it suffices to write both equations for
the two pairs of solutions, take the difference, and test (the difference of) (3.9) by \( \mathcal{N}(v_1 - v_2) \) and (the difference of) (3.10) by \( v_1 - v_2 \). Then, standard calculations and the monotonicity of \( \alpha \) and \( \beta_{\delta} \) allow to finish the proof of the Lemma.

Now, let us specify in more detail our discretization argument. Once the final time \( T \) is fixed, let us set, for \( N \in \mathbb{N}, \tau = \tau(N) := T/N, \) so that \( \tau \) goes to 0 as \( N \to \infty \). Then, we can proceed by induction and solve (3.7) recursively by means of Lemma 3.1, namely, setting \( v^0 := u_0 - u_\Omega \in V_0 \) and \( \overline{y} := g \), for \( i = 1, \ldots, N \), we can take \( \overline{v} := v^{i-1} \) and find a solution \( v = v^i \). Thus, we obtain (for any given \( N \)) a “discrete solution” \( v := (v^0, \ldots, v^N) \in V_0^{N+1} \) and, by (3.8), we can correspondingly define \( z := (z^0, \ldots, z^N) \in V^{N+1} \). Setting back \( u^i := v^i + u_\Omega \), for \( i = 1, \ldots, N \), we obtain (cf. (3.1)-(3.2))

\[
\frac{u^i - u^{i-1}}{\tau} + Bz^i = 0, \tag{3.12}
\]

\[
z^i = \alpha \left( \frac{u^i - u^{i-1}}{\tau} \right) + Bu^i + \beta_{\delta}(u^i) - 3\lambda u^{i-1} - g. \tag{3.13}
\]

To continue, given a vector \( \bar{y} = (y^0, \ldots, y^N) \), let us define the piecewise linear and backward constant interpolants of \( y \), respectively, as follows:

\[
\overline{y}_\tau(0) := y^0, \quad \tilde{y}_\tau(t) := a_i(t)y^i + (1 - a_i(t))y^{i-1},
\]

\[
\overline{y}_\tau(0) := y^0, \quad \overline{y}_\tau(t) := y^i, \quad \text{for } t \in ((i - 1)\tau, i\tau], \quad i = 1, \ldots, N, \tag{3.14}
\]

where \( a_i(t) := (t - (i - 1)\tau)/\tau \) for \( t \in ((i - 1)\tau, i\tau], \quad i = 1, \ldots, N \). Moreover, let us introduce the translation operator \( T_\tau \), related to the time step \( \tau \), by setting

\[
(T_\tau y)(t) := y(t) \quad \forall t \in [0, \tau) \quad \text{and} \quad (T_\tau y)(t) := y(t - \tau) \quad \forall t \in [\tau, T], \tag{3.15}
\]

where \( y \) is any function defined on \([0, T]\). Finally, for \( i = 1, \ldots, N \), we set \( \delta y^i := (y^i - y^{i-1})\tau^{-1} \). It is clear that \( \overline{\delta y}_\tau = \tilde{y}_\tau, t \) a.e. in \([0, T]\). With this notation, system (3.12)-(3.13) takes the equivalent form

\[
\tilde{u}_{\tau,t} + B\overline{\tau}_\tau = 0, \tag{3.16}
\]

\[
\overline{\tau}_\tau = \alpha(\tilde{u}_{\tau,t}) + B\overline{\tau}_\tau + \beta_{\delta}(\overline{\tau}_\tau) - 3\lambda T_{\tau}\overline{\tau}_\tau - g. \tag{3.17}
\]

Let us now derive some a priori estimates on the discrete solution. In the remainder of the section, the symbols \( c \) and \( c_j, j \geq 0 \), will denote positive constants depending on the data of the problem, but independent of the discretization and approximation parameters \( \tau \) and \( \delta \). The value of \( c \) is allowed to change even inside a single line.

**First estimate.** Let us test (3.12) by \( \mathcal{N}(u^i - u^{i-1}) \), (3.13) by \( (u^i - u^{i-1}) \), and take the difference. Using the definition of subdifferentials and the elementary equality (or variants)

\[
\tau(y^i, \delta y^i) = \frac{1}{2}(\|y^i\|^2 + \|y^i - y^{i-1}\|^2 - \|y^{i-1}\|^2), \tag{3.18}
\]

holding, e.g., when \( y \in H^{N+1} \), and recalling (2.6), we have

\[
\frac{\tau}{2} \|\delta u^i\|^2 + \tau(\alpha(\delta u^i), \delta u^i) + \frac{1}{2} \|\nabla u^i\|^2 - \frac{1}{2} \|\nabla u^{i-1}\|^2 + \beta_{\delta}(u^i) - \tilde{\beta}_{\delta}(u^{i-1})
\]

\[
\leq (g, u^i - u^{i-1}) + \frac{3\lambda}{2}(\|u^i\|^2 - \|u^i - u^{i-1}\|^2 - \|u^{i-1}\|^2), \tag{3.19}
\]

where \( \tilde{\beta}_{\delta} \) is the antiderivative of \( \beta_{\delta} \) such that \( \tilde{\beta}_{\delta}(0) = 0 \). Then, setting

\[
W_{\delta}(r) := \tilde{\beta}_{\delta}(r) - \frac{3\lambda}{2}r^2, \tag{3.20}
\]
it is clear from (2.10) that \( \delta_0 \) can be taken small enough so that \( W_\delta \) stays bounded from below, uniformly with respect to \( \delta \in (0, \delta_0] \).

Thus, taking the sum of (3.19) for \( i = 1, \ldots, n \), where \( n \leq N \), recalling assumption (2.11) (note also that \( W_\delta \leq W \), so that, defining \( \mathcal{E}_\delta \) as in (2.12), but with \( W_\delta \) in place of \( W \), it follows from (2.13) that \( \mathcal{E}_\delta(u_0) \leq c \), and using the notation (3.14) and (3.15), we readily obtain the estimates

\[
\| \bar{u}_{\tau,t} \|_{L^2(0,T;V')} \leq c, \tag{3.21}
\]
\[
\int_0^T \alpha(\bar{u}_{\tau,t}) \bar{u}_{\tau,t} \leq c, \tag{3.22}
\]
\[
|\alpha(\bar{u}_{\tau,t})|_{L^2(0,T;H)} \leq c, \tag{3.23}
\]
\[
|\bar{\pi}_\tau|_{L^\infty(0,T;V)} \leq c, \tag{3.24}
\]
\[
|\bar{\beta}_\delta(\bar{\pi}_\tau)|_{L^\infty(0,T;L^1(\Omega))} \leq c, \tag{3.25}
\]

where (3.23) has been deduced from (3.22) with the help of (2.24). Note that, in particular, the latter two estimates are equivalent to

\[
\| \mathcal{E}_\delta(\bar{\pi}_\tau) \|_{L^\infty(0,T)} \leq c. \tag{3.26}
\]

**Second estimate.** Let us test (3.12) by \( N(\beta_\delta(u^i) - (\beta_\delta(u^i))_\Omega), (3.13) \) by \( \beta_\delta(u^i) - (\beta_\delta(u^i))_\Omega \), and take the difference. It is then a standard matter to obtain, a.e. in \((0,T)\),

\[
\|\beta_\delta(\bar{\pi}_\tau) - (\beta_\delta(\bar{\pi}_\tau))_\Omega\|^2 \leq c(\|\bar{u}_{\tau,t}\|^2 + \|\alpha(\bar{u}_{\tau,t})\|^2 + 3\lambda_T \bar{\pi}_\tau + g^2), \tag{3.27}
\]

whence, integrating in time and taking advantage of (3.21), (3.23), (3.24), and (2.11), we find

\[
|\beta_\delta(\bar{\pi}_\tau) - (\beta_\delta(\bar{\pi}_\tau))_\Omega|_{L^2(0,T;H)} \leq c. \tag{3.28}
\]

Next, an application of (a suitable variant of) the argument in [21, Lemma 5.2] allows to estimate the spatial average of \( \beta_\delta(\bar{\pi}_\tau) \). More precisely, one can prove that, for a.e. \( t \in (0,T) \),

\[
|\beta_\delta(\bar{\pi}_\tau)|_\Omega \leq c(\|\beta_\delta(\bar{\pi}_\tau) - (\beta_\delta(\bar{\pi}_\tau))_\Omega\|_{\|\bar{\pi}_\tau - u_\Omega\| + 1}, \tag{3.29}
\]

whence, as a consequence of (3.28) and (3.24),

\[
|\beta_\delta(\bar{\pi}_\tau)|_{L^2(0,T;H)} \leq c. \tag{3.30}
\]

At this point, coming back to (3.16), recalling (3.21), and testing (3.17) by 1 to estimate the spatial average of \( \bar{\pi}_\tau \), we also have

\[
|\bar{\pi}_\tau|_{L^2(0,T;V)} \leq c, \tag{3.31}
\]

whence, applying standard elliptic regularity results to (3.17), there also follows

\[
|\bar{\pi}_\tau|_{L^2(0,T;H^2(\Omega))} \leq c. \tag{3.32}
\]

**Passage to the limit.** We now let the parameters \( \tau \) and \( \delta \) go to 0 and, for brevity, we assume that they tend to 0 simultaneously (one can think, e.g., that \( \delta = \delta(N) = \tau(N) := T/N \)). First, let us then notice that (2.15) and (3.22) imply

\[
\| \bar{u}_{\tau,t} \|_{L^2(0,T;H)} \leq c. \tag{3.33}
\]
Thus, up to the extraction of a subsequence, we deduce from (3.21)-(3.25), (3.30)-(3.32), and (3.33) that
\[
\pi_t \rightarrow u \quad \text{weakly star in } L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)),
\]
(3.34)
\[
\tilde{u}_{t,t} \rightarrow u_t \quad \text{weakly star in } L^2(0, T; H),
\]
(3.35)
\[
\pi_t \rightarrow z \quad \text{weakly star in } L^2(0, T; V),
\]
(3.36)
\[
\alpha(\tilde{u}_{t,t}) \rightarrow \xi \quad \text{weakly in } L^2(0, T; H),
\]
(3.37)
\[
\beta_0(\pi_t) \rightarrow w \quad \text{weakly star in } L^2(0, T; H),
\]
(3.38)
for suitable limit functions \(u, z, w, \xi\). It is now easy to infer from (3.34) and (3.35) that
\[
\pi_t \rightarrow u \quad \text{strongly in } L^\infty(0, T; H) \cap L^2(0, T; V) \quad \text{and a.e. in } \Omega \times (0, T),
\]
(3.39)
so that, from (3.39), (3.38), and a usual monotonicity argument (cf., e.g., [6, Prop. 1.1, p. 42]), we obtain \(w = \beta(u)\) a.e. in \(\Omega \times (0, T)\), as desired.

To conclude that we can take the limit of (3.16)-(3.17) and obtain (2.20)-(2.21), it then remains to prove that \(\xi = \alpha(u)\). To do so, we first notice that, by (3.35) and the analogue of (3.34) written for \(\tilde{u}_t\),
\[
\tilde{u}_t \rightarrow u \quad \text{in } C_w([0, T]; V),
\]
(3.40)
whence, in particular, \(\tilde{u}_t(t)\) tends to \(u(t)\) weakly in \(V\) for all (and not just a.e.) \(t \in [0, T]\). In particular, \(\pi_t(T)\) tends to \(u(T)\) weakly in \(V\), since \(\pi_t(T) = \tilde{u}_t(T)\) by construction.

Next, let us test (3.16) by \(N\tilde{u}_{t,t}\), (3.17) by \(\tilde{u}_{t,t}\), take the difference, and integrate over \((0, T)\). We then have the (integrated) analogue of (3.19), namely,
\[
\int_0^T (\alpha(\tilde{u}_{t,t}), \tilde{u}_{t,t}) \leq -\frac{1}{2} \|\nabla \pi_t(T)\|^2 + \frac{1}{2} \|\nabla u_0\|^2 - \int_\Omega (\tilde{\beta}_0(\pi_t(T)) - \tilde{\beta}_0(u_0)) + \frac{3\lambda}{2} (\|\pi_t(T)\|^2 - \|u_0\|^2) + \int_\Omega g(\pi_t(T) - u_0) - \int_0^T \|\tilde{u}_{t,t}\|^2.
\]
(3.41)

Let us now take the lim sup of the above relation. First, let us notice that, by relations (3.34)-(3.35) and the lower semicontinuity of norms with respect to weak and weak star convergences, we have
\[
\limsup_{\tau, \delta \searrow 0} \left( -\frac{1}{2} \|\nabla \pi_t(T)\|^2 + \frac{1}{2} \|\nabla u_0\|^2 + \frac{3\lambda}{2} \|\pi_t(T)\|^2 - \frac{3\lambda}{2} \|u_0\|^2 \right.
\]
\[
+ \int_\Omega g(\pi_t(T) - u_0) - \int_0^T \|\tilde{u}_{t,t}\|^2 + \frac{1}{2} \|\nabla u_0\|^2 + \frac{3\lambda}{2} \|u(T)\|^2 - \frac{3\lambda}{2} \|u_0\|^2
\]
\[
+ \int_\Omega g(u(T) - u_0) - \int_0^T \|u_0\|^2.
\]
(3.42)

We have also used here the fact that, as observed above, \(\pi_t(T) \rightarrow u(T)\) weakly in \(V\) and hence strongly in \(H\). Furthermore, exploiting the Mosco-convergence \(\tilde{\beta}_0 \rightarrow \tilde{\beta}\) (cf., e.g., [1, Chap. 3]), we obtain, owing to (3.39),
\[
\limsup_{\tau, \delta \searrow 0} (\tilde{\beta}_0(\pi_t(T)) - \tilde{\beta}_0(u_0)) \leq - \int_\Omega (\tilde{\beta}(u(T)) - \tilde{\beta}(u_0)).
\]
(3.43)
Thus, thanks to (3.42) and (3.43), a comparison between the lim sup of (3.41) and the limit relation (2.21) tested by \( u_t \) and integrated in time yields

\[
\limsup_{\tau, \delta \to 0} \int_0^T (\alpha(\tilde{u}_{\tau,t}, \tilde{u}_{\tau,t}) \leq \int_0^T (\xi, u_t)
\]

(3.44)

and we deduce that \( \xi = \alpha(u_t) \) by applying once more [6, Prop. 1.1, p. 42]. This finishes the proof. \( \square \)

3.2. Proof of Theorem 2.2. As a first step, we just come back to the discretization argument and repeat the first two estimates. To pass to the limit with the hypotheses of the theorem, we then need a new regularity bound.

Third estimate. Let us write (3.12) and (3.13) for \( i \) and for \( i - 1 \) and take the difference. This gives, for \( i \geq 2 \),

\[
\frac{u^i - u^{i-1}}{\tau} - \frac{u^{i-1} - u^{i-2}}{\tau} + B(z^i - z^{i-1}) = 0,
\]

(3.45)

Next, let us test (3.45) by \( \tau^{-1}N(u^i - u^{i-1}) \), (3.46) by \( \tau^{-1}(u^i - u^{i-1}) \), and take the difference. We do not write explicitly the resulting formula, but just estimate separately the quantities appearing there. First, by (3.48), we have

\[
\left\langle \frac{u^i - u^{i-1}}{\tau} - \frac{u^{i-1} - u^{i-2}}{\tau}, \frac{N(u^i - u^{i-1})}{\tau} \right\rangle 
\geq \frac{1}{2} \frac{\|u^i - u^{i-1}\|_r^2}{\tau} - \frac{1}{2} \frac{\|u^{i-1} - u^{i-2}\|_r^2}{\tau}.
\]

(3.47)

Now, let us set \( \xi^i := \alpha((u^i - u^{i-1})/\tau) \) and denote by \( \tilde{\alpha} \) the antiderivative of \( \alpha \) which vanishes at 0. Since \( \tilde{\alpha} \) is a convex function on \( \mathbb{R} \), we can indicate by \( \tilde{\alpha}^* \) its convex conjugate (see, e.g., [12]). Then, using the definition of subdifferentials, we have

\[
\left( \alpha \left( \frac{u^i - u^{i-1}}{\tau} \right) - \alpha \left( \frac{u^{i-1} - u^{i-2}}{\tau}, \frac{u^i - u^{i-1}}{\tau} \right) \right) \geq \int_\Omega (\tilde{\alpha}^*(\xi^i) - \tilde{\alpha}^*(\xi^{i-1})).
\]

(3.48)

Moreover, by monotonicity of \( \beta_\delta \) and standard integrations by parts,

\[
\tau^{-1} (B(u^i - u^{i-1}) + \beta_\delta(u^i) - \beta_\delta(u^{i-1}), (u^i - u^{i-1})) \geq \tau \|\nabla u^i\|^2.
\]

(3.49)

Finally, using the Young and Poincaré-Wirtinger inequalities, we find

\[
\tau^{-1} (3\lambda(u^{i-1} - u^{i-2}), (u^i - u^{i-1})) \leq \frac{\tau}{2} \|\nabla u^i\|^2 + c\tau \|\delta u^{i-1}\|^2_*.
\]

(3.50)

Thus, summing over \( i = 2, \ldots, N \) the expressions obtained from (3.45)-(3.46) and noting that the terms on the right-hand side of (3.50) are controlled thanks to (3.49) and (3.21), one sees that one just has to provide a bound on the quantity

\[
\frac{1}{2} \frac{\|u^1 - u^0\|_r^2}{\tau} + \int_\Omega \tilde{\alpha}^*(\xi^1), \quad \xi^1 = \alpha \left( \frac{u^1 - u^0}{\tau} \right),
\]

(3.51)

coming from (3.47)-(3.48) for \( i = 2 \). Now, let us notice that, if \( \alpha^{-1}(r) \) denotes the element of minimum absolute value in the convex set \( \alpha^{-1}(r) \) (cf., e.g., [12, Sec. 2.4]),
using monotonicity and recalling that $0 = \alpha_0^{-1}(0)$ (cf. (2.7)), we have
\[
\int_{\Omega} \tilde{\alpha}^*(\xi^1) = \int_{\Omega} \int_{0}^{\xi^1} \alpha_0^{-1}(r) \, dr \leq \int_{\Omega} \int_{0}^{\xi^1} \alpha_0^{-1}(\xi^1) \, dr
\]
\[
= \int_{\Omega} \xi^1 \alpha_0^{-1}(\xi^1) \leq \int_{\Omega} \alpha\left(\frac{u^1 - u^0}{\tau}\right) \left(\frac{u^1 - u^0}{\tau}\right).
\] (3.52)

Thus, to control (3.51), we can write (3.12)-(3.13) for $i = 1$ and test them, respectively, by $\tau^{-1}N(u^1 - u^0)$ and by $\tau^{-1}(u^1 - u^0)$. Taking as usual the difference, it is a standard matter to obtain
\[
\left\|\frac{u^1 - u^0}{\tau}\right\|^2_{\ast} + \int_{\Omega} \alpha\left(\frac{u^1 - u^0}{\tau}\right) \left(\frac{u^1 - u^0}{\tau}\right) + \left(Bu^1 + \beta_\delta(u^1) - 3\lambda u^0 - g, \frac{u^1 - u^0}{\tau}\right) = 0.
\] (3.53)

Then, noting that, again by monotonicity,
\[
\left(Bu^1 + \beta_\delta(u^1), \frac{u^1 - u^0}{\tau}\right) \geq \left(Bu^0 + \beta_\delta(u^0), \frac{u^1 - u^0}{\tau}\right),
\] (3.54)
it is clear that the right-hand side can be split in the duality $V' - V$ and controlled, provided that
\[
Bu^0 + \beta_\delta(u^0) \in V.
\] (3.55)

Actually, this property easily follows from (2.25), thanks to the properties of the Yosida approximations. Thus, we finally obtain the estimate
\[
\|\tilde{\alpha}_\tau, t\|_{L^\infty(0,T;V')} + \|\tilde{\alpha}_\tau, t\|_{L^2(0,T;V)} \leq c.
\] (3.56)

Moreover, since, by (a simple consequence of) (2.14),
\[
\int_{\Omega} \tilde{\alpha}^*(\xi^n) = \int_{\Omega} \int_{0}^{\xi^n} \alpha_0^{-1}(r) \, dr \geq 2\kappa\|\xi^n\|^2 - c \int_{\Omega} |\xi^n| \geq \kappa\|\xi^n\|^2 - c
\] (3.57)
for some $\kappa > 0$, we also have
\[
\|\alpha(\tilde{\alpha}_\tau, t)\|_{L^\infty(0,T;H)} \leq c.
\] (3.58)

Next, repeating the argument in the Second Estimate, but using this time (3.56) and (3.58) to control the terms on the right-hand side of (3.27), one can easily improve (3.30) up to
\[
\|\beta_\delta(\tilde{\gamma}_\tau)\|_{L^\infty(0,T;H)} \leq c,
\] (3.59)
whence the same arguments as the ones used there also give
\[
\|\tilde{\gamma}_\tau\|_{L^\infty(0,T;V)} \leq c,
\] (3.60)
\[
\|\tilde{\gamma}_\tau\|_{L^\infty(0,T;H^2(\Omega))} \leq c.
\] (3.61)

At this point, the passage to the limit is reached by using standard compactness arguments. In particular, the identification of the limits of the nonlinear terms is obtained exactly as above. We thus leave the details to the reader. \qed

**Remark 3.2.** Observe that, in case $\alpha$ is coercive (i.e., (2.15) holds, cf. Remark 2.3), then the second term on the left-hand side of (3.53) is larger than the square of the $H$-norm of $(u^1 - u^0)\tau^{-1}$ minus some constant $c$. Thus, the right-hand side of (3.53) can be dealt with by working in $H$ rather than in the $V' - V$ duality. Hence, (2.30) (which is, by the way, preserved for Yosida approximations) suffices, in place of (2.25).
Proof of Theorem 2.4. Let us notice that, although (2.14) does not hold here, the procedure in the First Estimate can still be carried out and (3.21)-(3.22) and (3.24)-(3.25) still hold. Of course, (3.23), which follows from (2.14), might no longer be valid. The Second Estimate, instead, cannot be used, since (3.23) is required in order to estimate the second summand on the right-hand side of (3.27). However, it is clear that (3.24) and (2.32) (the latter a fortiori holds for \( W_0^1 \)) give

\[
\| \beta_\delta(\vec{u}_\tau) \|_{L^\infty(0,T;L^{6/5}(\Omega))} \leq c. \tag{3.62}
\]

As far as the Third Estimate is concerned, one readily checks that everything can be repeated up to (3.56), while (3.58) and, consequently, (3.61) are no longer valid. Speaking of (3.60), by (3.56), one immediately has

\[
\| \nabla \tau \|_{L^\infty(0,T;H)} \leq c. \tag{3.63}
\]

Moreover, testing (3.13) by 1 and taking the absolute value,

\[
(\tau,\alpha)|_\Omega \leq ((\beta_\delta(\vec{u}_\tau))|_\Omega + 3\lambda(|T_{\tau}^\tau|)|_\Omega + |g_\delta| + |\alpha(\vec{u}_{\tau,t})|)|_\Omega \leq c. \tag{3.64}
\]

Actually, the terms on the right-hand side are controlled, respectively, by (3.62), (3.24), (3.11), and by noticing that (cf. (3.48))

\[
\int_{\Omega} \alpha^\tau(x^i) \geq \int_{\Omega} \left| \alpha \left( \frac{u^i - u^{i-1}}{\tau} \right) \right| - c. \tag{3.65}
\]

The above inequality can be proved as follows. Let us assume, for simplicity, that \( s > 1 \). Then,

\[
\alpha^\tau(\alpha(s)) = \int_0^s \frac{d}{dr} \alpha^\tau(\alpha(r)) \, dr = \int_0^s \alpha^{-1}(\alpha(r)) \alpha'(r) \, dr \]

\[
= \int_0^1 r \alpha'(r) \, dr + \int_1^s r \alpha'(r) \, dr \geq \alpha(s) - \alpha(1) \geq c(s). \tag{3.66}
\]

Notice that the third inequality holds because, for all \( r \) such that \( \alpha^{-1}(\alpha(r)) \) is multivalued, there holds \( \alpha'(r) = 0 \). Since (3.66) is easily adapted to the case when \( s < 1 \) and the inequality \( \alpha^\tau(\alpha(s)) \geq c(s) \) is trivial for \( s \in [-1, 1] \), then (3.65) holds. In particular, (3.60) still holds as a consequence of (3.63) and (3.65).

To conclude the proof, let us show that we can still take the limit, with respect to \( \tau, \delta \) simultaneously tending to 0, of (3.12)-(3.13). To this aim, we first notice that the continuity of the embedding \( L^{6/5}(\Omega) \subset V' \), and relations (3.60), (3.24), (3.62), and (2.11) entail

\[
\| \alpha(\vec{u}_{\tau}) \|_{L^\infty(0,T;V')} \leq c. \tag{3.67}
\]

Thus, we obtain, still up to the extraction of a subsequence,

\[
\vec{u}_\tau \rightharpoonup u \quad \text{weakly in } L^\infty(0,T;V), \tag{3.68}
\]

\[
\vec{u}_{\tau,t} \rightharpoonup u_t \quad \text{weakly in } L^\infty(0,T;V') \cap L^2(0,T;V), \tag{3.69}
\]

\[
\vec{u}_\tau \rightharpoonup z \quad \text{weakly in } L^\infty(0,T;V), \tag{3.70}
\]

\[
\alpha(\vec{u}_{\tau,t}) \rightharpoonup \xi \quad \text{weakly in } L^\infty(0,T;V'). \tag{3.71}
\]

Moreover, observing that \( \beta_\delta \rightharpoonup \beta \) uniformly on compact sets of \( \mathbb{R} \), and that, owing to (3.68) and (3.69), there follows

\[
\vec{u}_\tau \rightharpoonup u \quad \text{strongly in } L^\infty(0,T;H) \quad \text{and a.e. in } \Omega \times (0,T), \tag{3.72}
\]

by Lebesgue’s dominated convergence Theorem and (3.62), we then have

\[
\beta_\delta(\vec{u}_\tau) \rightharpoonup \beta(u) \quad \text{weakly star in } L^\infty(0,T;L^{6/5}(\Omega)). \tag{3.73}
\]
This shows that we can once more take the limit of (3.16)-(3.17) and find (2.20) and (2.37).

To conclude, let us show (2.38) and, to do so, let us proceed as in the last part of the proof of Theorem 2.1, namely, we test (3.16) by $\mathcal{N}\hat{u}_{\tau,t}$, (3.17) by $\hat{u}_{\tau,t}$, take the difference, and integrate over $(0,T)$. This gives (3.41). Now, the procedure is exactly the same as above till (3.43). Instead, the identification of the $\alpha$-term is slightly different. Indeed, the lim sup of the analogue of (3.41) now has to be compared with (2.37) which is tested by $u_t$ and integrated over $(0,T)$. Note that this makes sense in the duality between $V'$ and $V$ because $u_t \in L^2(0,T;V)$ and all single terms in (2.37) belong to $L^2(0,T;V')$. Thus, from (3.41)-(3.42), we obtain

$$\limsup_{\tau,\delta \searrow 0} \int_0^T (\alpha(\hat{u}_{\tau,t}), \hat{u}_{\tau,t}) = \limsup_{\tau,\delta \searrow 0} \int_0^T (\alpha(\hat{u}_{\tau,t}), \hat{u}_{\tau,t}) \leq \int_0^T \langle \xi, u_t \rangle. \quad (3.74)$$

Hence, noting that, for all $\tau,\delta > 0$ and a.e. in $(0,T)$, $\alpha(\hat{u}_{\tau,t})$ coincides in fact with $\alpha_w(\hat{u}_{\tau,t})$ (cf. the characterization of $\alpha_w$ reported just before the statement of Theorem 2.4), a further application of [6, Prop. 1.1, p. 42] to the (maximal monotone) operator $\alpha_w : V \to V'$ readily yields (2.38) and concludes the proof of the theorem. □

**Proof of Theorem 2.5.** Let us be given a pair $(u_1, z_1)$, $(u_2, z_2)$ of solutions and set $(u, z) := (u_1, z_1) - (u_2, z_2)$. Then, let us write the system for the two solutions and take the difference. Test the difference of (2.20) by $\mathcal{N}u_t$ and the one of (2.21) (or of (2.37)) by $u_t$. By the monotonicity of $\alpha$ (or $\alpha_w$), we then readily obtain

$$\|u_t\|_*^2 + \frac{d}{dt}\|\nabla u\|^2 + \int_\Omega u_t(W'(u_1) - W'(u_2)) \leq 0. \quad (3.75)$$

To estimate the last term on the left-hand side, we use the procedure devised in [18, Thm. 2.2]. More precisely, let us observe that, by (2.9), setting again $\beta(r) := W'(r) + 3\lambda r$, we obtain that $\beta'(r) \geq 2\lambda$ for all $r \in I$. Moreover, if $\tilde{\beta}(r) := \int_0^r \beta(s) \, ds$, a direct calculation shows that

$$\begin{align*}
(\beta(u_1) - \beta(u_2))u_t &= \frac{d}{dt}(\tilde{\beta}(u_1) - \tilde{\beta}(u_2) - \beta(u_2)u) - (\beta(u_1) - \beta(u_2) - \beta'(u_2)u)u_{2,t}.
\end{align*} \quad (3.76)$$

Thus, (3.75) can be rewritten as

$$\begin{align*}
\|u_t\|_*^2 + \frac{d}{dt}\left[\frac{1}{2}\|\nabla u\|^2 + \int_\Omega (\tilde{\beta}(u_1) - \tilde{\beta}(u_2) - \beta(u_2)u)\right] \\
\leq 3\lambda(u, u_t) + \int_\Omega (\beta(u_1) - \beta(u_2) - \beta'(u_2)u)u_{2,t}.
\end{align*} \quad (3.77)$$

Let us now prove (i). Then, noticing that $u$ has zero average for all times and using the Poincaré-Wirtinger inequality and Sobolev’s embeddings, we can estimate the right-hand side of (3.77) as follows:

$$\begin{align*}
\leq \frac{1}{2}\|u_t\|_*^2 + c\|\nabla u\|^2 \left(1 + \|u_{2,t}\|^2 + Q(||u_1||_{L^\infty(\Omega)}) + Q(||u_2||_{L^\infty(\Omega)})\right),
\end{align*} \quad (3.78)$$

where $Q$ is a suitable monotone function depending on the growth rate of $W$. Since $\beta' \geq 2\lambda$, we also have

$$\begin{align*}
\int_\Omega \left(\tilde{\beta}(u_1) - \tilde{\beta}(u_2) - \beta(u_2)u\right) \geq 2\lambda\|u\|^2.
\end{align*} \quad (3.79)$$
Thus, since \( u_1 \) and \( u_2 \) are uniformly bounded, thanks to the latter of (2.26) and the continuity of the embedding \( H^3(\Omega) \subset C^0(\overline{\Omega}) \), we can apply Gronwall’s Lemma to the functional in square brackets on the left-hand side of (3.77). Thanks to (3.78) and (3.79), this entails the uniqueness of \( u \). The uniqueness of \( z \) then follows by comparison in (2.21). This finishes the proof of (i).

To prove (ii), we go back to (3.77) and, in place of (3.78), we now have

\[
\leq \frac{1}{2} ||u_i||_\ast + c \int_\Omega (1 + |u_1|^3 + |u_2|^3) |u|^2 |u_{2,i}|
\]

\[
\leq \frac{1}{2} ||u_i||_\ast + c (1 + ||u_1||_V^3 + ||u_2||_V^3) \| \nabla u \|^2 \| u_{2,i} \|_V,
\]

and we can again apply Gronwall’s lemma, thanks to the (second of the) regularity (2.33). This finishes the proof. \( \square \)

4. Long time behavior. In what follows, we will change a little bit our convention on the constants. From now on, \( c, \kappa, \) and \( c_j, j \geq 0 \), will no longer be allowed to depend on time or on the initial datum \( u_0 \). However, they may depend on \( m_1, m_2 \) in (2.43). The symbol \( c_{Q_0} \) will denote embedding constants depending only on \( \Omega \).

Finally, \( c_j, j \geq 0 \), will stand for positive constants with additional dependences (e.g., on time or on the initial datum), specified on occurrence.

Proof of Theorem 2.6. We first prove the existence of a uniformly absorbing set which is bounded in the metric distance \( d_\mathcal{E} \) (cf. (2.41)). To this aim, it suffices to test (2.20) by \( z \) and (2.21) by \( u_t + \sigma (u - u_\Omega) \) for small \( \sigma > 0 \) and notice that

\[
\int_\Omega W'(u)(u - u_\Omega) \geq \int_\Omega W(u) - c
\]

and that, by (2.4),

\[
\int_\Omega z(u - u_\Omega) = \int_\Omega (z - z_\Omega)(u - u_\Omega) \leq \frac{1}{2} \| \nabla z \|^2 + \frac{1}{2} \| \nabla u \|^2.
\]

Moreover, using (2.44) and the Young and Poincaré-Wirtinger inequalities, it is easy to find

\[
\sigma \int_\Omega \alpha(u_i)(u - u_\Omega) \leq c \sigma \int_\Omega \alpha(u_i) u_t + c \sigma \| \nabla u \|^2.
\]

Hence, taking \( \sigma \) small enough and using properties (2.44) and (2.15), it is not difficult to obtain the relation

\[
\frac{d}{dt} \mathcal{E}(u) + \kappa \mathcal{E}(u) + \frac{1}{2} \| \nabla z \|^2 + \kappa \int_\Omega \alpha(u_i) u_t \leq c
\]

for some \( \kappa > 0 \). Thus, Gronwall’s Lemma entails dissipativity in the energy norm, namely, for all \( t \geq 0 \), we have

\[
\mathcal{E}(u(t)) + \kappa \int_0^{t+1} \int_\Omega \alpha(u_i) u_t + \frac{1}{2} \int_0^{t+1} \| \nabla z \|^2 \leq \mathcal{E}(u_0) e^{-\kappa t} + c.
\]

Notice that the above argument is not linked to the approximation scheme, since any solution in the setting of Theorem 2.1 has sufficient regularity by itself.

On the other hand, as a next step, we have to prove that the absorbing set is in fact compact and, to do this, we need to use the Second Estimate for strictly positive times. However, the regularity properties (2.16)-(2.19) do not allow to directly work on the limit solutions. Thus, we need to work on the approximations and then pass to the limit. This means that we have to restrict ourselves to the solutions which are
limits of the discretization procedure and, consequently, to interpret the attractor accordingly (cf. Remarks 2.7 and 2.8 for further details).

This said, we just proceed formally and observe that the continuous analogue of the Third Estimate can be rewritten as

\[
\frac{d}{dt} \left[ \frac{1}{2} \| \nabla z \|^2 + \int_{\Omega} A(u_t) + \| \nabla u_t \|^2 + \int_{\Omega} W''(u) u_t^2 \right] \leq 0, \quad (4.6)
\]

where we have set

\[
A(s) := \int_0^s \alpha'(r) r \, dr. \quad (4.7)
\]

We now claim that \( c_1 s^2 - c_2 \leq A(s) \leq c_3 s^2 \) for all \( s \in \mathbb{R} \). Actually, the latter inequality directly follows from (2.44). Concerning the first, we pick \( M > 0 \) and observe that

\[
A(s) \geq \int_{s/M}^s \alpha'(r) r \, dr \geq \frac{s}{M} (\alpha(s) - \alpha(s/M)) \geq \frac{\kappa s^2}{M} - \frac{K_s}{M} - \frac{c_o s^2}{M^2}, \quad (4.8)
\]

thanks to (2.15) and (2.44). Hence, the inequality follows by taking \( M \) large enough.

Thus, recalling (4.5) and using the uniform Gronwall Lemma in (4.6), we easily infer

\[
\frac{1}{2} \| \nabla z(t) \|^2 + \int_{\Omega} A(u_t(t)) + \int_0^{t+1} \| \nabla u_t \|^2 \leq c(\mathcal{E}(u_0) e^{-\kappa t} + 1) \quad (4.9)
\]

for all \( t \geq 1 \). Next, we notice that (2.21) can be rewritten as the time dependent family of elliptic problems

\[
Bu + W'(u) = z - \alpha(u_t) + g. \quad (4.10)
\]

Thus, repeating the argument of [21] used in the Second Estimate, it is not difficult to infer

\[
\| W'(u(t)) \|^2 + \| z(t) \|^2 \leq c(\mathcal{E}(u_0) e^{-\kappa t} + 1), \quad (4.11)
\]

whence we obtain in particular that the right-hand side of (4.10) is uniformly bounded with values in \( H \) in the sense of the right-hand side of (4.11) (here, also the sublinearity condition (2.44) is used). At this point, the required compactness property leading to (2.45) is obtained by applying to (4.10) the standard regularity results for elliptic problems with (essentially) monotone nonlinearities. Consequently, we obtain the existence of a uniformly absorbing set which is bounded in the sense of (2.45). The thesis then follows from the standard theory of attractors \([3, 38]\) in case uniqueness holds, or from some generalized theory \([2, 4, 5, 30]\) otherwise (cf. Remark 2.8).

**Proof of Theorem 2.9.** We now consider solutions \( u \) starting from initial data \( u_0 \in \mathcal{W} \) (cf. (2.26)). As a first step, we analyze the set of stationary states associated with (2.20), (2.37), and (2.38). Since we only consider trajectories originating from initial data \( u_0 \) in \( \mathcal{W}(m_1, m_2) \), it follows from the existence of Lyapunov functionals (see below) that the related \( \omega \)-limit sets only contain solutions to

\[
Bu + \beta(u) = z_\infty + g + 3\lambda u, \quad u_\Omega = m, \quad (4.12)
\]

where \( \beta = W' + 3\lambda I \), as above, so that \( \beta' \geq 2\lambda > 0 \). This statement needs some explanations. First, \( m \in [m_1, m_2] \) is the spatial average of the initial datum \( u_0 \). Then, \( z_\infty \) is some constant which is the “limit” of the spatial average of \( z \). More precisely, we cannot exclude that distinct elements of the \( \omega \)-limit set of one single trajectory \( u(t) \) could solve (4.12) for different values of \( z_\infty \).
We now study the properties of the solutions to (4.12). In what follows, \( c \) will possibly depend on \( m \) and \( g \), but will be independent of \( z_\infty \). First, testing (4.12) by \( u - m \), it is not difficult to find (cf. (4.1)), for some \( c_4, c_5 > 0 \),

\[
c_4 \|u\|_V^2 - c_5 \leq \|\nabla u\|^2 + \int_\Omega \beta(u) \leq c.
\]

(4.13)

Actually, we notice that, by (2.10), the monotone part of \( W' \) dominates the remainder one. Moreover, of course, there holds \((z_\infty, u - m) = 0\). Next, testing (4.12) by \( Bu \) and taking (4.13) into account, we obtain

\[
\|u\|_{H^2(\Omega)} \leq c.
\]

(4.14)

Thus, by the continuous embedding \( H^2(\Omega) \subset L^\infty(\Omega) \) and the growth condition (2.32), we also have

\[
\|\beta(u)\|_{L^\infty(\Omega)} \leq c,
\]

(4.15)

so that, by comparison, there also holds

\[
|z_\infty| \leq c.
\]

(4.16)

In other words, the set of the possible values \( z_\infty \) for which an element of the \( \omega \)-limit set of one trajectory \( u(t) \) starting from an initial datum \( u_0 \in W \) can solve (4.12) is bounded in \( \mathbb{R} \) in a way which only depends on \( m \) and, in fact, on the values \( m_1, m_2 \).

**Remark 4.1.** If \( g \) (and possibly \( W \)) is more regular, we can have higher regularity for the solutions to (4.12). For instance, if \( g \in V \), it is clear that the set of stationary states is bounded in \( H^3(\Omega) \).

Next, we can observe that two Lyapunov functionals are available for our system.

The first one is the energy \( \mathcal{E} \): indeed, testing (2.20) by \( Nu_t \), (2.37) by \( u_t \), and taking the difference, one has the energy equality (cf. (2.12))

\[
\frac{d}{dt} \mathcal{E} + \|u_t\|^2_V + \int_\Omega \alpha(u_t)u_t = 0.
\]

(4.17)

Consequently, we also have

\[
\mathcal{E}(t) + \|u(t)\|^2_V \leq Q \quad \forall \ t \geq 0.
\]

(4.18)

For brevity, here and in what follows, we only stress the dependence on time of functionals which actually depend on the solution (e.g., we write \( \mathcal{E}(t) \) in place of \( \mathcal{E}(u(t)) \)). Moreover, \( Q \) stands for a quantity \( Q(\|u_0\|_{H^2(\Omega)}) \), \( Q \) being a nonnegative and increasing monotone function possibly depending on \( m_1, m_2 \) (cf. (2.43)), but independent of time, whose expression may vary from case to case.

The second Lyapunov functional can just be read from (4.6). Actually, denoting by \( \mathcal{F} \) the term in square brackets, \( \mathcal{F} \) need not be a Lyapunov functional itself, since \( W \) can be nonconvex; however, it is clear that \( \mathcal{F} + k\mathcal{E} \) is, for sufficiently large \( k > 0 \), a Lyapunov functional. Indeed, there holds

\[
\frac{d}{dt}(\mathcal{F}(u) + k\mathcal{E}(u)) + \|\nabla u_t\|^2 + \int_\Omega W''(u)u_t^2 + k\|u_t\|^2 + k\int_\Omega \alpha(u_t)u_t \leq 0.
\]

(4.19)

We have to notice, however, that the argument leading to (4.19) is formal, even in the rather high regularity setting of Theorem 2.2, and one should in fact prove (4.19) by approximation and then pass to the limit. As noted above, we omit this procedure which leads to the use of the “limiting semiflows” of [30].
Our next aim is to show that, as \( u_0 \in \mathcal{W} \), then the solution \( u \) never leaves the phase space. We first observe that we can integrate (4.19) between 0 and an arbitrary \( t \geq 0 \). Noting that, for \( k \) large enough,
\[
\| \nabla u_t \|^2 + \int_{\Omega} W''(u)u_t^2 + k\| u_t \|^2 \geq \alpha \left( \| u_t \|^2 + \| \nabla z \|^2 \right),
\]
we obtain the following dissipation property:
\[
\int_0^t \| u_t \|^2 + \int_0^t \| \nabla z \|^2 + \int_0^t \int_{\Omega} \alpha(u_t)u_t \leq Q,
\]
provided that we can control the quantity \((F + k\xi)(0)\). Let us check this fact by applying a variant of the argument used in the Third Estimate. Actually, it is clear that it is enough to control
\[
\alpha(u_t(0))u_t(0) \geq \kappa_\alpha |u_t(0)|^2 - K_\alpha,
\]
thanks to (2.15), and that
\[
A(u_t(0)) = \int_0^{u_t(0)} \alpha'(r)r \leq \int_0^{u_t(0)} \alpha'(r)u_t(0) \leq \alpha(u_t(0))u_t(0).
\]
Thus, the right-hand side of the equality in (4.23) can be split by Young’s inequality and controlled by a quantity \( Q \). Next, let us test (2.37) by \( Bu_t \). We obtain
\[
\frac{1}{2} \frac{d}{dt} \| Bu_t \|^2 + \int_{\Omega} (W''(u)|\nabla u|^2) - 2(g, Bu_t) + \int_{\Omega} \alpha'(u_t)|\nabla u_t|^2
\leq \frac{1}{2} \| \nabla z \|^2 + \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \int_{\Omega} (W''(u)u_t|\nabla u|^2)
\]
and the nontrivial term on the right-hand side can be controlled as follows, using (2.40) and (4.18):
\[
\frac{1}{2} \int_{\Omega} (W''(u)u_t|\nabla u|^2) \leq c \left( 1 + \| u \|_{L^6}^3 \right)\| u_t \|_{L^6}\| \nabla u \|_{L^6}^2
\leq Q \left( \| \nabla u_t \|^2 + \| u \|_{H^2(\Omega)}^4 \right).
\]
We now notice that, thanks again to (4.18), the following relation holds for \( c_g > 0 \):
\[
\frac{d}{dt} \left( \| u \|^2 + 3\lambda \| \nabla u \|^2 + c_g \right) \leq Q + \| u_t \|^2.
\]
Adding it to (4.26), it is clear that, if we denote by \( \Sigma \) the sum of the terms in square brackets on the left-hand sides of (4.26) and (4.28) and suitably choose \( c_g > 0 \), we obtain a relation of the form
\[
\frac{d}{dt} \Sigma \leq Q(1 + \Sigma^2) + m,
\]
where
\[
\Sigma \geq c \| u \|_{H^2(\Omega)}^2 \quad \text{and} \quad m = Q(\| u_t \|_{V'}^2 + \| \nabla z \|^2).
\]
Let us notice that the integral of $m$ over $(0, +\infty)$ is controlled by a quantity $Q$, thanks to (4.21).

By the comparison principle, we then have

$$u \in L^\infty(0, T_0; H^2(\Omega)), \quad \|u\|_{L^\infty(0, T_0; H^2(\Omega))} \leq Q,$$

where $T_0 > 0$ is a decreasing function depending on $\Sigma(0)$ (and, in fact, on the $H^2$-norm of $u_0$). Of course, due to the quadratic growth of the right-hand side of (4.29), a global in time estimate cannot be obtained so far.

As a next step, let us observe that, by (4.21), there exists a time $\tau_0 \in (0, T_0)$ such that

$$\|u_t(\tau_0)\|^2 \leq Q.$$  \hspace{1cm} (4.32)

Indeed, here, the right-hand side is just the quotient of the function $Q$ in (4.21) by the value of $T_0$.

Now, let us test (2.20) and (2.37) and integrate by (4.20) and (4.32),

$$\mathcal{G}(\tau_0) \leq c(1 + \|u(\tau_0)\|_{L^4(\Omega)}^2 \|u_t(\tau_0)\|_{L^6(\Omega)}^2 + \|\nabla u_t(\tau_0)\|^2) \leq Q.$$  \hspace{1cm} (4.35)

Thus, integrating (4.33) over $(\tau_0, t)$ for a generic $t > \tau_0$ and using the dissipation property (4.21) and the Gronwall lemma, we have

$$\|u_t(t)\|^2 \leq Q \quad \forall t \geq \tau_0.$$  \hspace{1cm} (4.36)

Now, let us test (2.37) by $(Bu + W'(u))_t$. This readily gives

$$\frac{d}{dt} \left[ \frac{1}{2} \|Bu + W'(u)\|^2 - \langle g, Bu + W'(u) \rangle \right] + \int_{\Omega} \alpha'(u_t)|\nabla u_t|^2 + \int_{\Omega} \alpha(u_t)W''(u)u_t$$

$$\leq \frac{1}{2}\|\nabla z\|^2 + \frac{1}{2}\|\nabla u_t\|^2 + \int_{\Omega} W''(u)u_tz.$$  \hspace{1cm} (4.37)

Our next task is to control the latter integral. To this aim, we test (2.20) by $\mathcal{N}(W''(u)u_t - (W''(u)u_t)_{\Omega})$, thus obtaining

$$\int_{\Omega} W''(u)u_tz = \int_{\Omega} \langle W''(u)u_t \rangle z - \langle u_t, \mathcal{N}(W''(u)u_t - (W''(u)u_t)_{\Omega}) \rangle.$$  \hspace{1cm} (4.38)

Now, the latter term is readily controlled, noting that

$$\|u_t, \mathcal{N}(W''(u)u_t - (W''(u)u_t)_{\Omega})\| \leq c \|u_t\|_H \|W''(u)u_t\|_{L^2(\Omega)} \leq Q \|u_t\|_H \|u_t\|_V,$$

by the growth assumption (2.40) and standard interpolation inequalities and embeddings.
Next, we deal with the first term on the right-hand side of (4.38), which can be rewritten, by comparison with (2.37) (let us assume, for simplicity, that $|\Omega| = 1$), as
\[
\int_{\Omega} \left( W''(u) u_t \right) \Delta u = \left( \frac{d}{dt} \int_{\Omega} W'(u) \right) \int_{\Omega} u_t \Delta u
\]
\[= -\frac{d}{dt} \left[ \int_{\Omega} W'(u) \int_{\Omega} g \right] + \frac{1}{2} \frac{d}{dt} \left[ \int_{\Omega} W'(u) \right]^2 + \frac{d}{dt} \left[ \int_{\Omega} W'(u) \right] \int_{\Omega} \alpha(u_t). \quad (4.40)
\]
We will see later that the first two terms on the right-hand side can be easily controlled. Let us focus on the third one. We have
\[
\left| \frac{d}{dt} \left( \int_{\Omega} W'(u) \right) \int_{\Omega} \alpha(u_t) \right| \leq \|W''(u)\|_{L^{6/5}} \|u_t\|_V \left( \int_{\{|u| \leq 1\}} \alpha(u_t) \right) \int_{\{|u| \leq 1\}} \alpha(u_t) + \int_{\{|u| > 1\}} \alpha(u_t)
\]
\[\leq Q \|u_t\|_V \left( \int_{\{|u| \leq 1\}} \alpha(u_t) \right) + \int_{\Omega} \alpha(u_t) u_t
\]
\[\leq Q \|u_t\|^2_V + \int_{\Omega} \alpha(u_t) u_t, \quad (4.41)
\]
where, to derive the second inequality, we have used (2.40) and (4.18); as far as the third inequality is concerned, we have used (4.36), the condition $\alpha(0) = 0$ and the trivial fact that $|\alpha(u_t)| \leq c$ whenever $|u_t| \leq 1$, and, finally, the fourth one follows from standard embeddings.

At this point, we come back to (4.37) and denote by $\mathcal{P}$ the difference between the terms in square brackets in (4.37) and the ones in (4.40), namely,
\[
\mathcal{P} = \frac{1}{2} \|Bu + W'(u)\|^2 - (g, Bu + W'(u)) + \int_{\Omega} W'(u) \int_{\Omega} g - \frac{1}{2} \left( \int_{\Omega} W'(u) \right)^2. \quad (4.42)
\]
Then, using (4.38)-(4.41), (2.9), and the monotonicity of $\alpha$, we infer
\[
\frac{d}{dt} \mathcal{P} \leq \frac{1}{2} \|\nabla z\|^2 + \frac{1}{2} \|\nabla u_t\|^2 + \lambda \int_{\Omega} \alpha(u_t) u_t
\]
\[+ Q \|u_t\|_H \|u_t\|_V + Q \left( \|u_t\|^2_V + \int_{\Omega} \alpha(u_t) u_t \right), \quad (4.43)
\]
where it is clear that, for some $c_6, c_7 > 0$,
\[
c_6 \|u_t\|^2_{H^2(\Omega)} - Q \leq \mathcal{P} \leq c_7 \|u_t\|^2_{H^2(\Omega)} + Q. \quad (4.44)
\]
Now, we notice that the right-hand side of (4.43) is summable over $(\tau_0, +\infty)$, thanks to (4.21). Integrating it over $(\tau_0, t)$ for a generic $t > \tau_0$ and using (4.31), we then finally obtain
\[
u \in L^\infty(0, +\infty; H^2(\Omega)), \quad \|u\|_{L^\infty(0, +\infty; H^2(\Omega))} \leq Q, \quad (4.45)
\]
i.e., $u$ never exits from $W$, as desired.

At this point, the proof of the theorem is readily completed by noting that the trajectories are eventually bounded in $W$ and, hence, weakly compact in the $H^2$-metric. Since the set of stationary states is also bounded in $W$, the proof follows by applying, e.g., [30, Thm. 5].

**Remark 4.2.** Assuming that $g \in V$ and recalling Remark 4.1, we would conclude that $\mathcal{A}$ is bounded in $H^3(\Omega)$, provided that we can prove that any solution starting
from $u_0 \in \mathcal{W}$ is eventually bounded in $H^3(\Omega)$. This could probably be done by rewriting (2.37)-(2.38) in the form of the second-order (in space) equation
\begin{equation}
\alpha(u_t) + Bu + W'(u) = G,
\end{equation}
where $G := g + z \in L^\infty(0, +\infty; V)$, and suitably adapting the Alikakos-Moser iteration argument in [34, Thm. 3.5] in order to prove that, eventually, $u_t$ (and, hence, also $\alpha(u_t)$ and $\alpha'(u_t)$, at least if $\alpha$ is a bit smoother) lies in $L^\infty(\Omega)$. However, the procedure might not be straightforward because here the conditions on the “driving force” $G$ are different from those in [34]. Notice also that this argument should in fact be performed on some approximated or regularized system.

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