A Cahn–Hilliard model in a domain with non-permeable walls

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A B S T R A C T

We consider in this article a Cahn–Hilliard model in a bounded domain with non-permeable walls, characterized by dynamic-type boundary conditions. Dynamic boundary conditions for the Cahn–Hilliard system have recently been proposed by physicists in order to account for the interactions with the walls in confined systems and are obtained by writing that the total bulk mass is conserved and that there is a relaxation dynamics on the boundary. However, in the case of non-permeable walls, one should also expect some mass on the boundary. It thus seems more realistic to assume that the total mass, in the bulk and on the boundary, is conserved, which leads to boundary conditions of a different type. For the resulting mathematical model, we prove the existence and uniqueness of weak solutions and study their asymptotic behavior as time goes to infinity.

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1. Introduction

We consider in this article the following Cahn–Hilliard system describing the evolution of the order parameter of a binary material associated with a particular class of dynamic boundary conditions:

\begin{align*}
\rho_t - \Delta \mu &= 0, \quad \text{in } \Omega, \quad (1.1) \\
\mu &= -\Delta \rho + f(\rho), \quad \text{in } \Omega, \quad (1.2) \\
w \rho_t - \delta \Delta f(\mu) &= -\partial_n \mu, \quad \text{on } \Gamma, \quad (1.3) \\
w \mu &= -\sigma \Delta f(\rho) + g(\rho) + \partial_n \rho, \quad \text{on } \Gamma. \quad (1.4)
\end{align*}

A physical derivation of this model, based on a variational principle, is detailed in the next section. Here, \( \Omega \subset \mathbb{R}^3 \) is a smooth and bounded domain with boundary \( \Gamma \) corresponding to the binary material under consideration. Moreover, \( \delta \) and \( \sigma \) are nonnegative parameters (which can vanish, not necessarily simultaneously, in the absence of boundary diffusion), \( w \) is a bounded and nondegenerate weight function, \( \Delta_f \) is the Laplace–Beltrami operator, and the nonlinear function \( f \) is the derivative of a \( \lambda \)-convex and possibly singular “bulk” potential \( F \). Finally, \( g \) is a smooth function with controlled growth at infinity. The terminology “singular potential” (cf. Remark 3.1 below) refers to the fact that \( F \) is allowed to take identically the value \( +\infty \) outside a bounded interval \( I \subset \mathbb{R} \), while \( \lambda \)-convex means convex up to a quadratic perturbation.

The Cahn–Hilliard system describes important qualitative features of two-phase systems related to phase separation processes. This phenomenon can be observed, e.g., when a binary alloy is cooled down sufficiently. One then observes a partial nucleation (i.e., the appearance of nucleides in the material) or a total nucleation, the so-called spinodal decomposition: the material quickly becomes inhomogeneous, forming a fine-grained structure in which each of the two components appears more or less alternatively. In a second stage, which is called coarsening and which occurs at a slower time scale and is less understood, these microstructures coarsen. Such phenomena play an essential role in the mechanical properties of the material, e.g., strength. We refer the reader to, e.g., [1–3] for more details.

Dynamic boundary conditions have recently been proposed by physicists, in the context of the Cahn–Hilliard equation, in order to account for the interactions with the walls in confined systems (cf. [4–6] and the references therein). In particular, such boundary conditions have mainly been studied for polymer mixtures (although this should also be important in other systems, such as binary metallic alloys): from a technological point of view, binary polymer mixtures are particularly interesting, since the structures occurring during the phase separation process may be frozen by a rapid quench into the glassy state; microstructures at surfaces on very small length scales can be produced in this way.
More precisely, the following boundary conditions have been proposed:

\[ \partial_{n\mu} u = 0, \text{ on } \Gamma, \]
\[ \rho_1 - \sigma \Delta_{\Gamma} \rho + g(\rho) + \partial_\mu \rho = 0, \text{ on } \Gamma, \sigma > 0, \]

the second boundary condition being called a dynamic boundary condition, in the sense that the kinetics, i.e., \( \rho_1 \), appears explicitly. The Cahn–Hilliard system, endowed with these boundary conditions, has been studied in [7–13] (see also [14–20] for similar boundary conditions for the Caginalp phase-field system).

Now, the first boundary condition, namely, \( \partial_{n\mu} u = 0, \text{ on } \Gamma \), means that there is no mass flux on the boundary (the second one being obtained by assuming a relaxation dynamics on the boundary and yielding that the total free energy decreases) and implies, integrating (formally) (1.1) over \( \Omega \), the conservation of the total mass in \( \Omega \). However, it also seems reasonable, to fully account for the interactions of the material’s components with the walls, to actually write that the total mass, in \( \Omega \) and on \( \Gamma \), is conserved; indeed, it seems reasonable to assume that there is also some mass on the boundary. This leads (see the next section for details) to the boundary conditions (1.3)–(1.4).

Our first aim in the present article is to give a physical derivation of system (1.1)–(1.4) on the basis of a variational principle. Actually, similar boundary conditions (with \( \delta = 0 \), \( w \) constant and \( g \) linear) have been derived in [21], though in a different context, namely, in the case of permeable walls, and studied in [21–24] for a regular nonlinear term \( f \).

Then, we analyze, from a mathematical point of view, the well-posedness issue for this model and try to deal with the most general conditions on the nonlinear terms. In particular, in order to treat singular functions \( f \), we are forced to set the problem in a suitable weak framework by considering duality techniques in negative order Sobolev spaces. Indeed, it is expected here that, when the bulk potential \( F \) is bounded (\( F' = f \)), we can have nonexistence of classical (i.e., in the sense of distributions) solutions. This can be seen, as in [11] for the usual dynamic boundary conditions, by considering the scalar ODE

\[ y'' - f(y) = 0, \quad x \in (-1, 1), \quad y'(\pm 1) = C > 0, \]

for large \( C \)'s (which corresponds to the one-dimensional stationary problem with \( \mu = 0 \) and \( g \equiv -C \); note that, in one space dimension, the Laplace–Beltrami operator does not make sense and does not appear in the boundary conditions). This type of approach, based on duality, has been proved to be effective for other similar models (see, e.g., [20,25]; see also [11] for a different, though related, notion of a weak solution, based on variational inequalities).

Once the well-posedness of the system has been established in the proper weak setting, we discuss further properties of the solutions, such as parabolic regularization effects, dissipativity, and long-time behavior. In particular, we show that, if the functions \( f \) and \( g \) satisfy suitable growth conditions (which, in particular, restrict the class of admissible functions \( f \) in the singular case), then the solutions satisfy instantaneous regularization properties and can be intended in a stronger sense (i.e., pointwise) for all strictly positive times (for all times if the initial datum is also more regular). We also prove the existence of a compact global attractor for the dynamical process generated by the system. Finally, we show the existence of \( \omega \)-limit sets of solution trajectories and prove that they only consist of stationary states. We also prove that, if \( f \) and \( g \) are analytic functions and some other technical conditions hold, then the \( \omega \)-limit set of any trajectory consists of one single point. This result is shown by appealing to the well-known Łojasiewicz–Simon method (cf., e.g., [7]). In the case of a singular function \( f \), the key point in the proof consists in showing that any solution is, at least for large times, uniformly separated from the singular points of \( f \) (i.e., from the endpoints of \( I \)).

This article is organized as follows. In Section 2, we give a derivation of (1.3)–(1.4). Then, in Section 3, we give our main assumptions and state and prove our results.

2. Derivation of the model

First, we set \( H := L^2(\Omega) \) and denote by \( \langle \cdot, \cdot \rangle \) the scalar product both in \( H \) and in \( H^2 \) and by \( \| \cdot \| \) the related norm. Moreover, we set \( V := H^1(\Omega) \) and denote by \( V' \) the (topological) dual of \( V \). The duality between \( V' \) and \( V \) is indicated by \( \langle \cdot, \cdot \rangle \).

We also set \( H_f := L^2(\Gamma) \) and \( V_f := H^1(\Gamma) \) and denote by \( \langle \cdot, \cdot \rangle_f \) the scalar product in \( H_f \), by \( \| \cdot \|_f \) the corresponding norm, and by \( \langle \cdot, \cdot \rangle_{\Gamma} \) the duality between \( V_f \) and \( V_f \). In general, \( \| \cdot \|_X \) indicates the norm in the generic (real) Banach space \( X \) and \( \langle \cdot, \cdot \rangle_X \) stands for the duality between \( X' \) and \( X \).

We can then define the Hilbert spaces

\[ \mathcal{H} := H \times H_f \quad \text{and} \quad \mathcal{V} := \{ z \in V : z|_{\Gamma} \in V_f \}, \quad (2.1) \]

endowed with the natural scalar products and norms. Unless otherwise specified, in what follows, we will impose the following convention: when we write \( h \in \mathcal{H} \), \( h \) will be interpreted as a pair of functions belonging, respectively, to \( H \) and to \( H_f \) and both denoted by the same letter \( h \). Analogously, when we consider \( v \in \mathcal{V} \) (or even \( v \in V \)), the symbol \( v \) will be intended, depending on the context, either as a function defined in \( \Omega \), or as a pair formed by a function in \( \Omega \) and its trace on \( \Gamma \).

We will also need some weighted spaces. We take \( w \in L^\infty(\Gamma) \), \( 0 < w_* \leq w(x) \leq w^* \) for a.e. \( x \in \Gamma \), (2.2) where \( w_*, w^* \) are given constants. Then, we introduce, on \( \overline{\Omega} = \Omega \cup \Gamma \), the measure \( dm \) given by

\[ \int_{\overline{\Omega}} v \, dm := \int_{\Omega} v \, dx + \int_{\Gamma} v \, w \, d\Sigma, \quad (2.3) \]

where \( v \) represents a generic function in \( L^1(\Omega) \times L^1(\Gamma) \). We also define

\[ m(v) := \left( \int_{\Omega} v \, dx + \int_{\Gamma} v \, w \, d\Sigma \right) \left( |\Omega| + \int_{\Gamma} w \, d\Sigma \right)^{-1}, \quad (2.4) \]

i.e., the average of \( v \) with respect to the measure \( dm \).

Thanks to (2.2), \( dm \) is equivalent to the Lebesgue measure \( dx \otimes d\Sigma \). In particular, the scalar product

\[ \langle h, k \rangle_m := \int_{\overline{\Omega}} h k \, dm = \int_{\Omega} h k \, dx + \int_{\Gamma} h k w \, d\Sigma \quad (2.5) \]

generates on \( \mathcal{H} \) a norm \( \| \cdot \|_m \) which is equivalent to the standard norm \( \| \cdot \|_H \).

We can now provide a mathematically accurate derivation of system (1.1)–(1.4), starting from mechanical principles.

First, we assume the validity of Eq. (1.1) which describes the balance of mass in \( \Omega \). Indeed, integrating it over \( \Omega \), we have

\[ \frac{d}{dt} \int_{\Omega} \rho \, dx = \int_{\Gamma} \partial_n \mu \, d\Sigma \quad (2.6) \]

and we have to specify the evolution of \( \mu \). To this end, we define the free energy of the system as

\[ \mathcal{E}(\rho) := \int_{\Omega} \left( \frac{\| \nabla \rho \|^2}{2} + F(\rho) \right) \, dx + \int_{\Gamma} \left( \frac{\sigma}{2} |\nabla_{\Gamma} \rho|^2 + G(\rho) \right) \, d\Sigma, \quad (2.7) \]
i.e., the sum of the Ginzburg–Landau (bulk) free energy and of a surface free energy, where $\nabla_{\text{f}}$ is the tangential gradient on $\Gamma$ and $\sigma$ may vanish, due to a possible lack of boundary diffusive effects.

Here, $F$ and $G$ are suitable antiderivatives of the functions $f$ and $g$ in (1.2) and (1.4).

We then assume that, a.e. in the reference time interval $(0, T)$,

$$
\mu = \partial_{m}E(\rho),
$$

(2.8)

$\partial_{m}$ denoting the subdifferential in the space $\mathcal{H}$ with respect to the scalar product (2.5). As we will see below, by introducing the weight $m$, we are able to consider the situation when the “boundary mass” is linked to the variable $\rho$ in a different way with respect to the “bulk mass”. For instance, one can think of the case when the dynamic boundary conditions arise as an approximation of a thin diffusive layer occupied by a different material, as in the so-called “concentrated capacity” models; see, e.g., [26–29].

It is then not difficult to check that (2.8) can be rephrased as the system

$$
\mu = -\Delta\rho + f(\rho), \quad \text{in } \Omega,
$$

(2.9)

$$
\frac{\partial}{\partial t} \mu = \frac{1}{w}(\sigma \Delta_{f} \rho + g(\rho) + \partial_{m}(\rho)), \quad \text{on } \Gamma,
$$

(2.10)

where $\Delta_{f}$ is the Laplace–Beltrami operator. Actually, to see that (2.9)–(2.10) is equivalent to (2.8), it is sufficient to take a sufﬁciently regular test function $z$, multiply (2.9) by $z - \rho$ and (2.10) by $w(z - \rho)$, and integrate (i.e., we test (2.9)–(2.10) with the test function $z - \rho$ for the scalar product (2.5)). Integrations by parts then lead to

$$
(\mu, z - \rho)_{m} \leq \mathcal{E}(z) - \mathcal{E}(\rho),
$$

(2.11)

which coincides with (2.8) by the definition of subdifferentials.

Of course, coupling (1.1) with (2.9)–(2.10) does not give a closed system, since a boundary condition for $\mu$ is still missing (while (2.10) or, equivalently, (1.4) provides, in fact, a boundary condition for $\mu$).

The main novelty of the present approach is to consider, as mentioned in the introduction, the class of those boundary conditions which ensure the conservation of the total (i.e., bulk plus boundary) mass. More precisely, we ask that

$$
\frac{d}{dt} \int_{\Omega} \rho \, dm = \frac{d}{dt} \left( \int_{\Omega} \rho \, dx + \int_{\Gamma} \rho \, w \, d\Sigma \right) = 0.
$$

(2.12)

This is in contrast with what happens with the usual Cahn–Hilliard models for which only the bulk mass is conserved. However, in the framework of dynamic boundary problems, it seems reasonable to assume that the boundary gives a non-negligible contribution to diffusion. Hence, (2.12) may appear as a more realistic condition.

Then, using (2.6), (2.12) yields the compatibility condition

$$
\int_{\Gamma} (\rho_{t} w + \partial_{m}h(t)) \, d\Sigma = 0.
$$

(2.13)

A class of boundary conditions guaranteeing the validity of (2.13) (and, hence, of (2.12)) is given by

$$
\rho_{t} + \frac{1}{w} \left( -\delta \Delta_{f} \rho + \mu + \partial_{m}h(t) \right) = 0, \quad \text{on } \Gamma,
$$

(2.14)

where $\delta \geq 0$. Of course, (2.14) can be rephrased as the equivalent form (1.3).

**Remark 2.1.** In [21], as mentioned in the introduction, similar boundary conditions are derived in the context of permeable walls. Compared with our derivation, (2.14) (with $w \equiv \text{Const.}$ and $\delta = 0$; such a boundary condition is then called a Wenzell boundary condition) is somehow assumed and yields the conservation of mass (2.12), whereas, here, (2.8) (and, thus, (2.10)) is assumed.

### 3. Weak formulation and main results

We introduce here our main assumptions, together with a number of tools which are needed in order to reformulate system (1.1)–(1.4) in a mathematically precise way and give a rigorous statement of our results.

We first need some further discussion on the functional spaces. Actually, we notice that we have the chain of continuous embeddings

$$
\mathcal{V} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V} \subset \mathcal{V}.
$$

(3.1)

More precisely, identifying $\mathcal{H}$ with its dual through the scalar product (2.5), we obtain the Hilbert triplets $(\mathcal{V}, \mathcal{H}, \mathcal{V})$ and $(\mathcal{V}, \mathcal{H}, \mathcal{V})$. Indeed, it is not difficult to prove that the space $\mathcal{V}$ is dense in $\mathcal{H}$.

For later convenience, we set, for $\eta \geq 0$,

$$
\mathcal{V}_{\eta} := \mathcal{V} \quad \text{if } \eta > 0 \quad \text{and} \quad \mathcal{V}_{0} := \mathcal{V} \quad \text{if } \eta = 0
$$

(3.2)

(in applications, $\eta = \delta$ or $\alpha$). In particular, the norm $\| \cdot \|_{\mathcal{V}_{\eta}}$ coincides with the norm of $\mathcal{V}$ for $\eta = 0$ and with that of $\mathcal{V}$ for $\eta > 0$.

We also define the elliptic operators

$$
A : \mathcal{V} \rightarrow \mathcal{V}', \quad (A_{\eta}z_{1}, z_{2}) := \int_{\Omega} \nabla z_{1} \cdot \nabla z_{2},
$$

(3.3)

$$
A_{f} : \Gamma \rightarrow \Gamma', \quad (A_{f}z_{1}, z_{2})_{\Gamma} := \int_{\Gamma} \nabla_{f} z_{1} \cdot \nabla_{f} z_{2},
$$

(3.4)

$$
A : \mathcal{V} \rightarrow \mathcal{V}', \quad (A_{\eta}z_{1}, z_{2})_{\mathcal{V}} := (A_{\eta} z_{1}, z_{2}) + (A_{f} z_{1}, z_{2})_{\Gamma},
$$

(3.5)

$$
A_{\eta} : \mathcal{V}_{\eta} \rightarrow \mathcal{V}_{\eta}', \quad (A_{\eta} z_{1}, z_{2})_{\mathcal{V}_{\eta}} := (A_{\eta} z_{1}, z_{2}) + \eta (A_{f} z_{1}, z_{2})_{\Gamma}.
$$

(3.6)

Let us now present our basic hypotheses on the nonlinear terms. We assume that $I$ is an open, possibly bounded, interval of $\mathbb{R}$ containing $0$ and that

$$
\begin{align*}
&f \in C^{0,1}_{\text{loc}}(I, \mathbb{R}), \quad g \in C^{0,1}_{\text{loc}}(\mathbb{R}, \mathbb{R}), \quad f(0) = g(0) = 0, \quad (\text{fg1}) \\
\text{Moreover, we assume that} & \\
&\lim_{r \rightarrow 0} f(r) \text{sign } r = \lim_{r \rightarrow +\infty} g(r) \text{sign } r = +\infty, \quad (\text{fg2}) \\
&f'(r), g'(r) \geq -\lambda, \quad \text{for some } \lambda \geq 0, \quad (\text{fg3})
\end{align*}
$$

and for a.e. $r \in I$ and in $\mathbb{R}$, respectively. Then, we define, whenever they make sense, the potentials

$$
F(r) := \int_{0}^{r} f(s) \, ds, \quad G(r) := \int_{0}^{r} g(s) \, ds
$$

(3.7)

and it follows from (fg2) that both $F$ and $G$ are bounded from below. If $I \neq \mathbb{R}$, $F$ is extended by continuity to the closure $\bar{I}$ and then extended by $+\infty$ outside $\bar{I}$.

**Remark 3.1.** Actually, we will speak of singular potentials $F$ in the case when $I$ is a bounded interval. A physically relevant case is given by the so-called logarithmic potential

$$
F(r) = (1 + r) \ln(1 + r) - (1 - r) \ln(1 - r) - \lambda r^{2}, \quad \lambda \geq 0,
$$

(3.8)

where $I = (-1, 1)$, of course. The recent literature on Cahn–Hilliard problems (see, e.g., [30, 11]) shows that singular potentials are considerably more difficult to deal with than “regular” ones, especially in connection with dynamic boundary conditions.

To deduce a weak formulation of (1.1)–(1.4), we multiply (1.2) and (1.4) by a test function $\phi \in \mathcal{V}_{\eta}$ and (1.1) and (1.3) by $\psi \in \mathcal{V}_{\eta}$ and then integrate. This leads to the relations...
\[
\int_{\Omega} \mu \phi + \int_{\Gamma} \mu \phi w = \int_{\Omega} \nabla \rho \cdot \nabla \phi + \int_{\Omega} f(\rho)\phi \\
+ \sigma \int_{\Gamma} \nabla \Gamma \rho \cdot \nabla \phi + \int_{\Gamma} g(\rho)\phi.
\]  
(3.9)

\[
\int_{\Omega} \rho_1 \psi + \int_{\Gamma} \rho_1 \psi w = - \int_{\Omega} \nabla \mu \cdot \nabla \psi \\
- \delta \int_{\Gamma} \nabla \Gamma \mu \cdot \nabla \psi.
\]  
(3.10)

We also set
\[
F : \mathcal{H} \to \mathcal{H}, \quad F(\nu) := f(\nu), \quad \text{in } \Omega;
\]
\[
F(\nu) := \frac{g(\nu)}{w}, \quad \text{on } \Gamma;
\]
\[
B : \mathcal{H} \to \mathcal{H}, \quad B(\nu) := f(\nu) + \lambda \nu, \quad \text{in } \Omega;
\]
\[
B(\nu) := \frac{g(\nu) + \lambda \nu}{w}, \quad \text{on } \Gamma.
\]
(3.11)

Actually, the operator \(F\) is seen as a linear perturbation of the maximal monotone operator \(B\) acting from \(H\) to itself. Notice the occurrence of the function \(w\) in the denominator, due to the fact that \(H\) is identified with \(H^r\) through the scalar product \((\cdot, \cdot)\), in relation with the way the Hilbert triplet \(((V, H, V')^r)\) is built. Of course, the effective domains of \(B\) and \(F\) are in general strict subsets of \(H\).

Using (3.3)–(3.6) and (3.11), system (3.9)–(3.10) can then be rewritten in the more compact form
\[
\rho_t = -A_{\lambda} \mu, \quad \text{in } V'_s,
\]
(3.13)
\[
\mu = A_{\rho} \rho + F(\rho), \quad \text{in } V'_s.
\]
(3.14)

However, under general conditions on the functions \(f, g\), it is very difficult, if not impossible, to prove an existence result for system (3.13)–(3.14) in its present form. More precisely, due to the dynamic boundary conditions, an \(H\)-uniform bound on the term \(F(\rho)\) is generally out of reach. One way to overcome this difficulty is to introduce a weaker notion of a solution, by suitably relaxing the functional \(F\) (a different, albeit closely related, approach has been proposed in [11]).

To this end, we first define the convex (cf. (fg3)) functional
\[
J : \mathcal{H} \to \mathbb{R} \cup [+\infty],
\]
\[
J(\rho) := \int_{\Omega} \left( F(\rho_1) + \frac{\lambda}{2} \rho_1^2 \right) + \int_{\Omega} (G(\rho_2) + \frac{\lambda}{2} \rho_2^2).
\]
(3.15)

where, here, \( \rho = (\rho_1, \rho_2) \). Furthermore, for \( \rho = (\rho_1, \rho_2) \in \mathcal{H} \), we say that \( \rho \in D(J) \) if and only if \( \rho_1 \in L^2(\Omega, \rho_2 \in L^2(\Gamma), F(\rho_1) \in L^1(\Omega) \) and \( G(\rho_2) \in L^1(\Gamma) \). If \( \rho \in \mathcal{H} \setminus D(J) \), then we set \( J(\rho) = +\infty \). Then, \( J \) is proper, convex, and lower semicontinuous on \( \mathcal{H} \). Notice that, for \( \rho, \kappa \in D(J) \), it holds that
\[
\kappa = (k_1, k_2) \in \partial_{H} J(\rho) \iff k_1 = f(\rho_1) + \lambda \rho_1, \quad \text{a.e. in } \Omega
\]
and
\[
k_2 = \frac{g(\rho_2) + \lambda \rho_2}{w}, \quad \text{a.e. on } \Gamma.
\]
(3.16)

In other words, \( B \) coincides with the \( H \)-subdifferential of \( J \) with respect to the scalar product (2.5). Note that, if \( \kappa = (k_1, k_2) \in V'_s \), then \( k_1 = k_1|_\Omega \) and \( k_2 = k_1|_\Gamma \). Then, since \( V'_s \subset \mathcal{H} \) for any \( \sigma \geq 0 \), we can consider the restriction of \( J \) to \( V'_s \) and denote by \( B_{\omega} \) (i.e., the “weak” form of \( B \)) its subdifferential with respect to the duality between \( V_s \) and \( V'_s \). More precisely, for \( \rho \in V_s \) and \( h \in V'_s \), we have
\[
h \in \mathcal{B}_{\omega}(\rho) \iff (h, z - \rho)_V \leq J(\rho) - J(\rho) \quad \forall z \in V_s.
\]
(3.17)

It is not difficult to prove that, considering the Hilbert triplet \( V_s \subset \mathcal{H} \subset V'_s \), graph(\( B \) \( \cap (V_s \times V'_s) \) is included in graph(\( B_{\omega} \)). In other words,
\[ \text{dist}_X(\rho_1, \rho_2) := \| \rho_1 - \rho_2 \|_{L^1} + \| F(\rho_1) - F(\rho_2) \|_{L^1(\Omega)} + \| G(\rho_1) - G(\rho_2) \|_{L^1(I)} \]  

\text{(3.29)}

It is then not difficult to prove (cf., e.g., [34, Lemma 3.8]) that \((X, \text{dist}_X)\) is a complete metric space. To account for the conservation of the spatial average (cf. (2.12) and (3.21)), given \(m_0 \in I = \text{dom}(f)\), we also set

\[ X_0 := \{ \rho \in X : m(\rho) = m_0 \}. \]  

\text{(3.30)}

**Theorem 3.5.** Let the assumptions of Theorem 3.2 hold. In particular, let us assume that, for a given \(m_0\) as above, \(\rho_0 \in X_0\) and let us set \(E_0 := E(\rho_0)\). Then, for all \(t \geq 0\), \(\rho(t) \in X_0\). Moreover, there exists a number \(R_0\) which is independent of the initial datum (but may depend on \(m_0\)) and a time \(T_0\) depending only on \(E_0\) such that \(E(\rho(t)) \leq R_0\) for all \(t \geq T_0\). In other words, the set \(B_0 := \{ \rho \in X_0 : E(\rho) \leq R_0 \}\) is an absorbing set for the dynamical system \(S(\cdot)\) associated with (the \(\rho\)-components of) the solutions of system (3.24)-(3.25).

As a next step, without assuming any further property on the initial datum, we show that any weak solution satisfies some instantaneous regularization effect.

**Theorem 3.6.** Let the assumptions of Theorem 3.5 hold and let, as above, \(E_0 := E(\rho_0)\). Let \((\rho, \mu)\) be a weak solution. Then, for any \(\tau > 0\), the following properties hold:

\[ \| \rho \|_{L^2(1+\tau; V_0)} \leq Q(\tau^{-1}, E_0) \text{ for all } \tau > 0, \]  

\text{(3.31)}

\[ \| \mu \|_{L^1(1+\tau; V_0 \Delta V_0)} \leq Q(\tau^{-1}, E_0), \]  

\text{(3.32)}

where \(Q\) is a computable nonnegative function and is monotone increasing with respect to each argument.

Restricting the class of admissible nonlinearities, we can show that properties (3.31)-(3.32) also entail further regularity for \(\rho\), both in \(\Omega\) and on \(\Gamma\).

**Theorem 3.7.** Let the assumptions of Theorem 3.5 hold and let, for a.e. \(t \in I\),

\[ f'(r) \leq c_I(1 + |f'(r)|^{\delta'}) \text{ for some } p_0 \in (0, 2) \text{ if } \delta > 0, \]  

\text{(3.33)}

\[ f'(r) \leq c_I(1 + |f'(r)|^{\delta'}) \text{ if } \delta = 0, \]  

\text{(3.34)}

for some constant \(c_I > 0\). Let us also assume that

\[ I = \mathbb{R}, \text{ then } \exists \delta > 0 : |g(r)| \leq c_I g(1 + |f'(r)|) \text{ for all } r \in \mathbb{R}. \]  

\text{(3.35)}

Let \((\rho, \mu)\) be a weak solution. Then, for any \(\tau > 0\),

\[ \| \rho \|_{L^2(1+\tau; H^1(\Omega))} + \frac{1}{2} \| \rho \|_{L^2(1+\tau; H^1(\Gamma))} \leq Q(\tau^{-1}, E_0), \]  

\text{(3.36)}

\[ \| \mu \|_{L^2(1+\tau; H^1(\Omega))} \leq Q(\tau^{-1}, E_0), \]  

\text{(3.37)}

where \(Q\) is again a computable nonnegative monotone function. Moreover,

\[ \| F(\rho) \|_{L^2(1+\tau; L^2(\Gamma))} \leq Q(\tau^{-1}, E_0), \]  

\text{(3.38)}

for some \(p_1 > 1\). Finally, for almost all \(t > 0\), the system can be rewritten in the “strong” form (1.1)-(1.4). More precisely, (1.1)-(1.2) hold as equalities in \(H\) and (1.3)-(1.4) are satisfied in \(H'\) for a.e. \(t > 0\).

**Remark 3.8.** Unfortunately, (3.33) (or (3.34)) is not satisfied by the physically relevant logarithmic potential (3.8).

**Remark 3.9.** In other problems characterized by singular terms and dynamic boundary conditions (cf., e.g., [8,20]), the growth assumption (3.33) (or (3.34)) can be replaced by sign conditions on \(g\) near the boundary of \(I\) (for instance, if \(I = (-1, 1)\), then \(g\) has to be nonnegative near 1 and nonpositive near \(-1\)). This does not seem to be possible here, due to the term \(w_\mu\) in (1.4) which may be unbounded with respect to the spatial variables and “kill” the sign conditions.

**Remark 3.10.** The compatibility assumption (3.35) simply states that \(g\) cannot grow strictly faster than \(f\), which is indeed true in all relevant cases. Actually, (3.35) could be strongly relaxed by paying the price of technical complications in the proof.

As a consequence of the above result, we easily obtain the existence of a compact absorbing set for the dynamical process \(S(\cdot)\). Moreover, thanks to the contractive estimate (3.27), one can see that \(S(\cdot)\) is a closed semigroup in the sense of Pata and Zelik [35]. Thus, by exploiting [35, Thm. 2], we immediately deduce:

**Corollary 3.11.** Let the assumptions of Theorem 3.7 hold. Then, the dynamical system \(S(\cdot)\) possesses a compact global attractor \(\mathcal{K}\). Moreover, there exists \(c_\mathcal{K} > 0\) such that

\[ \| \rho \|_{L^2(I; \Omega)} + \sigma(\rho) \| \mu(I; \Omega) \| + \| F(\rho) \|_{L^1(I; \Omega)} \leq c_\mathcal{K} \]  

\text{(3.39)}

for all \(\rho \in \mathcal{K}\).

**Remark 3.12.** Of course, (3.39) follows as a direct consequence of (3.36) and (3.38). It is also clear that it is not an optimal condition and that it could be further improved, depending on the regularity properties of \(f\) and \(g\).

In the next subsection, we will give the proofs of the results stated so far. The analysis and characterization of \(\omega\)-limit sets of single trajectories will then be presented separately in Section 3.3.

### 3.1. Proof of Theorem 3.2

We start by deriving several a priori estimates in order to establish the weak sequential stability of the solutions. For the sake of simplicity, we will perform these estimates by working directly, albeit formally, on the “strong” formulation (3.9)-(3.10) (or, equivalently, (3.13)–(3.14)) of the system. Actually, one should rather consider a proper approximation of system (1.1)-(1.4) (obtained, e.g., by suitably regularizing the functions \(f\) and \(g\) and the initial datum), prove that the approximate system possesses sufficiently smooth solutions, and then show that the family of these solutions satisfies the a priori estimates uniformly with respect to the approximation parameter. This has been done, in fairly complete detail, in the recent article [20], devoted to the Caginalp phase-field system with dynamic boundary conditions and singular potentials. Since the procedure required for the present system would be rather similar, we prefer to use here the formal approach and refer the reader to [20] for further details.

**The energy estimate.** Take \(\phi = \rho_1\) in (3.9) and \(\psi = \mu \in (3.10)\). We then easily find

\[ \frac{d}{dt} E(\rho) + \frac{1}{2} \| \nabla \rho \|_2^2 + \delta \| \nabla \rho \|_2^2 = 0, \]  

\text{(3.40)}

where the “energy” \(E\) was defined in (2.7). Using (fg2) to recover the full V-norm of \(\rho\), it is then easy to infer

\[ \| \rho \|_{L^2(0, T; V_0)} \leq c, \]  

\text{(3.41)}

\[ \| F(\rho) \|_{L^2(0, T; L^1(\Gamma))} + \| G(\rho) \|_{L^2(0, T; L^1(\Gamma))} \leq c, \]  

\text{(3.42)}

\[ \| \mu \|_{L^2(0, T; H^1)} + \delta^{3/2} \| \nabla \mu \|_{L^2(0, T; H^1)} \leq c. \]  

\text{(3.43)}

Here and below, \((\rho, \mu)\) should be considered as a sequence of “approximate” solutions and the letter \(c > 0\) denotes constants which are independent of the approximation parameters.
In fact, the $c$’s can depend here on $f, g$ and on the “magnitude” of the initial datum. We will use the letter $\kappa$ to denote constants used in estimates from below, with the same dependences as $c$. The constants $c, \kappa$ are not allowed to depend on the final time $T$. Specific constants will be denoted by $c_i, \kappa_i \geq 1$.

A further estimate. Taking $\gamma \equiv 1$ in (3.10), we observe that the $dm$-average $m(\rho) \equiv (\cdot, \cdot)_m$ is conserved in time, namely,

$$m(\rho(t)) = m(\rho_0) = m_0 \quad \forall t \geq 0.$$  \hfill (3.44)

Then, we can take $\phi := \rho - m(\rho)$ in (3.9) and obtain

$$\langle \mathcal{A}_n \rho, \rho \rangle + \langle F(\rho), \rho - m(\rho) \rangle_m = (\mu, \rho - m(\rho))_m,$$  \hfill (3.45)

where we recall that $(\cdot, \cdot)_m$ is the inner product on $\mathcal{H}$ with respect to the measure $dm$. Next, we observe that

$$(\mu, \rho - m(\rho))_m = (\mu - \mu_\Omega, \rho - m(\rho))_m$$

\leq \|\mu - \mu_\Omega\|_m \|\rho - m(\rho)\|_m$$

\leq c \|\mu - \mu_\Omega\|_m,$$  \hfill (3.46)

where $\mu_\Omega$ is the standard spatial average of $\mu$ in $\Omega$ and, to deduce the last inequality, we have used (3.41) and the conservation property (3.44). Moreover,

$$\|\mu - \mu_\Omega\|_m \leq c \|\mu - \mu_\Omega\|_V \leq c \|\nabla \mu\|,$$  \hfill (3.47)

thanks to the trace theorem and the Poincaré–Wirtinger inequality. Next, proceeding as in [30, Appendix], it is not difficult to prove that

$$\langle F(\rho), \rho - m(\rho) \rangle_m$$

\geq \kappa \left( \int_\Omega (|f(\rho)| + F(\rho)) + \int_\Omega (|g(\rho)| + G(\rho)) \right) - c,$$  \hfill (3.48)

where the constant $\kappa$ depends on $\rho$ only through the assigned value $m(\rho) = m_0$ (cf. (3.44)). Notice that condition (3.21) is also used here. Then, squaring (3.45) and using (3.43), we obtain

$$\|f(\rho)\|_{L^2(0, T; L^1(\Omega))} + \|g(\rho)\|_{L^2(0, T; L^1(\Gamma))} \leq c.$$  \hfill (3.49)

The estimate of $\mu$. Test (3.14) using 1. Performing standard integrations and squaring, we find

$$\int_0^T \left(\int_\Omega |\mu|^2 \right) \leq c \int_0^T \left(\int_\Omega |\mu|^2 + c \|f(\rho)\|^2_{L^2(0, T; L^1(\Omega))} + c \|g(\rho)\|^2_{L^2(0, T; L^1(\Gamma))} \right)$$

\leq c \int_0^T \left(\int_\Omega |\mu|^2 \right) + c,$$  \hfill (3.50)

thanks to (3.49). Then, integrating in time, adding the squared $L^2(0, T; H)$-norm of $\nabla \mu$ to both sides, and using (3.43), we infer

$$\int_0^T \left(\int_\Omega |\mu|^2 \right) + \|\nabla \mu\|^2_{L^2(0, T; H)} \leq 2 \int_0^T \left(\int_\Omega |\mu|^2 \right) + c.$$  \hfill (3.51)

Next, noting that the quantity on the left-hand side is an equivalent norm on $L^2(0, T; V)$ and using the trace theorem and standard interpolation inequalities, we have, for $\varepsilon \in (0, 1/2),$

$$\|\mu\|^2_{L^2(0, T; V)} \leq c \left(1 + \|\mu\|^2_{L^{(\frac{1}{2}, H; L^2(\Gamma))}} \right)$$

\leq c \left(1 + \|\mu\|^2_{L^{(\frac{1}{2}, H; L^2(\Gamma))}} \right)$$

\leq c + c \|\mu\|^2_{L^{(1/2, \frac{1}{2}, H; L^2(\Gamma))}} \|\mu\|^2_{L^{(1/2, H; L^2(\Gamma))}}$$

\leq c + c \|\mu\|^2_{L^{(1/2, \frac{1}{2}, H; L^2(\Gamma))}} \|\mu\|^2_{L^{(1/2, H; L^2(\Gamma))}}$$

\leq c + \frac{1}{2} \|\mu\|^2_{L^{(1/2, H; L^2(\Gamma))}} + c \|\mu\|^2_{L^{(1/2, H; L^2(\Gamma))}}.$$  \hfill (3.52)

Now, let us take

$$\phi \in H^2_0(\Omega) \subset V$$

in (3.9). Then, the $\Gamma$-components vanish, since $\phi$ has zero trace. Thus, using also the continuous embedding $H^2_0(\Omega) \subset \mathcal{L}^\infty(\Omega)$, we obtain

$$\|\langle \mu, \phi \rangle\|_{L^2(\Omega)} \leq c \|\nabla \rho\|_H + \|f(\rho)\|_{L^1(\Omega)} \|\phi\|_{L^2(\Omega)}.$$  \hfill (3.53)

Squaring, integrating over $(0, T)$, passing to the supremum with respect to $\phi$ with unit norm, and using (3.41) and (3.49), we then deduce that the last term on the right-hand side of (3.52) is controlled. Recalling also (3.43), we finally infer

$$\|\mu\|^2_{L^2(0, T; V)} \leq c.$$  \hfill (3.54)

Passing to the limit. In order to prove the existence of a weak solution, we exploit estimates (3.41)–(3.43), (3.49), and (3.54) to show that any sequence of approximate solutions (let us denote it by, say, $(\rho_n, \mu_n)$, where $n$ is the approximation parameter intended to go to $+\infty$) admits at least a subsequence which converges to a weak solution of our system. Of course, this part is again formal, since the approximation has not been specified. That said, what we obtain is the existence of a pair $(\rho, \mu)$ of a nonrelabeled subsequence of $n$ such that

$$\rho_n \to \rho \quad \text{weak star in } L^\infty(0, T; V_\sigma),$$

$$\mu_n \to \mu \quad \text{weakly in } L^2(0, T; V_\sigma).$$  \hfill (3.55)

In addition, using the continuity of the linear operator $\mathcal{A}_\sigma$ in (3.13), we see that

$$\rho_{n,t} \to \rho_t \quad \text{weakly in } L^2(0, T; V'_\sigma),$$  \hfill (3.56)

whence, recalling (3.55) and using the Aubin–Lions compactness lemma,

$$\rho_n \to \rho \quad \text{strongly in } C^0([0, T]; H^{1-\varepsilon}(\Omega))$$  \hfill (3.57)

for all $\varepsilon > 0$. In particular, we have the a.e. convergence of $\rho_n$ both in $\Omega$ and on $\Gamma$.

Next, by (3.55), (3.56), and the continuity of the linear operator $\mathcal{A}_\sigma$ in (3.14), we end up with

$$\mathcal{B}(\rho_n) \to \mathcal{B}(\rho) \quad \text{weakly in } L^2(0, T; V'_\sigma)$$  \hfill (3.59)

and we have to identify the limit function $\mathcal{B}$ in some way. To do this, we first notice that (3.55)–(3.56) and (3.59) are sufficient for taking the limit of (3.14) which reads

$$\mu = \mathcal{A}_\sigma \rho + \mathcal{B} - \lambda \mathcal{T}_\sigma(\rho).$$  \hfill (3.60)

Actually, it is immediate to take the limit of $\mathcal{T}_\sigma(\rho)$, since $\mathcal{T}_\sigma$ is linear and continuous. Then, we test (3.14) (written at the $n$-approximation level) using $\rho_n$ in the $V_\sigma$-duality and integrate. This gives

$$\int_0^T \langle \mathcal{B}(\rho_n), \rho_n \rangle = \int_0^T \langle \mathcal{B}(\rho_n), \rho_n \rangle$$

\quad = \int_0^T \langle \mu_n, \rho_n \rangle - \int_0^T \left(\|\nabla \rho_n\|^2 + \sigma \|\nabla \rho_n\|^2 \right)$$

\quad + \lambda \int_0^T \left(\|\rho_n\|^2 + \|\rho_n\|^2 \right),$$  \hfill (3.61)

where we have used the identification of $\mathcal{H}$ with $\mathcal{H}'$ through the scalar product (2.5). Then, taking the lim sup as $n \to +\infty$ of (3.61), using relations (3.55)–(3.59), and comparing with (3.60) tested by using $\rho$, it is not difficult to infer

$$\limsup_{n \to +\infty} \int_0^T \langle \mathcal{B}(\rho_n), \rho_n \rangle \leq \int_0^T \langle \mathcal{B}(\rho), \rho \rangle$$  \hfill (3.62)
whence, recalling (3.18) and using, e.g., [36, Prop. 1.1, p. 42], we obtain
\[ \mathcal{F} \in B_w(\rho) \quad \text{a.e. in } (0, T) \]  
(3.63)
and (3.60) actually coincides with (3.25). Observing that passing to the limit in (3.13) and the initial condition is immediate, thanks to (3.56)–(3.58), we have the validity of (3.24) and (3.26). Since (3.22)–(3.23) are obvious consequences of (3.55)–(3.59), this proves the proof of existence.

**Uniqueness.** We will prove a contractive estimate. Let us consider two initial data \( \rho_{0,1}, \rho_{0,2} \) both belonging to the space \( \mathcal{X}_0 \) (and, hence, having in particular the same \( \mu \)-average \( m_0 \)). Let \( (\rho_1, \mu_1), (\rho_2, \mu_2) \) be the solutions corresponding to these initial data and let us denote by \( (\rho, \mu) \) their difference. We take the difference of (3.24) [written for both solutions], integrate with respect to time, and test the resulting equation with \( \mu \) in the \( V_2 \)-duality. This gives
\[
(\rho, \mu)_m = -\frac{1}{2} \frac{d}{dt} \left( \int_\Omega |\nabla (1 + \mu)|^2 \right) + \delta \int_F |\nabla \Gamma (1 + \mu)|^2 - 2(\rho_0, 1 + \mu)_m \]
(3.64)
and, using the fact that \( m(\rho_0) = 0 \) and the Poincaré-Wirtinger inequality, we obtain
\[
2|(\rho_0, 1 + \mu)_m| \leq 2\|\rho_0\|_{V_2} \|\nabla (1 + \mu)\|_{\Omega} \leq c\|\rho_0\|_{V_2} + \frac{1}{2} \int_\Omega (1 + \mu) \|\nabla (1 + \mu)\|^2 + \frac{\delta}{2} \|\nabla \Gamma (1 + \mu)\|_{\Omega}^2. \]
(3.65)
Next, testing the difference of (3.25) (written, again, for both solutions) with \( \rho \) in the \( V_2 \)-duality and using the monotonicity of \( B_w \), we infer
\[
(\rho, \mu)_m \geq \|\nabla \rho\|^2 + \sigma \|\nabla \Gamma \rho\|_{\Omega}^2 - \lambda (\|\rho\|^2 + \|\rho\|_{\Omega}^2) \]
(3.66)
and we have to control the last term on the right-hand side. To do this, we first notice that, by standard trace theorems, for \( \epsilon \in (0, 1/2) \), it holds that
\[
\lambda (\|\rho\|^2 + \|\rho\|_{\Omega}^2) \leq c\|\rho\|^2 + c\|\rho\|_{\Omega}^2 \]
(3.67)
the latter inequality following from the chain of compact embeddings \( V \subset H^{1/2+\epsilon}(\Omega) \subset H^{-1}(\Omega) \). Thus, collecting (3.64)–(3.67), we have
\[
\frac{1}{2} \frac{d}{dt} \left( \int_\Omega |\nabla (1 + \mu)|^2 + \delta \int_F (1 + \mu)^2 - 2(\rho_0, 1 + \mu)_m \right) + \|\nabla \rho\|^2 + \sigma \|\nabla \Gamma \rho\|_{\Omega}^2 \leq c_1\|\rho\|_{\Omega}^2. \]
(3.68)
We now consider the linear operator \( B : H^2(\Omega) \cap H_0^1(\Omega) \to H \) defined by
\[ Bu = h \iff -\Delta u = h \quad \text{and } u|_\Gamma = 0. \]
(3.69)
Then, we integrate the difference of (3.24) [written for both solutions] in time and test the resulting equality with \( KB^{-1} \rho, \text{ for } K > 0 \) to be chosen, to obtain
\[
K\|\rho\|^2_{H^{-1}(\Omega)} \leq K \int_\Omega |\nabla (1 + \mu) \cdot \nabla B^{-1} \rho| + K(\rho_0, B^{-1} \rho)_m \]
\[
\leq K \|\rho\|^2_{H^{-1}(\Omega)} + cK\|\nabla (1 + \mu)\|^2 + cK\|\rho_0\|^2_{H^{-1}(\Omega)}. \]
(3.70)
Thus, summing (3.68) and (3.70), choosing \( K \geq 4c_1 \), applying Gronwall’s lemma, and recalling (3.65), we finally obtain the contractive estimate
\[
\|\rho\|^2_{L^2(0,T;V_2)} \leq c(\|\rho\|^2_{L^2(0,T,H^{-1}(\Omega))} + \|\nabla \rho\|^2_{L^2(0,T,H)} + \sigma \|\nabla \Gamma \rho\|^2_{L^2(0,T,H)}) \]
\[
\leq cT (\|\rho_0\|^2_{H^{-1}(\Omega)} + \|\rho_0\|^2_{V_2}), \]
(3.71)
whence (3.27) follows by observing that \( \|\cdot\|_{H^{-1}(\Omega)} \leq \epsilon \|\cdot\|_{V_2} \). In particular, in the case when \( \rho_{0,1} = \rho_{0,2} \), then \( \rho_1 = \rho_2 \) almost everywhere, i.e., we have the uniqueness of \( \rho \). On the other hand, our argument only implies that \( \nabla (1 + \mu) = 0 \) almost everywhere, whence \( \mu_1 \) differs from \( \mu_2 \) by a function which is constant in space, but may be time dependent. However, the possible multivalued character of \( F_w \) prevents us from using Eq. (3.25) to prove that such a constant vanishes.

## 3.2. Regularization properties of the solutions

**Proof of Theorem 3.5.** The procedure is standard. It is however worth noting that it is still necessary to proceed by working on the approximate statement and then passing to the limit. The reason for this is that we need to consider the operator \( F \) in its “strong” form in order to use pointwise arguments. That said, it is sufficient to combine the “energy estimate” and the “further estimate”. More precisely, we sum (3.40) and \( \eta \) times (3.45), where \( \eta \in (0,1) \) will be chosen later. Then, proceeding as in (3.46)–(3.48), it is not difficult to find
\[
\frac{d}{dt} \left( \epsilon + c_1 \|\rho\|_{\Omega}^2 + \sigma \|\nabla \Gamma \rho\|_{\Omega}^2 \right) \]
\[
+ \sigma \|\nabla \Gamma \rho\|_{\Omega}^2 \leq c_2 \|\nabla \Gamma \rho\|_{\Omega}^2 \]
(3.72)
for some \( c_2 > 0 \). From this point on, the constants \( \theta, c, \text{ and } c_1 \text{ are allowed to depend on the initial datum through } m_0 = m(\rho_0) \text{ only}. \text{ Then, the right-hand side of (3.72) can be estimated as follows:}
\[
\leq c\eta\|\nabla \mu\|_{\Omega}^2 + c\|\rho\|_{\Omega}^2 \leq c_2 \eta^2\|\nabla \mu\|_{\Omega}^2 + c_2^2 \eta^2\|\rho\|_{\Omega}^2 + c\|\nabla \rho\|_{\Omega}^2 \]
(3.73)
Now, thanks to assumption (1g2), \( F \) is superquadratic and, consequently, there exists \( c_3 \) depending on \( c_2 \) and \( \theta \), but not on \( \eta \), such that
\[
\frac{\kappa \eta}{2} \int_\Omega F(\rho) - c_3 \eta^2 \|\rho\|_{\Omega}^2 \geq c_3 \]
(3.74)
for all \( \eta \in (0,1) \). Thus, choosing \( \eta \) small enough that \( c_2 \eta^2 \leq 1/2 \), we finally end up with
\[
\frac{d}{dt} \left( \epsilon + c_1 \|\rho\|_{\Omega}^2 + \sigma \|\nabla \Gamma \rho\|_{\Omega}^2 \right) \]
\[
+ \frac{\kappa \eta}{2} \left( \int_\Omega |f(\rho)| + F(\rho) \right) \leq c. \]
(3.75)
Recalling (2.7), we obtain, in particular,
\[
\frac{d}{dt} \left( \epsilon + \frac{\kappa \eta}{2} \|\rho\|_{\Omega}^2 + \kappa \|\nabla \mu\|_{\Omega}^2 + \kappa \delta \|\nabla \Gamma \rho\|_{\Omega}^2 \right) \]
\[
+ \kappa \left( \int_\Omega |f(\rho)| + \int_\Omega |g(\rho)| \right) \leq c \]
(3.76)
for some (new) value of \( \kappa > 0 \). Then, Gronwall’s lemma immediately gives the dissipative estimate.
\begin{align*}
\mathcal{E}(\rho(t)) & + \kappa \int_t^{t+1} \left( \left\| \nabla \mu \right\|^2 + \delta \left\| \nabla \mu \right\|^2 \right) \\
& + \int_{\Omega} \left[ f'(\rho) \rho - g(\rho) \right] \leq \mathbb{E}_0 \kappa \rho^3 + c, \quad \forall t \geq 0,
\end{align*}
whence the thesis of the theorem. \hfill \Box

**Proof of Theorem 3.6.** The main ingredient is a new a priori estimate (as above, we derive it formally, although we should still work on some kind of approximation). We compute the time derivative of (3.9),

\begin{align*}
\int_{\Omega} \mu \phi + \int_{\Omega} \mu \rho w &= \int_{\Omega} \nabla \rho \cdot \nabla \phi + \int_{\Omega} f'(\rho) \rho \phi \\
& + \sigma \int_{\Omega} \nabla \rho \cdot \nabla \mu \\
& + \int_{\Omega} g'(\rho) \rho \phi.
\end{align*}

Then, we take \( \phi = \rho_t \) in the above formula and also set \( \psi = \mu_t \) in (3.10). This clearly gives

\begin{align*}
\frac{d}{dt} \left( \frac{1}{2} \left\| \nabla \mu \right\|^2 + \frac{\delta}{2} \left\| \nabla \rho \right\|^2 \right) &+ \left\| \nabla \rho_t \right\|^2 + \sigma \left\| \nabla \rho \right\|^2 \\
& \leq -\lambda \left( \left\| \rho_t \right\|^2 + \left\| \rho_t \right\|^2 \right).
\end{align*}

To estimate the right-hand side, we proceed as in the proof of uniqueness. More precisely, we first observe that, at least at the approximate level, \( \rho_t \in H \), say, a.e. in time. Thus, we can test (3.13) with \( B^{-1} \rho_t \) (cf. (3.69)). As in (3.70), we obtain

\begin{align*}
\left\| \rho_t \right\|^2_{H^{-1}(\Omega)} &= -\int_{\Omega} \nabla \mu \cdot \nabla B^{-1} \rho_t \leq \left\| \nabla \mu \right\| \left\| \nabla \rho_t \right\| \\
& \leq \frac{1}{2} \left\| \rho_t \right\|^2_{H^{-1}(\Omega)} + c \left\| \nabla \mu \right\|^2.
\end{align*}

Summing now (3.79) and K times (3.80), with \( K > 0 \) to be chosen, and noting that (cf. (3.67))

\begin{align*}
\lambda \left\| \rho_t \right\|^2 + \lambda \left\| \rho_t \right\|^2 \leq \frac{1}{2} \left\| \nabla \rho_t \right\|^2 + c_4 \left\| \rho_t \right\|^2_{H^{-1}(\Omega)},
\end{align*}

we can take \( K \geq 4c_4 \). Then, the uniform Gronwall lemma (see, e.g., [37, Lemma 1.1.1]), also on account of the dissipative estimate (3.77), gives (3.31). Moreover, we obtain a locally uniform bound on the gradient of \( \mu \). To have the full \( V_q \)-norm of \( \mu \), we first have to come back to the “further estimate” and notice that, thanks to the above procedure, (3.49) can now be improved (and rephrased) as

\begin{align*}
\left\| f'(\rho) \right\|_{L^\infty(\Omega \times (\tau, \tau + 1))} + \left\| g(\rho) \right\|_{L^\infty(\Omega \times (\tau, \tau + 1))} \leq Q (\tau^{-1}, \mathbb{E}_0).
\end{align*}

Then, we go back to the “estimate of \( \mu \)” and repeat (3.50)–(3.53), but without integrating in time. We then obtain, pointwise in time,

\begin{align*}
\left\| \mu \right\|_V & \leq c \left\| f'(\rho) \right\|_{V(\Omega)} + c \left\| g(\rho) \right\|_{V(\Omega)} \\
& + c \left( \left\| \nabla \rho_t \right\| + \left\| \mu \right\|_{V(\Omega)} \right).
\end{align*}

Finally, for any \( t \geq \tau > 0 \), we can compute the essential supremum of the right-hand side of (3.83) over \( (t, t+1) \). By (3.77) and (3.82), this gives (3.32) and concludes the proof. \hfill \Box

**Proof of Theorem 3.7.** Let us now fix \( \tau > 0 \) and take \( \tau \geq \tau \) so that properties (3.31)–(3.32) hold. We then consider, for \( \tau \geq \tau \), the elliptic system

\begin{align*}
-\Delta \rho + b(\rho) = \mu + \lambda \rho, \\
-\sigma \Delta \rho + g(\rho) = w \mu - \delta \rho,
\end{align*}

where we have set \( b(r) := f(r) + \lambda r, \) for \( r \in I \). Notice that, in the framework of weak solutions, we are not allowed to intend (3.84)–(3.85) in the present (strong) form. However, this may be done at the approximate level. For simplicity, let us proceed formally and look for (pointwise in time) estimates of the nonlinear terms. In what follows, we denote by \( C \) a constant behaving as the right-hand side of (3.31)–(3.32), i.e., monotonically depending on \( \mathbb{E}_0 \) and \( \tau^{-1} \).

Then, we test both (3.84) and (3.85) by \( b(\rho) \) and integrate by parts. We obtain

\begin{align*}
\left\| b(\rho) \right\|^2 + \int_{\Omega} g(\rho) b(\rho) + \int_{\Omega} b'(\rho) \nabla \rho \cdot \nabla \rho \leq \int_{\Omega} (\mu + \lambda \rho) b(\rho) + \int_{\Omega} w \mu b(\rho).
\end{align*}

Let us first provide an estimate for the right-hand side. We first note that

\begin{align*}
\int_{\Omega} (\mu + \lambda \rho) b(\rho) & \leq \frac{1}{4} \left\| b(\rho) \right\|^2 + c \left( \left\| \mu \right\|^2 + \left\| \rho \right\|^2 \right) \\
& \leq \frac{1}{4} \left\| b(\rho) \right\|^2 + C,
\end{align*}

thanks to (3.77) and (3.32). Next, we observe that

\begin{align*}
\int_{\Omega} w \mu b(\rho) & \leq \left\| w \mu \right\|_{L^p(\Omega)} \left\| b(\rho) \right\|_{L^q(\Omega)} \\
& \leq c \left( \left\| b(\rho) \right\|^2 + \left\| b'(\rho) \right\|_{L^q(\Omega)} \right).
\end{align*}

Next, we notice that

\begin{align*}
\left\| b'(\rho) \nabla \rho \right\|_{L^q(\Omega)} \leq \left\| b'(\rho) \right\|_{L^2(\Omega)} \left\| b(\rho) \right\|_{L^q(\Omega)} + 1
\end{align*}

and, of course, we also have

\begin{align*}
\left\| w \mu \right\|_{L^q(\Omega)} \left\| b(\rho) \right\| \leq \frac{1}{4} \left\| b(\rho) \right\|^2 + \left\| w \mu \right\|^2_{L^q(\Omega)}.
\end{align*}

Thus, collecting (3.88)–(3.91), we end up with

\begin{align*}
\int_{\Omega} \left( \mu + \lambda \rho \right) b(\rho) \leq \frac{1}{2} \left\| b'(\rho) \right\|^2 + \frac{1}{4} \left\| b(\rho) \right\|^2 \\
+ c \left\| \mu \right\|_{L^q(\Omega)} \left( \left\| b'(\rho) \right\|_{L^q(\Omega)} + 1 \right) + C
\end{align*}

and, expressing \( q \) in terms of \( p \), we finally have

\begin{align*}
\left\| b(\rho) \right\|^2 + \int_{\Omega} g(\rho) b(\rho) + \int_{\Omega} b'(\rho) \nabla \rho \cdot \nabla \rho \\
+ \sigma \int_{\Omega} b'(\rho) \nabla \rho^2 \\
\leq c \left\| w \mu \right\|_{L^q(\Omega)} \left( \left\| b'(\rho) \right\|_{L^q(\Omega)} + 1 \right) + C
\end{align*}

where \( \eta > 0 \) is a (small) constant to be chosen. Now, the procedure differs, depending on the value of \( \delta \). Indeed, if \( \delta > 0 \), we obtain, from (2.2) and (3.32),

\begin{align*}
\left\| w \mu \right\|_{L^q(\Omega)} \leq C_\rho, \quad \forall p \in (4/3, +\infty),
\end{align*}
the constant $\mathcal{C}_p$ depending on $p$. Thus, by (3.33) (which holds, of course, also for $b$, since it is a linear perturbation of $f$), we can take $p$ large enough so that
\[
\eta \int_\Omega |b'(\rho)| \, d\rho \leq \mathcal{C}_p \int_\Omega |b(\rho)| \, d\rho + c
\]
and the right-hand side is controlled by the first term in (3.93). Instead, if $\delta = 0$, we just have
\[
\|w\|_{H^1(T)} \leq c,
\]
thanks to the continuity of the trace operator from $V$ to $L^2(T)$. Thus, we take $p = 4$ and obtain, using (3.34) (which still holds for $b$),
\[
\eta \int_\Omega |b'(\rho)|^2 \leq \mathcal{C}_p \int_\Omega |b(\rho)|^2 + c \leq \frac{1}{4} \int_\Omega |b(\rho)|^2 + c,
\]
provided that we choose $\eta$ small enough. Therefore, in both cases, we can control the right-hand side of (3.86) (or, equivalently, of (3.93)).

To obtain an estimate from (3.93), we still have to control the second term on the left-hand side. Actually, if $f$ is a “regular” function (i.e., $f = \mathbb{R}$), the monotonicity of $b$ and the second of (fg9) show that $b$ and $g$ have the same sign for $|r|$ large enough. Thus, clearly, thanks also to (3.35),
\[
\int_\Omega g(r)b(r) \leq \kappa \|g(r)\|^2 - c.
\]
If $f \neq \mathbb{R}$, we have to estimate this term directly. Noting, that by (fg1) $g$ is uniformly bounded on $\bar{T}$ and using Gagliardo’s trace theorem (cf., e.g., [38]), we obtain
\[
\int_\Omega g(r)b(r) \leq c\|b(r)\|_{L^1(\Gamma)} + c\|b(\rho)\|_{W^{1,1}(\Omega)}
\]
and the last term on the right-hand side is estimated, for $\eta$ small enough, by the third term on the left-hand side of (3.93), while the first two terms, as above, can be controlled by the first term in (3.93), thanks to (3.33) or (3.34) (we leave the details to the reader).

Since the above estimates hold pointwise in time, passing to the supremum with respect to $t \in [\tau, +\infty)$ and recalling the “meaning” of the constant $C$, we have
\[
\|b(\rho)\|_{L^\infty(t, +\infty; H^1(\Omega))} \leq Q(\mathbb{E}_0, \tau^{-1}).
\]
Moreover, thanks to the boundedness of $\rho$ for “singular” $f$ or to (3.98) (for “regular” $f$), we also have
\[
\|g(\rho)\|_{L^\infty(t, +\infty; H^1(\Omega))} \leq Q(\mathbb{E}_0, \tau^{-1}).
\]
In particular, with (3.100)-(3.101) at our disposal, at least for $\tau > 0$ we can identify the nonlinear terms with respect to the topology of $\mathcal{H}$ (rather than in the duality between $V^*_\tau$ and $V_\tau$ as we did in (3.59)-(3.63)). In other words, (3.37) holds and we can replace $F_\tau(\rho)$ with $\mathcal{F}_\tau(\rho)$ in (3.25). Hence, (3.24)-(3.25) can be rewritten as (3.13)-(3.14) or, equivalently, (3.9)-(3.10).

In particular, (3.9) can be seen as a weak formulation of (3.84)-(3.85). Then, we can apply standard elliptic regularity results (see, e.g., [20, Lemma 2.2 and Rem. 2.3] for a precise statement) to obtain an $H^1$-estimate for $\rho$ if $\sigma > 0$ and an $H^{1/2}$-estimate if $\sigma = 0$. Namely, we have proved (3.36). At this point, if we integrate by parts the gradient terms in (3.9)-(3.10), estimates (3.31)-(3.32), (3.100)-(3.101), and (3.36) allow us to prove, by comparison arguments, that
\[
\|\Delta \rho\|_{L^\infty(t, +\infty; H^1)} + \sup_{t \geq \tau} \|\Delta \mu\|_{L^2(t, +\infty; V)} \leq Q(\mathbb{E}_0, \tau^{-1})
\]
and to give sense to the normal derivatives. More precisely, by the first of (3.102), (3.36), and proper trace theorems, we have
\[
\|\partial_\nu \rho\|_{L^\infty(t, +\infty; H^{-1/2}(\Gamma))} + \eta^{1/2} \|\partial_\nu \rho\|_{L^\infty(t, +\infty; H^{1/2}(\Gamma))} \leq Q(\mathbb{E}_0, \tau^{-1}),
\]
From this relation, we can see that, for all $t > 0$, (1.4) makes sense as an equality in $H^{-1/2} \cap H^{1/2}$ (actually, in the case $\sigma = 0$, we first obtain it in the space $H^{-1/2} \cap H^{1/2}$). Then, we see that it holds in $H^{-1/2} \cap H^{1/2}$ by simply comparing terms). Analogously, the second of (3.102) gives
\[
\sup_{t \geq \tau} \|\partial_\nu \mu\|_{L^2(t, +\infty; H^{-1/2}(\Gamma))} \leq Q(\mathbb{E}_0, \tau^{-1}),
\]
so (1.3) can be interpreted at least in $H^{-1/2} \cap H^{1/2}$. Actually, we can say more. Indeed, in the case $\delta = 0$, $\partial_\nu \mu \equiv -w_\rho \mu$; thus, (1.3) holds by (3.36). The same holds for $\delta > 0$, since we can view (1.1) + (1.3) as a coupled (time-dependent) elliptic system and apply once more [20, Lemma 2.2]. Hence, we have, in all cases,
\[
\sup_{t \geq \tau} \|\partial_\nu \mu\|_{L^2(t, +\infty; H^{-1/2}(\Gamma))} \leq Q(\mathbb{E}_0, \tau^{-1})
\]
and, summarizing, (3.9)-(3.10) can be interpreted in the strong form (1.1)-(1.4).

To conclude the proof, we have to show (3.38), which is, however, immediate. Actually, in the singular case, it is clear that $|F(r)| \leq c(1 + |b(r)|)$ for any $r \in (1, \infty)$, whence (3.38) holds for $p_1 = 2$, thanks to (3.100). In the regular case, i.e., if $f = \mathbb{R}$, using the monotonicity of $b$, we have, instead,
\[
F(r) = \int_0^r (b(s) - \lambda s)ds \leq |\lambda r| + cr^2,
\]
whence (3.38) now follows from (3.100), (3.36), and interpolation (again, for $p_1 = 2$ if $\sigma > 0$ and for all $p_1 \in (1, 2)$ if $\sigma = 0$). The proof of the theorem is complete. □

**Remark 3.13.** If the initial datum is more regular, one can use the standard (instead of the uniform) Gronwall lemma in the proof of Theorem 3.6 and see that estimates (3.31)-(3.32) (as well as all the subsequent ones) also hold for $\tau = 0$, with the right-hand sides replaced by a quantity suitably depending on some better norm of the initial datum. To be more precise, what is needed is that the term under the time derivative on the left-hand side of (3.79) is controlled at $t = 0$, namely,
\[
\rho|_{t=0} \in V_\tau.
\]
Notice, however, that $\rho|_{t=0}$ is not prescribed as an initial datum and, in fact, (3.107) has to be obtained by asking for more regularity on $\rho_0$ (both in $\Omega$ and on $\Gamma$) and deducing the corresponding regularity on $\mu|_{t=0}$ by comparison in (1.2) and in (1.4) evaluated at $t = 0$. We leave the details to the reader.

**Remark 3.14.** The growth condition (3.33) or (3.34) could actually be relaxed a bit. Indeed, in the above procedure, one could test the equations with $|b(\rho)|^s \mathrm{sgn}(b(\rho))$ for a suitable $s \in (0, 1)$ in order to look for an estimate of $b(\rho)$ in $L^{s+1}(\Omega)$, rather than in $H$. Such a weaker estimate is compatible with a (slightly) more general class of nonlinearities; it is anyway still sufficient, of course, to interpret the system in the stronger form. We do not give the details, since the procedure would involve a number of boring technical complications to provide just a small extension of the result.
3.3. Characterization of $\omega$-limit sets

In this final subsection, we deal with the long-time behavior of single trajectories and use the well-posedness and regularization results to characterize their $\omega$-limit sets. We will always assume that the hypotheses of Theorem 3.7 (among which, in particular, (3.33)-(3.35)) hold, so that the evolutionary system can be interpreted in the strong form (1.1)-(1.4), at least for $t > 0$.

Our first step consists in analyzing the stationary problem associated with (1.1)-(1.4). As far as the $\mu$-component is concerned, it reads

$$-\Delta \mu_\infty = 0, \quad \text{in } \Omega,$$

$$-\delta \Delta_G \mu_\infty = -\delta_a \mu_\infty, \quad \text{on } \Gamma, \quad (3.109)$$

and, clearly, $\mu_\infty$ solves the system if and only if it is a constant function. Thus, the $\mu$-system becomes

$$-\Delta \rho_\infty + f(\rho_\infty) = \mu_\infty, \quad \text{in } \Omega,$$

$$-\sigma \Delta_G \rho_\infty + g(\rho_\infty) = w \rho_\infty - \delta_a \rho_\infty, \quad \text{on } \Gamma, \quad (3.111)$$

where $\mu_\infty$ is a constant which is, at least a priori, undetermined.

We can now prove the existence of (nonempty) $\omega$-limit sets of weak solutions.

**Theorem 3.15.** Let the assumptions of Theorem 3.7 hold and let $(\rho, \mu)$ be a solution. Then, the function $t \mapsto \rho(t)$ possesses a nonempty $\omega$-limit set which only consists of solutions $\rho_\infty$ of the stationary problem (3.110)-(3.111). More precisely, for any sequence $(t_n)$ diverging to $+\infty$, there exist a nonrelabeled subsequence and a limit $\rho_\infty$ (3.112) such that

$$\rho(t_n) \to \rho_\infty, \quad \mu(t_n) \to \mu_\infty \quad (3.112)$$

in suitable topologies. Moreover, any such $\rho_\infty$ satisfies, in addition,

$$m(\rho_\infty) = m_0, \quad (3.113)$$

where $m_0$ is the dm-average of the initial datum $\rho_0$.

**Proof.** Let $\{t_n\} \subset [0, +\infty)$ be any sequence of times diverging to $+\infty$. Then, thanks to (3.36), we see that, up to a (nonrelabeled) subsequence, $(\rho(t_n))$ tends to some limit $\rho_\infty$ strongly in $V$. We can set, for $t \in [0, 1)$, $(\rho_n, \mu_n)(t) \equiv (\rho, \mu)(t_n + t)$ and notice that $(\rho_n, \mu_n)$ solves, for $t \in [0, 1)$, problem (1.1)-(1.4). Moreover, it satisfies the "initial" condition $\rho_n|_{t=0} = \rho(t_n)$. Furthermore, due to (3.36), it holds that

$$\rho_n \to \widehat{\rho} \quad \text{weakly star in } L^\infty(0, 1; V), \quad (3.114)$$

while (3.32) entails

$$\mu_n \to \widehat{\mu} \quad \text{weakly star in } L^\infty(0, 1; V). \quad (3.115)$$

Additionally, integrating (3.40) over $(0, +\infty)$, we find

$$\int_0^{+\infty} \|\nabla \mu\|^2 + \delta \|\nabla_G \mu\|^2 \leq E_0, \quad (3.116)$$

which, using the continuity of the linear operator $A_3$ in (3.24), also entails

$$\int_0^{+\infty} \|ho_n\|_{V_0}^2 \leq Q(E_0). \quad (3.117)$$

Consequently, we have

$$\rho_n \to 0 \quad \text{strongly in } L^2(0, 1; V_0), \quad (3.118)$$

which yields that $\widehat{\rho}$ in (3.114) is constant in time and, hence, coincides with $\rho_\infty = \lim \rho_n(0)$.

At this point, we can take the limit of (1.1)-(1.4), written for $(\rho_n, \mu_n)$, over the time interval $(0, 1)$. To do this, we first notice that, by (3.114), (3.100)-(3.101),

$$b(\rho_n) \to \overline{b} \quad \text{weakly star in } L^\infty(0, 1; H), \quad (3.119)$$

$$g(\rho_n) \to \overline{g} \quad \text{weakly star in } L^\infty(0, 1; H_{\Gamma}). \quad (3.120)$$

Moreover, by (3.114), (3.118), and the Aubin–Lions lemma,

$$\rho_n \to \rho_\infty \quad \text{strongly in } L^2(0, 1; H). \quad (3.121)$$

Thanks to the standard monotonicity argument [36, Prop. 1.1, p. 42], we then have

$$\overline{b} = b(\rho_\infty), \quad \text{a.e. in } (0, 1) \times \Omega \quad (3.122)$$

and, analogously, $\overline{g}$ coincides a.e. on $\Gamma$ with $g(\rho_\infty)$.

Summarizing, we see that $\rho_\infty$ and $\widehat{\mu}$ solve, a.e. in the time interval $(0, 1)$, the system

$$-\Delta \widehat{\mu} = 0, \quad \text{in } \Omega, \quad (3.123)$$

$$-\delta \Delta_G \widehat{\mu} = -\delta_a \rho_\infty, \quad \text{on } \Gamma, \quad (3.124)$$

$$-\Delta \rho_\infty + f(\rho_\infty) = \widehat{\mu}, \quad \text{in } \Omega, \quad (3.125)$$

$$-\sigma \Delta_G \rho_\infty + g(\rho_\infty) = w \rho_\infty - \delta_a \rho_\infty, \quad \text{on } \Gamma. \quad (3.126)$$

From (3.125)-(3.126), it is then clear that also $\mu_\infty$ is constant with respect to time. Thus, we can denote it by $\rho_\infty$. Moreover, (3.123)-(3.124) imply that $\mu_\infty$ is also constant with respect to the space variables. Then, (3.124)-(3.125) actually reduces to (3.109)-(3.110). Finally, since $m(\rho(t_n)) = m_0$ for all $n$, thanks to (3.44), (3.113) immediately follows by letting $n \to +\infty$, which concludes the proof. \qed

As a next step, we analyze a bit more carefully the set of the solutions of the stationary problem (3.110)-(3.111) which can be reached in the $\omega$-limit. We first have:

**Lemma 3.16.** Let the assumptions of Theorem 3.7 hold and let $(\rho, \mu)$ be a solution. Then, there exists a constant $c_5$ depending on the initial datum only such that, for any $\mu_\infty$ in the $\omega$-limit set of $\rho$, it holds

$$|\mu_\infty| \leq c_5. \quad (3.127)$$

**Proof.** We test both (3.110) and (3.111) with $\rho_\infty - m_0$, where $\rho_\infty$ lies in the $\omega$-limit set of $\rho$ (and, hence, satisfies, in addition, (3.113)). Proceeding as in (3.48), we then have

$$\|\nabla \rho_\infty\|^2 + \sigma \|\nabla_G \rho_\infty\|_{\Gamma}^2 + \int_\Omega (|f(\rho_\infty)| + F(\rho_\infty)) + \int_\Gamma (|g(\rho_\infty)| + G(\rho_\infty)) \leq \mu_\infty - m_0 + c = c. \quad (3.128)$$

Indeed, the latter scalar product vanishes, since $\mu_\infty$ is a constant and (3.113) holds. Consequently, we have

$$\|\rho_\infty\|_{V_0}^2 + \|f(\rho_\infty)\|_{H(I)} + \|g(\rho_\infty)\|_{H_{\Gamma(I)}} \leq c. \quad (3.129)$$

The thesis then follows by integrating both (3.110) and (3.111) with respect to the space variables and using (3.129). \qed

Our next aim is to show that there exists a compact subinterval $I_1 \subset I$ such that any element $\rho_\infty$ of the $\omega$-limit set can only take values in $I_1$. In other words, $\rho_\infty$ is bounded in the regular case and is "separated from the singularities" in the singular one. However, in the singular case this requires some additional hypotheses.

**Lemma 3.17.** Let the assumptions of Theorem 3.7 hold and let $(\rho, \mu)$ be a solution. Let us also assume that either $f$ is regular (i.e., $f \in \mathbb{R}$)
or there exists $\epsilon > 0$ such that
\[
g(r) - w^* c_0 \geq 0 \quad \forall r \geq r^* - \epsilon,
\]
\[
g(r) + w^* c_0 \leq 0 \quad \forall r \leq r_0 + \epsilon,
\]
where $c_0$ is the constant in (3.127) and we have written $I = (r_0, r^*)$.
Then, there exists a compact subinterval $I_1 \subset I$ such that any element
\[
\rho \in \text{the } \omega\text{-limit set of } \rho \text{ takes values in } I_1,
\]
Proof. We test (3.110) with $(\rho - M)^+$ for $M > 0$ to be chosen. Using (3.111), we then have
\[
\|\nabla (\rho - M)^+ \|^2 + \sigma \|\nabla F (\rho - M)^+ \|_2^2
+ \int_I (f(\rho^*)) - \mu_\infty (\rho - M)^+ + \int_I (g(\rho^*)) - w \mu_\infty (\rho - M)^+ = 0.
\]
It is now clear that, if $f$ is regular, then, thanks to (2.2), (fg2) and (3.127), we can always choose $M$ large enough that the last two terms in the left-hand side are nonnegative. On the other hand, if $f$ is singular, then $M$ has to stay in $I$. Thus, nothing changes in what relates to the third term in (3.131) which can still be made nonnegative by taking $M$ close enough to $r^*$. Concerning, instead, the boundary term, in order to ensure the same, we need the first of (3.130). The lower bound is then obtained in a similar way by testing (3.110) with $-(\rho^* + M)^-$ and using the second inequality of (3.130). \qed

Remark 3.18. Since the constant $c_0$ depends, in particular, on $m_0$, (3.130) can be seen as an assumption of compatibility between $g$, $m_0$, and $w$ which is more likely to be satisfied when $m_0$ is close to 0. Of course, it may very well happen that there exists some nonlinearity $g$ satisfying all other hypotheses, but for which (3.130) holds for no value of $m_0$. In such a case, the elements of the $\omega$-limit set may not be separated from the singularities of $f$ and the subsequent analysis cannot be performed.

We now show that, under the same hypotheses on the nonlinear terms, also the solutions of the evolution problem are separated from the singularities of $f$, at least for sufficiently large times.

Lemma 3.19. Let the assumptions of Lemma 3.17 (among which, in particular, is (3.130), in the case of a singular function $f$) hold and let $(\rho, \mu)$ be a solution. Then, there exist a compact subinterval $I_2 \subset I_1$ and a time $T_1$ depending on the solution such that $\rho(t, x) \in I_2$ for almost all $x \in \Omega$ and $t \geq T_1$.

Proof. We only give the proof in the case $\sigma > 0$, which is more immediate. Actually, if $f$ is regular, i.e., $I = \mathbb{R}$, then the result directly follows from (3.36) and the continuous embedding
\[
H^2(\Omega) \subset C^{0,\alpha}(\overline{\Omega}), \quad \alpha \in (0, 1/2).
\]
In the case of a singular function $f$, the assertion follows from the uniform separation of any element of the $\omega$-limit set (cf. Lemma 3.17), from relation (3.36) again, and from the embedding (3.132) which entails that the elements of the $\omega$-limit set are reached with respect to the uniform convergence. So, one can take any compact subinterval $I_2$ of $I$ such that $I_1$ is contained in the interior of $I_2$.

As regards the case $\sigma = 0$, the same argument does not apply, since, from (3.36), we can only obtain precompactness in the $H^{1/2}$-norm, which does not entail uniform convergence. Nonetheless, this problem can be overcome by proving a further estimate on $f(\rho)$, whose details can be found in [9, Lemma 6.3]. \qed

Lemma 3.20. Let the assumptions of Lemma 3.17 hold and let $(\rho, \mu)$ be a solution. Additionally, assume that
\[
f \in C^{1,1}_{bc}(I, \mathbb{R}), \quad g \in C^{1,1}_{bc}(\mathbb{R}, \mathbb{R}), \quad w \in W^{1,\infty}(\Gamma).
\]
Then, there exist a time $T_2$ and a constant $C$ depending on the solution such that
\[
\|\rho\|_{L^\infty(T_2, +\infty; \mathcal{H}^{1/2}(\Omega))} + \sigma^{1/2}\|\rho\|_{L^\infty(T_2, +\infty; \mathcal{H}^{1/2}(\Omega))} \leq C.
\]
Proof. We take, for $t \geq T_1$, the time derivatives of (3.9) and of (3.10) and set $\phi = \rho$ and $\psi = \mu$. Using the previous lemma and the local Lipschitz continuity of $f'$ and $g'$ given by (3.133), we then have
\[
\frac{1}{2} \int_\Omega \left(\|\rho_t\|_2^2 + \|\nabla F (\rho)^+\|_2^2 + \int_\Omega f(\rho) \rho_t^2 + \int_\Omega g'(\rho) \rho_t^2\right)
+ \frac{1}{2} \int_\Omega f''(\rho) \mu_\infty^2 + \frac{1}{2} \int_\Omega g''(\rho) \mu_\infty^2
+ C(\|\rho\|_{L^2(I_2)} + \|\rho\|_{L^2(I_2)}) \leq C \|\rho\|_2^2.
\]
Since the term under the time derivative may be unbounded from below, due to the nonmonotonicity of $f$ and $g$, we add to (3.135) the inequality
\[
K \frac{d}{dt} \|\rho\|_{H^{-1}(\Omega)} \leq c K \|\rho\|_{H^{-1}(\Omega)}^2 + \eta \|\nabla \mu\|_2^2.
\]
where the last passage follows by computing the $H^{-1}$-norm of $\rho_t$ by using (the time derivative of) Eq. (1.1). Then, taking $\eta > 0$ small enough and $K > 0$ large enough, the sum of the terms under time derivatives in (3.135) and (3.136) becomes nonnegative. Thus, recalling (3.31) and using the uniform Gronwall lemma, we obtain
\[
\|\rho_t\|_{L^\infty(T_2, +\infty; V_t)} \leq C.
\]
e.g., for $T_2 = T_1 + 1$. Next, we consider (1.1) + (1.3) as a coupled time-dependent linear elliptic system, where, even in the worse case $\sigma = 0$, the forcing terms satisfy, thanks also to the last of (3.133),
\[
\|\rho_t\|_{L^\infty(T_2, +\infty; V_t)} + \|w\rho_t\|_{L^\infty(T_2, +\infty; H^2(\Omega))} \leq C.
\]
Hence, by elliptic regularity (cf. [20, Lemma 2.2]) and standard bootstrap arguments, we have, at least,
\[
\|\mu\|_{L^\infty(T_2, +\infty; H^2(\Omega))} + \delta^{1/2}\|\mu\|_{L^\infty(T_2, +\infty; H^1(\Omega))} \leq C.
\]
We then turn to (1.2) + (1.4) and note that, even in the worse case $\sigma = \delta = 0$, the forcing terms satisfy
\[
\|\mu - f(\rho)\|_{L^\infty(T_2, +\infty; V_t)} + \|w\mu - g(\rho)\|_{L^\infty(T_2, +\infty; V_t)} \leq C.
\]
A further application of elliptic regularity results then yields (3.134). \qed

Next, let us consider a solution $(\rho, \mu)$ under the assumptions of Lemma 3.20. Due to this lemma, $\rho$ eventually is precompact in $H^1(\Omega)$.

Then, we denote by $\mathbb{S}$ the family of the stationary solutions $\rho_\infty$, satisfying the constraint (3.113). In the case of a singular function $f$, all elements of $\mathbb{S}$ are separated from the singularities, thanks to Lemma 3.17. Actually, one easily sees that the conclusions of Lemma 3.17 hold for all stationary solutions $(\rho_\infty, \mu_\infty)$ under the constraint (3.113) (and not only for those belonging to the $\omega$-limit set of some weak solution). Moreover, deriving further estimates on the elements of $\mathbb{S}$ as in the proof of Lemma 3.20, it is not difficult to prove that $\mathbb{S}$ is bounded (at least) in $H^{1/2}(\Omega)$. Furthermore, the elements of $\mathbb{S}$ are the only stationary solutions which can belong to the $\omega$-limit set of $\rho$. For simplicity, we restrict ourselves to the case
Notice that the integrations by parts are rigorous, since \( \rho \) eventually lies in \( \mathbb{W} \).

Thus, using (1.2) and (1.4), we obtain
\[
\langle \mathcal{E} (\rho), k \rangle_{\mathbb{W}} = (\mu, k)m = (\mu - \mu_2, k)m.
\]
(3.145)

Indeed, we can subtract \( \mu_2 \) (the "standard" average of \( \mu \) over \( \Omega \)), since \( m(k) = 0 \). Thus, using the Poincaré–Wirtinger inequality and passing to the supremum with respect to \( k \in \mathbb{V}_\mu \) with unit norm, we have
\[
\| \mathcal{E} (\rho) \|_{\mathbb{W}} \leq c \| \mathcal{E} (\rho) \|_{H} \leq c \| \mu - \mu_2 \|_{H} \leq c \| \nabla \mu \|.
\]
(3.146)

Next, estimating the right-hand side of (3.143) with the help of (3.40) and (3.146), we infer the relation
\[
- \frac{\partial_t (\mathcal{E} (\rho) - \varepsilon_\infty)^\theta}{\varepsilon_\infty} \geq \frac{\theta}{\Lambda} \| \nabla \mu \|_2^2 + \delta \| \nabla^2 \mu_2 \|_2^2 - \frac{\Lambda c \| \nabla \mu \|}{c(\theta, \Lambda) \| \nabla \mu \| + \delta^{1/2} \| \nabla \mu \|_r},
\]
(3.147)

which holds for any \( t \geq T_3 \). Integrating over \( (T_3, +\infty) \), we obtain
\[
\nabla \mu \in L^2 (T_3, +\infty; H), \quad \delta^{1/2} \| \nabla \mu \|_r \in L^1 (T_3, +\infty; H_r).
\]
(3.148)

Now, from (1.1) and (1.3), we have, for any \( \phi \in V_\rho \),
\[
\langle \rho_t, \phi \rangle_{\mathbb{V}_\rho} = \langle \rho_t, \phi \rangle_m = \int_\Omega \nabla \mu \cdot \nabla \phi + \delta \int_\Gamma \nabla \mu \cdot \nabla \phi.
\]
(3.149)

Thus, taking \( \phi \in V_\rho \) with unit norm and passing to the supremum, it follows from (3.148) that
\[
\rho_t \in L^1 (T_3, +\infty; V_\rho^g).
\]
(3.150)

which shows that the whole trajectory \( \rho \) converges to a single element \( \rho_\infty \in \mathbb{S} \), as desired.

References


